Once more on Riemann dzeta functions cancellations and manifestations in the multiloop calculations

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Intoductional coments

- What have happened four years later after the personal talk at AQFT-21 "Fixation of definite $\overline{\text{MS}}$-scheme α_s^5 QCD contributions to Adler function and R(s) in the β-expanded representation "
- Work was published as Goriachuk,Kataev, Molokoedov JHEP 05 (2022) 028 (hep-ph/2111.12060)
- Work was cited 13 times and by Pich; Ayala et al; Baikov-Mikhailov; 2 PhD Thesis by Wellman, Calderon; Davier et al; Deur et al; Alam Khan; Mason-Gracey ; Mikhailov; Chistie and AK, Molokoedov (23)
- K.G. Chetyrkin, S. A. Larin and A.K. got 2025 INR Markov Prize

2025 Markov Conference, INR

https://yandex.ru/video/preview/10823430920463103816

Problems to be touched

- Tree branches and Conformal symmetry level related identical (!) cancellations of the perturbative QCD effects ;
- High orders perturbative QCD and Renormalization Group related signals of the conformal symmetry violations ;
- Perturbative QCD (scale-scheme) ambiguities ;
- Axial-Vector-Vector anomaly as the bridge between Deep Inelastic and Annihilation procecceses ;
- Bridges between DI sum rules and e^+e^--annihilation to hadrons D-function (and R-ratio) PT QCD expressions ;
- Form of maxssless analytical expressions of PT coefficients through Riemann ζ-functions $\zeta(n) = \sum_{k\geq 1} k^{-n} = \frac{1}{\Gamma(n)} \int_0^\infty dt \frac{t^{n-1}}{e^t - 1}$ **see also AQFT-25 more theory Kotikov talk**

Processes to be considered

- The process e^+e^- - annihilation to hadrons. It is tested at different colliders. It is possible to consider e^+e^- annihilation into γ^* or Z^0, creating then hadrons. Novosibirsk, Bejing, KEK existing colliders and CEPC, CERN FCC etc.
- DIS processes give possibility to understand better the content of nucleon; JLAB, EIC (BNL)
- Is it possible to relate characteristics of definite annihilation (s-channel) and DIS (t-channel) processes ? The answer is - yes, through applying OPE to the AVV triangle diagram. What are the outcomes ?

Definitions of basic quantities

The e^+e^- to hadrons D function in QCD with $a_s = \alpha_s/\pi$

$$D(a_s(Q^2)) = Q^2 \int_0^\infty ds \frac{R^{th}_{e^+e^-}(a_s(s))}{(s+Q^2)^2} \to Q^2 \int_0^\infty ds \frac{R^{exp+th}_{e^+e^-}(s)}{(s+Q^2)^2},$$

$$R^{th}_{e^+e^-} = \sigma_{tot}^{e^+e^- \to hadrons}(a_s)/(\sigma_0(e^+e^- \to \mu^+\mu^-) = 4\pi\alpha^2/(3s)).$$

$$\left(\frac{\partial}{\partial \ln \mu^2} + \beta(a_s)\frac{\partial}{\partial a_s}\right) D(a_s) = 0, \frac{\partial a_s}{\partial \ln \mu^2} = \beta(a_s)$$

$$D(a_s) = (\mathbf{N_c = 3}) \left(\sum_i q_i^2\right) D^{NS}(a_s) + \left(\sum_i q_i\right)^2 D^{SI}(a_s)$$

While considering $\sigma_0(e^+e^- \to \gamma^* \to \mu^+\mu^-) = 4\pi\alpha^2/(3s)$ we fix RS procedure $\alpha = Z_3(\alpha)\alpha_B$ with $Z_3 = 1$. Then the expression for $R^{th}(s)$ is defined in the model $SU_c(3) + U(1)$ and not $SU_c(3) \times U(1)$ (in THEORY : γ^* (or decaying Z^0) is virtual

$\overline{\text{MS}}$-scheme results for $D^{NS}(a_s) = 1 + \sum_{n\geq 1} d_n a_s^n$

$$d_1 = \frac{3}{4}C_F; d_2 = -\frac{3}{32}C_F^2 - \left(\frac{11}{8} - \zeta_3\right)C_F T_f + \left(\frac{123}{32} - \frac{11}{4}\zeta_3\right)C_F C_A;$$

$$= 1.9857 - 0.1153 n_f; T_f = n_f/2$$

Chetyrkin, Kataev, Tkachov (79); numerical Dine, Sapirstein (1979); analytical Celmaster, Gonsalves (1980); unpublished Ross, Terrano, Wolfram (1978-1980-corrected by Ch,K,T)

$$d_3 = -\frac{69}{128}C_F^3 - \left(\frac{29}{64} - \frac{19}{4}\zeta_3 + 5\zeta_5\right)C_F^2 T_f + \left(\frac{151}{54} - \frac{19}{9}\zeta_3\right)C_F T_f$$
$$-\left(\frac{127}{64} + \frac{143}{16}\zeta_3 - \frac{55}{4}\zeta_5\right)C_F^2 C_A + \left(\frac{90445}{3456} - \frac{2737}{144}\zeta_3 - \frac{55}{24}\zeta_5\right)C_F C_A^2$$
$$-\left(\frac{485}{27} - \frac{112}{9}\zeta_3 - \frac{5}{6}\zeta_5\right)C_F C_A T_f = 18.243 - 4.216 n_f + 0.086 n_f^2$$

Gorishny,K,Larin (87-bug in SCHOONSCHIP programs); corrected and recalculated further in Gorishny,K,Larin (91); Surguladze, Samuel (91); Chetyrkin (97)

d_4 in the $\overline{\text{MS}}$-scheme

$$d_4 = \frac{d_F^{abcd}d_A^{abcd}}{d_R}\left(\frac{3}{16} - \frac{1}{4}\zeta_3 - \frac{5}{4}\zeta_5\right) + n_f\frac{d_F^{abcd}d_F^{abcd}}{d_R}\left(-\frac{13}{16} - \zeta_3 - \frac{5}{2}\zeta_5\right)$$
$$+C_F^4\left(\frac{4157}{2048} + \frac{3}{8}\zeta_3\right) + C_F^3T_f\left(\frac{1001}{384} + \frac{99}{32}\zeta_3 - \frac{125}{4}\zeta_5 + \frac{105}{4}\zeta_7\right)$$
$$+C_F^2T_f^2\left(\frac{5713}{1728} - \frac{581}{24}\zeta_3 + \frac{125}{6}\zeta_5 + 3\zeta_3^2\right)$$
$$+C_FT_f^3\left(-\frac{6131}{972} + \frac{203}{54}\zeta_3 + \frac{5}{3}\zeta_5\right)$$
$$+C_F^3C_A\left(-\frac{253}{32} - \frac{139}{128}\zeta_3 + \frac{2255}{32}\zeta_5 - \frac{1155}{16}\zeta_7\right)$$
$$+C_F^2T_fC_A\left(\frac{32357}{13824} + \frac{10661}{96}\zeta_3 - \frac{5155}{48}\zeta_3 - \frac{33}{4}\zeta_3^2 - \frac{105}{8}\zeta_7\right)+$$

d_4-continuation

$$
\begin{aligned}
&+C_F C_A T_f^2\left(\frac{340843}{5184}-\frac{10453}{288}\zeta_3\right)\\
&+C_F^2 C_A^2\left(-\frac{592141}{18432}-\frac{49325}{384}\zeta_3+\frac{6505}{48}\zeta_5+\frac{1155}{32}\zeta_7\right)\\
&+C_F C_A^2 T_f\left(-\frac{4379861}{20736}+\frac{8609}{77}\zeta_3+\frac{18805}{288}\zeta_5-\frac{11}{2}\zeta_3^2+\frac{35}{16}\zeta_7\right)\\
&+C_F C_A^3\left(\frac{52207039}{248832}-\frac{426223}{3456}\zeta_3-\frac{77995}{1152}\zeta_5+\frac{605}{32}\zeta_3^2-\frac{385}{64}\zeta_7\right)\\
&=135.792-34.440n_f+1.875n_f^2-0.010n_f^3
\end{aligned}
$$

Baikov, Chetyrkin, Kuhn (2008-2010); Confirmed by Herzog, Ruijl, Ueda, Vermaseren, Vogt (2017)

Sum rules of lN and νN deep-inelastic scattering

$$S_{Bjp}(Q^2) = \int_0^1 dx[g_1^{(lp)}(x, Q^2) - g_1^{(ln)}(x, Q^2)] = \frac{1}{6}|\frac{g_A}{g_V}|C_{Bjp}(a_s(Q^2))$$

$$S_{GLS}(Q^2) = \frac{1}{2}\int_0^1 dx[F_3^{(\nu p)}(x, Q^2) + F_3^{(\nu n)}(x, Q^2)] = 3C_{GLS}(a_s)$$

$$C_{Bjp}(a_s) = C^{NS}(a_s) + C^{SI}_{Bjp}(a_s)$$

$$C_{GLS}(a_s) = C^{NS}(a_s) + C^{SI}_{GLS}(a_s)$$

Sum rules are used for extraction of α_s values and study of the contributions of high-twist non-perturbativec effects Non-singlet and singlet coefficient functions should be analysed separately

$\overline{\mathrm{MS}}$-scheme analytical results for $C^{NS}(a_s) = 1 + \sum_{n\geq 1} c_n a_s^n$

$$c_1 = -\frac{3}{4}C_F$$

$$c_2 = \frac{21}{32}C_F^2 + \frac{1}{2}C_F T_f - \frac{23}{16}C_F C_A = -4.583 + 0.333 n_f$$

Gorishny, Larin (1986)

$$c_3 = -\frac{3}{128}C_F^3 - \left(\frac{133}{576} + \frac{5}{12}\zeta_3\right)C_F^2 T_f - \frac{115}{216}C_F T_f^2$$
$$+\left(\frac{1241}{576} - \frac{11}{12}\right)C_F^2 C_A + \left(-\frac{5437}{864} + \frac{55}{24}\zeta_5\right)C_F C_A^2$$
$$+\left(\frac{3535}{864} + \frac{3}{4}\zeta_3 - \frac{5}{6}\zeta_5\right)C_F C_A T_f = -41.4399 + 7.6077 n_f - 0.1775 n_f^2$$

Larin, Vermaseren (1991)

c_4 coefficient in the $\overline{\text{MS}}$-scheme

$$
\begin{aligned}
c_4 = {} & \frac{d_F^{abcd} d_A^{abcd}}{d_R}\left(-\frac{3}{16}+\frac{1}{4}\zeta_3+\frac{5}{4}\zeta_5\right)+n_f\frac{d_F^{abcd} d_F^{abcd}}{d_R}\left(\frac{13}{16}+\zeta_3-\frac{5}{2}\zeta_5\right) \\
& +C_F^4\left(-\frac{4157}{2048}-\frac{3}{8}\zeta_3\right)+C_F^3T_f\left(\frac{839}{2304}+\frac{451}{96}\zeta_3-\frac{145}{24}\zeta_5\right) \\
& +C_F^2T_f^2\left(-\frac{265}{576}+\frac{29}{24}\zeta_3\right)+C_FT_f^3\left(\frac{605}{972}\right) \\
& +C_F^3C_A\left(-\frac{3707}{4608}-\frac{971}{96}\zeta_3+\frac{1045}{48}\zeta_5\right) \\
& +C_F^2T_fC_A\left(-\frac{37403}{13824}-\frac{1289}{144}\zeta_3+\frac{275}{144}\zeta_5+\frac{105}{8}\zeta_7\right)
\end{aligned}
$$

c_4-continuation

$$
\begin{aligned}
&+C_F C_A T_f^2\left(-\frac{165283}{20736}-\frac{43}{144}\zeta_3+\frac{5}{12}\zeta_5-\frac{1}{6}\zeta_3^2\right)\\
&+C_F^2 C_A^2\left(\frac{1071641}{55296}+\frac{1591}{144}\zeta_3-\frac{1375}{144}\zeta_5+\frac{385}{16}\zeta_7\right)\\
&+C_F C_A^2 T_f\left(-\frac{1238827}{41472}+\frac{59}{64}\zeta_3-\frac{18855}{288}\zeta_5+\frac{11}{12}\zeta_3^2-\frac{35}{16}\zeta_7\right)\\
&+C_F C_A^3\left(-\frac{8004277}{248832}+\frac{1069}{576}\zeta_3+\frac{12545}{1152}\zeta_5-\frac{121}{96}\zeta_3^2+\frac{385}{64}\zeta_7\right)
\end{aligned}
$$

Baikov, Chetyrkin, Kuhn (2010) evaluated directly ; presented first in this forn by us with Mikhailov (10-12)

Special analytical structure of the CBK relation

Conformal symmetry based study Crewther (1972) and respecting QCD asymptotic freedom effects at $O(a_s^3)$ level Broadhurst, Kataev (1993) CBK relation . Property of the factorization of the QCD β-function is outlined. Confirmed at the $O(a_s^4)$-level by Chetyrkin, Baikov, Kuhn (2010)

$$C_{NS}(a_s)D^{NS}(a_s) = 1 + ZERO(C_F^n a_s^n) + ZERO(C_F^k C_A^m a_s^{k+m}) +$$

$$+\frac{\beta^{(3)}(a_s)}{a_s}\Bigg[S_1 C_F a_s + \Bigg(S_2 T_f N_f + S_3 C_A + S_4 C_F\Bigg) C_F a_s^2$$

$$+a_s^3\Bigg(S_5 C_F^3 + S_6 C_F^2 T_f + S_7 C_F T_f^2 + S_8 C_F C_A^2 + S_9 C_F C_A T_f + S_{10} C_F C_A^2$$

$$S_1 = -\frac{21}{8} + 3\zeta_3, S_2 = \frac{163}{24} - \frac{19}{3}\zeta_3$$

$$S_3 = -\frac{629}{32} + \frac{221}{12}\zeta_3 \ , S_4 = \frac{397}{96} + \frac{17}{2}\zeta_3 - 15\zeta_5, S_5 - S_{10} - analytical$$

In view of conformal symmetry applied to AVV scale-independent coefficients in C_{NS} and D^{NS} are cancelling

Factorization property of the QCD β-function

This property is non-trivial even in QED when $C_A = 0$. In general β-function is responsible for effects of conformal symmetry violation. Expressions in MS-like schemes

$$\beta(a_s) = -\sum_{n\geq 0} \beta_n a_s^n \ , \ \beta_0 = (\frac{11}{3}C_A - \frac{4}{3}Tf)\frac{1}{4}$$

$$\beta_1 = (\frac{34}{3}C_A^2 - \frac{20}{3}C_A T_f - 4C_F T_f)\frac{1}{16} \ ,$$

$$\beta_2 = (\frac{2857}{54}C_A^3 + 2C_F^2 T_f - \frac{205}{9}C_F C_A T_f - \frac{1415}{27}C_A^2 T_f + \frac{44}{9}C_F T_f^2 + \frac{158}{27}C_A T_f^2\Big)\frac{1}{64}$$

O.V. Tarasov,A.A. Vladimirov, A.A. Zharkov (1980); S.A.Larin, J.A.M. Vermaseren (1993); Available 4 and 5 loop QCD β-function terms will be not considered.

Massless $\overline{\mathrm{MS}}$ Broadhurst-Kataev (93) renormalon related exp

$$\sum_{n<10} R_n x^n = 3x + \left[-22 + 16\zeta_3\right]x^2 + \left[\frac{4832}{27} - \frac{1216}{9}\zeta_3\right]x^3 + \left[-\frac{392384}{243} + \frac{25984}{27}\zeta_3 + \frac{1280}{3}\zeta_5\right]x^4$$
$$+ \left[\frac{11758720}{729} - \frac{5073920}{729}\zeta_3 - \frac{194560}{27}\zeta_5\right]x^5 + \left[-\frac{3499697920}{19683} + \frac{357201920}{6561}\zeta_3 + \frac{20787200}{243}\zeta_5 + \frac{71680}{3}\zeta_7\right]x^6$$
$$+ \left[\frac{381559797760}{177147} - \frac{9308446720}{19683}\zeta_3 - \frac{2029568000}{2187}\zeta_5 - \frac{5447680}{9}\zeta_7\right]x^7$$
$$+ \left[-\frac{5056220794880}{177147} + \frac{2445582254080}{531441}\zeta_3 + \frac{200033075200}{19683}\zeta_5 + \frac{814858240}{81}\zeta_7 + \frac{194969600}{81}\zeta_9\right]x^8$$
$$+ \left[\frac{5908327309475840}{14348907} - \frac{239732713062400}{4782969}\zeta_3 - \frac{20850920652800}{177147}\zeta_5 - \frac{318236262400}{2187}\zeta_7 - \frac{59270758400}{729}\zeta_9\right]x^9$$

$$\sum_{n<10} K_n x^n = -3x + 8x^2 - \frac{920}{27}x^3 + \frac{38720}{243}x^4 - \frac{238976}{243}x^5 + \frac{130862080}{19683}x^6 - \frac{10038092800}{177147}x^7$$
$$+ \frac{274593587200}{531441}x^8 - \frac{82519099473920}{14348907}x^9$$

$$\sum_{n<10} S_n x^n = \left[-\frac{21}{2} + 12\zeta_3\right]x + \left[\frac{326}{3} - \frac{304}{3}\zeta_3\right]x^2 + \left[-\frac{9824}{9} + \frac{6496}{9}\zeta_3 + 320\zeta_5\right]x^3$$
$$+ \left[\frac{2760448}{243} - \frac{1268480}{243}\zeta_3 - \frac{48640}{9}\zeta_5\right]x^4 + \left[-\frac{280736320}{2187} + \frac{89300480}{2187}\zeta_3 + \frac{5196800}{81}\zeta_5 + 17920\zeta_7\right]x^5$$
$$+ \left[\frac{10320047360}{6561} - \frac{2327111680}{6561}\zeta_3 - \frac{507392000}{729}\zeta_5 - \frac{1361920}{3}\zeta_7\right]x^6$$
$$+ \left[-\frac{3723517199360}{177147} + \frac{611395563520}{177147}\zeta_3 + \frac{50008268800}{6561}\zeta_5 + \frac{203714560}{27}\zeta_7 + \frac{48742400}{27}\zeta_9\right]x^7$$
$$+ \left[\frac{485484017500160}{1594323} - \frac{59933178265600}{1594323}\zeta_3 - \frac{5212730163200}{59049}\zeta_5 - \frac{79559065600}{729}\zeta_7 - \frac{14817689600}{243}\zeta_9\right]x^8$$
$$+ \left[-\frac{7616109282344960}{1594323} + \frac{726735764193280}{1594323}\zeta_3 + \frac{195646580326400}{177147}\zeta_5 + \frac{1120185221120}{729}\zeta_7\right.$$
$$\left. + \frac{316630630400}{243}\zeta_9 + \frac{7821721600}{27}\zeta_{11}\right]x^9$$

Few words on factorization of the β-function

- Is important feature. Measure of violation of the CS in renormalized QFT models. Step to proof in all orders Crewther (96).
- Step to independent confirmation V.Braun, Korchemsky, Muller (03)
- Factorization is valid in gauge-independent schemes Garkusha, AK,Molokoedov (18) and gauge-dependent schemes Gracey, Mason (23)
- Considerations of static potential vs cusp anomalous dimension= Wilson loops related Grozin,Henn, Korchemsky , Merquard (16); Grozin (23)

In t 'Hooft scheme no property of factorization, though effects of β_0 and β_1 are seen Garkusha, AK (11). **In the diagrammatic $\overline{MS}$-relatesd schemes** CBK relation can be re-written as

$$C_{NS}(a_s)D^{NS}(a_s) = 1 + \sum_{n\geq 0}\left(\frac{\beta(a_s)}{a_s}\right)^n P_n(a_s)$$

$P_0(a_s) = 0$ Effect of conformal symmetry. AK,Mikhailov (10-12)

The $\{\beta\}$-expansion for the RG-invariant quantities

Consider the PT expansion

$$D^{ns}(a_s) = 1 + d_1 a_s + d_2(n_f) a_s^2 + d_3(n_f) a_s^3 + d_4(n_f) a_s^4 + O(a_s^5)$$

In the MS-like schemes β-expansion prescription is:

$$d_1 = d_1[0]$$

$$d_2(n_f) = \beta_0 d_2[1] + \mathbf{d_2[0]} - \textbf{ the Basis of BLM procedure}$$

$$d_3(n_f) = \beta_0^2 d_3[2] + \beta_1 d_3[0,1] + \beta_0 d_3[1] + \mathbf{d_3[0]},$$

$$d_4(n_f) = \beta_0^3 d_4[3] + \beta_2 d_4[0,0,1] + \beta_1\beta_0 d_4[1,1] + \beta_0^2 d_4[2] + \beta_1 d_4[0,1] + \beta_0 d_4[1] + \mathbf{d_4[0]}; \ldots$$

Suggested by **Mikhailov (Quarks2004, hep-ph.0411397 ; JHEP(07))** Further on Kataev, Mikhalov (12,15,16) ; Brodsky,Wu, Mojaza et al(12-25) ; Cvetic,Kataev(16) ; Kataev,Molokoedov (22,23) ; Baikov, Mikhailov (22-23) ; Mikhailov (24) **diagrammatic but or (!) extanded QCD and in part SQCD related study**

The $\{\beta\}$ expanded QCD terms for D^{ns} in $SU(N_c)$

Using the Cvetic,Kataev (16) $\overline{MS}$-scheme factorized model , motivated by Kataev,Mikhailov (12) CBK-related consideration

$$D^{ns}(a_s) = 1 + \sum_{n\geq 0} \left(\frac{\beta(a_s)}{a_s}\right)^n P_n(a_s)$$

we obtain the results, which differ in part from obtained in **QCD+gluino theory Mikhailov (07), Kataev,Mikhailov (14;15;16) diagrammatic** analysis

$$d_1[0] = \frac{3}{4}C_F \;\; d_2[0] = \left(-\frac{3}{32}C_F^2 + \frac{1}{16}C_F C_A\right) \;\; d_2[1] = \left(\frac{33}{8} - 3\zeta_3\right)C_F$$

$$d_3[0] = -\frac{69}{128}C_F^3 - \left(\frac{101}{256} - \frac{33}{16}\zeta_3\right) \neq +\mathbf{\frac{71}{64}} \;\; C_F^2 C_A$$

$$-\left(\frac{53}{192} + \frac{33}{16}\zeta_3\right) \neq +\left(\mathbf{\frac{523}{768}} - \mathbf{\frac{27}{8}}\zeta_{\mathbf{3}}\right) \;\; C_F C_A^2$$

As the result one has $d_3[0] = -23.227 \neq -\mathbf{35.87}$,

The $\{\beta\}$ expanded QCD expression for d_4 for Adler e^+e^--function

$$d_4[0] = \left(\frac{3}{16} - \frac{1}{4}\zeta_3 - \frac{5}{4}\zeta_5\right)\frac{d_F^{abcd}d_A^{abcd}}{d_R} - \left(\frac{13}{16} + \zeta_3 - \frac{5}{2}\zeta_5\right)\frac{d_F^{abcd}d_F^{abcd}}{d_R}n_f$$

$$+\left(\frac{4157}{2048} + \frac{3}{8}\zeta_3\right)C_F^4$$

$$-\left(\frac{3509}{1536} + \frac{73}{128}\zeta_3 + \frac{165}{32}\zeta_5\right) \neq -\left(\mathbf{\frac{2409}{512}} + \mathbf{\frac{27}{16}}\boldsymbol{\zeta_3}\right) \quad C_F^3 C_A$$

$$+\left(\frac{9181}{4608} + \frac{299}{128}\zeta_3 + \frac{165}{64}\zeta_5\right) \neq -\left(\mathbf{\frac{3105}{1024}} + \mathbf{\frac{81}{32}}\boldsymbol{\zeta_3}\right) \quad C_F^2 C_A^2$$

$$\left(-\frac{30863}{36864} - \frac{147}{128}\zeta_3 + \frac{165}{64}\zeta_5\right) \neq \left(\mathbf{\frac{68047}{12288}} + \mathbf{\frac{8113}{512}}\boldsymbol{\zeta_3} - \mathbf{\frac{3555}{128}}\boldsymbol{\zeta_5}\right)C_F C_A^3$$

$d_4[0] = +81.157 \neq \mathbf{-98}$ Differ from **diagrammatic related expression of Mikhailov (22-24), obtained using Chetyrkin (22) and Zoller (16) eQCD calculations** supported by diagrammatic QCD Ball, Beneke, V. Braun (95) .

D-function at the $O(a_s^4)$-level and PMC/BLM - representations

Compare MS-scheme D-function with the **scale independent** coefficients, but running a_* and a_{**} Cvetic,K (16) -model . Independent from RG-quantity non-diagrammatic with Mikhailov (07;..;24) diagrammatic **though** quantity dependent Extended QCD

$$D_{\overline{\rm MS}}(a_s) = 3\sum_f Q_f^2\big(1 + a_s + (1.9857 - 0.1153 n_f) a_s^2$$

$$+(18.243 - 4.216 n_f + 0.086 n_f^2) a_s^3$$

$$+(135.792 - 34.440 n_f + 1.875 n_f^2 - 0.010 n_f^3) a_s^4\big)$$

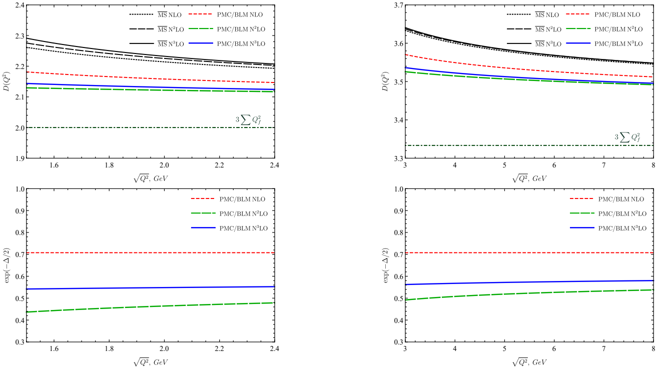
$$D^{CK}_{PMC/BLM}(a_s) = 3\sum_f Q_f^2\big(1 + a_* + \frac{1}{12} a_*^2 - 23.227 a_*^3$$

$$+(81.157 - 0.0080 n_f) a_*^4\big)$$

$$D^{M}_{PMC/BLM}(a_s) = 3\sum_f Q_f^2\big(1 + a_{**} + \frac{1}{12} a_{**}^2 - 35.87 a_{**}^3$$

$$+(-98 - 0.0080 n_f) a_{**}^4\big)$$

PMC/BLM vs massless $\overline{\text{MS}}$: AK,Molokoedov PRD 108 (23)



Experimental related data are higher then $\overline{MS}$ Eidelman, Jegerlehner, AK, Veretin (98); Davier et al (23).

Conclusion from K,Molokoedov PRD (23)

As seen from Fig. 1a, the application of the PMC-BLM procedure to the massless MS Adler function PT approximations is leading to moving the considered curves lower away fromthe experimentally based results for the Adler function in the considered kinematical region. Therefore, better not to use the PMC-BLM in the process of comparison with the existing experimental data and, in particular, the ones provided by the e^+e^- *colliders. PMC/BLM are qualitatively closer to "Finite QED" . Leads to UNDERESTIMATE of theory QCD ambiguity.*

.

See also Arbuzov AQFT-25 talk

Available 5-loops contributions in QCD to Adler function; may be useful in future

$$d_5[4] = \frac{C_F(T_F N_F)^4}{4^5}\left(\frac{11758720}{729} - \frac{5073920}{729}\zeta_3 - \frac{194560}{27}\zeta_5\right)$$

Broadhurst,Kataev (93)

$$d_5^{\zeta_4} = -c_5^{\zeta_4} = d_5^{\zeta_4}[0] + \beta_0 d_5^{\zeta_4}[1] + \beta_0^2 d_5^{\zeta_4}[2]$$

$$d_5^{\zeta_4}[0] = -c_5^{\zeta_4}[0] = \zeta_4\left(\frac{693}{2048}C_F C_A^4 + \frac{99}{512}C_F^2 C_A^3 - \frac{1089}{2048}C_F^3 C_A^2\right.$$

$$\left.+\frac{33}{128}\frac{d_A^{abcd}d_A^{abcd}}{d_R}T_f - \frac{429}{256}\frac{d_F^{abcd}d_A^{abcd}}{d_R}C_A + \frac{33}{64}\frac{d_F^{abcd}d_F^{abcd}}{d_R}n_f C_A\right)$$

$$d_5^{\zeta_4}[1] = -c_5^{\zeta_4}[1] = \zeta_4\left(- \frac{615}{1024}C_F C_A^3 - \frac{339}{512}C_F^2 C_A^2 + \frac{165}{128}C_F^3 C_A\right.$$

$$\left.+\frac{117}{64}\frac{d_F^{abcd}d_A^{abcd}}{d_R} - \frac{9}{16}\frac{d_F^{abcd}d_F^{abcd}}{d_R}n_f\right)$$

$$d_5^{\zeta_4}[2] = \zeta_4\left(\frac{63}{256}C_F c_A^2 + \frac{63}{128}C_F^2 C_A - \frac{99}{128}C_F^3\right)$$

Goriachuk,Kataev (20) and Goriachuk,Kataev,Molokoedov (22)

OTHER CONTRIBUTIONS ARE WELLCOMED
Though already the progress in multiloops calculations AQFT-25 Onischenko and Lee talks
Thats it with mine talk