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Renormalization group running of couplings
in various supersymmetric theories

Quantum corrections as a tool for understanding nature

Sometimes the structure of the surrounding world can be understood by studying quantum corrections. For instance, the very precise agreement of the theoretical prediction of the electron anomalous magnetic moment with the experimental data tells us that the nature is described by quantum field theory.

The unification of running couplings and absence of quadratically divergent quantum corrections to the Higgs boson mass may be considered as indirect indications to the existence of supersymmetry and Grand Unification.

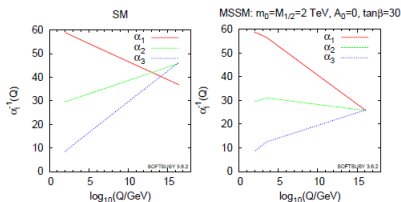


Figure 94.1: Running couplings in SM and MSSM using two-loop RG evolution. The SUSY threshold at 2 TeV is clearly visible on the MSSM side. (We thank Ben Allanach for providing the plots created using SOFTSUSY [61].)

Some important information about new physics can be obtained from the detailed analysis of quantum corrections to the (light) Higgs boson mass in supersymmetric theories, anomalous magnetic moment of muon, etc.

Supersymmetric gauge theories

It is convenient to describe $\mathcal{N} = 1$ supersymmetric gauge theories in $\mathcal{N} = 1$ superspace. Then, for the renormalizable theories, the action can be written as

$$S = \frac{1}{2e_0^2} \text{Re tr} \int d^4x d^2\theta W^a W_a + \frac{1}{4} \int d^4x d^4\theta \phi^{*i} (e^{2V})_i{}^j \phi_j \\ + \left\{ \int d^4x d^2\theta \left(\frac{1}{4} m_0^{ij} \phi_i \phi_j + \frac{1}{6} \lambda_0^{ijk} \phi_i \phi_j \phi_k \right) + \text{c.c.} \right\}.$$

Here V is the gauge superfield, ϕ_i are the chiral matter superfields in the representation R of the gauge group G , and

$$W_a = \frac{1}{8} \bar{D}^2 \left(e^{-2V} D_a e^{2V} \right)$$

is the supersymmetric gauge field strength.

The gauge invariant theory is obtained if the Yukawa couplings and masses satisfy the constraints

$$m_0^{im} (T^A)_m{}^j + m_0^{mj} (T^A)_m{}^i = 0; \\ \lambda_0^{ijm} (T^A)_m{}^k + \lambda_0^{imk} (T^A)_m{}^j + \lambda_0^{mjk} (T^A)_m{}^i = 0,$$

where $(T^A)_i{}^j$ are the generators of the gauge group G in the representation R .

The NSVZ β -function for $\mathcal{N} = 1$ supersymmetric theories

In $\mathcal{N} = 1$ supersymmetric theories there is an all-loop relation between the β -function and the anomalous dimension of the matter superfields called the exact Novikov, Shifman, Vainshtein, and Zakharov (NSVZ) β -function

V. A. Novikov, M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Nucl. Phys. B **229** (1983), 381; Phys. Lett. **166B**(1986), 329;
D. R. T. Jones, Phys. Lett. **123B** (1983), 45;
M. A. Shifman and A. I. Vainshtein, Nucl. Phys. B **277** (1986), 456.

For a general $\mathcal{N} = 1$ supersymmetric gauge theory with a single gauge coupling it can be written in the form

$$\beta(\alpha, \lambda) = - \frac{\alpha^2 \left(3C_2 - T(R) + C(R)_i^j (\gamma_\phi)_j^i(\alpha, \lambda)/r \right)}{2\pi(1 - C_2\alpha/2\pi)}.$$

Here α and λ are the gauge and Yukawa coupling constants, respectively, and we use the notation

$$\begin{aligned} \text{tr}(T^A T^B) &\equiv T(R) \delta^{AB}; & (T^A)_i^k (T^A)_k^j &\equiv C(R)_i^j; \\ f^{ACD} f^{BCD} &\equiv C_2 \delta^{AB}; & r &\equiv \delta_{AA} = \dim G. \end{aligned}$$

For the pure $\mathcal{N} = 1$ SYM theory

$$S = \frac{1}{2e_0^2} \operatorname{Re} \operatorname{tr} \int d^4x d^2\theta W^a W_a$$

from the NSVZ β -function we obtain the equation

$$\frac{1}{\alpha^2} \left(1 - \frac{C_2 \alpha}{2\pi} \right) \frac{d\alpha}{d \ln \mu} = -\frac{3C_2}{2\pi}.$$

Integrating it, we obtain the **all-loop** RGI

$$\left(\frac{\mu^3}{\alpha} \right)^{C_2} \exp \left(-\frac{2\pi}{\alpha} \right) = \text{RGI}.$$

Such expressions appear in calculating the instanton contributions to the effective action, and **the NSVZ β -function was first obtained by requiring their renormalization group invariance.**

V. A. Novikov, M. A. Shifman, A. I. Vainshtein and V. I. Zakharov,
Nucl. Phys. B **229** (1983), 381.

For theories with chiral matter superfields the analogous invariants contain the renormalization constants for the matter superfields or masses.

Scheme dependence of the NSVZ equation

Nevertheless, it is necessary to remember that the NSVZ equation is valid only for certain renormalization prescriptions. Therefore, the all-loop renormalization group invariance of the above expressions does not hold for a general subtraction scheme.

Note that in the $\overline{\text{DR}}$ scheme the NSVZ equation is not valid starting from the order $O(\alpha^4)$ (the three-loop approximation for the β -function and the two-loop approximation for the anomalous dimension)

I. Jack, D. R. T. Jones and C. G. North, Phys.Lett. B **386** (1996) 138;
Nucl.Phys. B **486** (1997) 479;
R. V. Harlander, D. R. T. Jones, P. Kant, L. Mihaila and M. Steinhauser,
JHEP **0612** (2006) 024.

However, in this case it is possible to make a special redefinition of the coupling constant which restores the NSVZ relation.

The all-loop NSVZ schemes have been constructed with the help of the higher covariant derivative regularization

A. A. Slavnov, Nucl. Phys. B **31** (1971), 301;
Theor. Math. Phys. **13** (1972), 1064; **33** (1977), 977.

In the supersymmetric case the higher covariant derivative regularization can be formulated in terms of superfields and, therefore, does not break supersymmetry

V. K. Krivoshchekov, Theor. Math. Phys. **36** (1978), 745;
P. C. West, Nucl. Phys. B **268** (1986), 113.

In this case (logarithmic) divergences are given by powers of $\ln \Lambda/\mu$, where Λ is a dimensionful regularization parameter, and μ is the renormalization point.

The NSVZ β -function is valid in all loops if a supersymmetric theory is regularized by Higher covariant Derivatives and the renormalization is made by Minimal Subtractions of Logarithms (the so-called HD+MSL scheme), see

K.S., Eur. Phys. J. C **80** (2020) no.10, 911

and references therein.

Minimal subtractions of logarithms mean that only powers of $\ln \Lambda/\mu$ (where Λ is the cutoff parameter and μ is the renormalization point) are included into renormalization constants,

A. L. Kataev, K.S., Nucl. Phys. B **875** (2013), 459.

Theories with multiple gauge couplings

In what follows we will mostly investigate theories with multiple gauge couplings. In this case the gauge group is a direct product

$$G = G_1 \times G_2 \times \dots \times G_n,$$

where any G_i is either a simple group or $U(1)$. In this case there are n gauge coupling constants $\alpha_1, \alpha_2, \dots, \alpha_n$.

Such theories can be interesting for phenomenology because they include

- QCD+QED
- The Standard Model
- The MSSM
- Some Grand Unified Theories, e.g., the flipped $SU(5)$ theory.

We argue that in some $\mathcal{N} = 1$ supersymmetric theories with multiple couplings the couplings do not run independently, and (sometimes) it is possible to construct all-loop RGIs from the gauge and Yukawa couplings. We will also discuss under what renormalization prescriptions the renormalization group invariance of the corresponding expressions is valid in all orders.

The simplest example of a theory with two gauge coupling constants $\alpha_s \equiv g^2/4\pi$ and $\alpha = e^2/4\pi$ is QCD+QED. In the massless limit this theory is described by the Lagrangian

$$\mathcal{L} = -\frac{1}{2g^2} \text{tr} F_{\mu\nu}^2 - \frac{1}{4e^2} F_{\mu\nu}^2 + \sum_{a=1}^{N_f} i\bar{\psi}_a \gamma^\mu \mathcal{D}_\mu \psi_a,$$

which is invariant under the transformations of the gauge group $G \times U(1)$. The Dirac spinors ψ_a (where the subscript a numerates flavors) lie in a certain irreducible representation R of the group G and have the electromagnetic charges q_a . In this case the covariant derivatives are written in the form

$$\mathcal{D}_\mu \psi_a = \partial_\mu \psi_a + iA_\mu \psi_a + iq_a \mathbf{A}_\mu \psi_a,$$

where A_μ and $\mathbf{A}_\mu$ are the non-Abelian and Abelian gauge fields, respectively. The corresponding gauge field strengths are given by the expressions

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]; \quad \mathbf{F}_{\mu\nu} = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu.$$

In quantum field theory the couplings α_s and α depend on scale,

$$\frac{d\alpha}{d \ln \mu} = \beta(\alpha, \alpha_s); \quad \frac{d\alpha_s}{d \ln \mu} = \beta_s(\alpha_s, \alpha).$$

It is convenient to formulate the supersymmetric version of the above model in terms of superfields

$$S = \frac{1}{2g^2} \text{Re tr} \int d^4x d^2\theta W^a W_a + \frac{1}{4e^2} \text{Re} \int d^4x d^2\theta \mathbf{W}^a \mathbf{W}_a \\ + \sum_{a=1}^{N_f} \frac{1}{4} \int d^4x d^4\theta \left(\phi_a^+ e^{2V+2q_a \mathbf{V}} \phi_a + \tilde{\phi}_a^+ e^{-2V^T-2q_a \mathbf{V}} \tilde{\phi}_a \right),$$

because in this case $\mathcal{N} = 1$ supersymmetry is manifest. Here V and $\mathbf{V}$ are the gauge superfields corresponding to the subgroups G and $U(1)$, respectively. The chiral matter superfields ϕ_a and $\tilde{\phi}_a$ belong to the (conjugated) representations R and $\bar{R}$, respectively, and have opposite $U(1)$ charges.

Two supersymmetric gauge superfield strengths are written in the form

$$W_a = \frac{1}{8} \bar{D}^2 \left(e^{-2V} D_a e^{2V} \right); \quad \mathbf{W}_a = \frac{1}{4} \bar{D}^2 D_a \mathbf{V}.$$

Is it possible to relate running of two gauge coupling constants in this model?

A. L. Kataev, K.S., JETP Lett. **121** (2025) no.5, 315.

The NSVZ equations for theories with multiple gauge couplings

The NSVZ equations can also be written for theories with multiple gauge couplings,

M. A. Shifman, Int. J. Mod. Phys. A **11** (1996), 5761;
D. Korneev, D. Plotnikov, K.S. and N. Tereshina, JHEP **10** (2021), 046.

In the particular case $q_a = 1$ for $\mathcal{N} = 1$ SQCD+SQED they have the form

$$\frac{\beta_s(\alpha_s, \alpha)}{\alpha_s^2} = -\frac{1}{2\pi(1 - C_2\alpha_s/2\pi)} \left[3C_2 - 2T(R)N_f \left(1 - \gamma(\alpha_s, \alpha) \right) \right];$$
$$\frac{\beta(\alpha, \alpha_s)}{\alpha^2} = \frac{1}{\pi} \dim R N_f \left(1 - \gamma(\alpha_s, \alpha) \right),$$

where we took into account that if the representation for the matter superfields is irreducible, then in the case under consideration

$$\gamma(\alpha_s, \alpha)_i^j = \gamma(\alpha_s, \alpha) \cdot \delta_i^j,$$

where i and j include both the indices numerating chiral matter superfields ϕ_a and $\tilde{\phi}_a$ and the indices corresponding to the representation R (or $\bar{R}$).

Comparing the above expressions for the β -functions we see that the anomalous dimension of the matter superfields can be eliminated.

The RGI for $\mathcal{N} = 1$ SQCD+SQED

After eliminating the anomalous dimension of the matter superfields we obtain that the β -functions satisfy the all-order exact equation

$$\left(1 - \frac{C_2 \alpha_s}{2\pi}\right) \frac{\beta_s(\alpha_s, \alpha)}{\alpha_s^2} = -\frac{3C_2}{2\pi} + \frac{T(R)}{\dim R} \cdot \frac{\beta(\alpha, \alpha_s)}{\alpha^2}.$$

Evidently, this equation is valid in the HD+MSL scheme, because the original NSVZ equations are satisfied for this renormalization prescription.

Taking into account the boundary conditions for the HD+MSL scheme, it is possible to integrate the relation between the β -functions over μ . Then we obtain the equation which relates running of the strong and electromagnetic couplings in the theory under consideration.

$$\frac{1}{\alpha_s} - \frac{1}{\alpha_{s0}} + \frac{C_2}{2\pi} \ln \frac{\alpha_s}{\alpha_{s0}} = -\frac{3C_2}{2\pi} \ln \frac{\Lambda}{\mu} + \frac{T(R)}{\dim R} \left(\frac{1}{\alpha} - \frac{1}{\alpha_0} \right).$$

This in particular implies that the expression

$$\left(\frac{\alpha_s}{\mu^3}\right)^{C_2} \exp\left(\frac{2\pi}{\alpha_s} - \frac{T(R)}{\dim R} \cdot \frac{2\pi}{\alpha}\right) = \text{RGI}$$

is the renormalization group invariant, i.e. the expression which vanishes after differentiating with respect to $\ln \mu$.

The three-loop verification

The three-loop verification of this general result has been done in

O. Haneychuk, K. S., Eur. Phys. J. C **85** (2025) no.5, 540.

It is important that in this approximation the dependence on **regularization** and **renormalization** parameters becomes essential. We will use a **renormalization prescription** for that

$$\begin{aligned}\frac{1}{\alpha_0} &= \frac{1}{\alpha} - \frac{N_f \dim R}{\pi} \left(\ln \frac{\Lambda}{\mu} + d_1 \right) - \frac{\alpha_s}{\pi^2} N_f C(R) \dim R \left(\ln \frac{\Lambda}{\mu} + d_2 \right) - \frac{\alpha}{\pi^2} N_f \dim R \\ &\times \left(\ln \frac{\Lambda}{\mu} + \tilde{d}_2 \right) + O(\alpha_s^2, \alpha_s \alpha, \alpha^2); \\ \frac{1}{\alpha_{s0}} &= \frac{1}{\alpha_s} + \frac{3C_2}{2\pi} \left(\ln \frac{\Lambda}{\mu} + b_{11} \right) - \frac{N_f T(R)}{\pi} \left(\ln \frac{\Lambda}{\mu} + b_{12} \right) + \frac{3\alpha_s}{4\pi^2} (C_2)^2 \left(\ln \frac{\Lambda}{\mu} + b_{21} \right) \\ &- \frac{\alpha_s}{2\pi^2} N_f C_2 T(R) \left(\ln \frac{\Lambda}{\mu} + b_{22} \right) - \frac{\alpha_s}{\pi^2} N_f C(R) T(R) \left(\ln \frac{\Lambda}{\mu} + b_{23} \right) - \frac{\alpha}{\pi^2} N_f T(R) \left(\ln \frac{\Lambda}{\mu} \right. \\ &\left. + \tilde{b}_{21} \right) + O(\alpha_s^2, \alpha_s \alpha, \alpha^2).\end{aligned}$$

The HD+MSL scheme is obtained if $b_{11} = b_{12} = b_{21} = b_{22} = b_{23} = \tilde{b}_{21} = 0$, $d_1 = d_2 = \tilde{d}_2 = 0$. The **regularization parameters** essential in the three-loop approximation are

$$A \equiv \int_0^\infty dx \ln x \frac{d}{dx} \frac{1}{R(x)}; \quad a_\varphi \equiv \frac{M_\varphi}{\Lambda}; \quad a_G \equiv \frac{M_G}{\Lambda}; \quad a_1 \equiv \frac{M_1}{\Lambda}.$$

The three-loop RGFs for $\mathcal{N} = 1$ SQCD+SQED

The three-loop β -functions have the form

$$\frac{\beta(\alpha_s, \alpha)}{\alpha^2} = \frac{N_f \dim R}{\pi} \left\{ 1 + \frac{\alpha}{\pi} + \frac{\alpha_s}{\pi} C(R) - \frac{1}{2\pi^2} (\alpha + \alpha_s C(R))^2 - \frac{\alpha^2}{\pi^2} N_f \dim R (\ln a_1 + 1 + \frac{A}{2} + \tilde{d}_2 - d_1) + \frac{3\alpha_s^2}{2\pi^2} C_2 C(R) (\ln a_\varphi + 1 + \frac{A}{2} + d_2 - b_{11}) - \frac{\alpha_s^2}{\pi^2} N_f C(R) T(R) \times (\ln a_G + 1 + \frac{A}{2} + d_2 - b_{12}) + O(\alpha_s^3, \alpha_s^2 \alpha, \alpha_s \alpha^2, \alpha^3) \right\};$$

$$\frac{\beta_s(\alpha_s, \alpha)}{\alpha_s^2} = -\frac{1}{2\pi} (3C_2 - 2N_f T(R)) + \frac{\alpha}{\pi^2} N_f T(R) + \frac{\alpha_s}{4\pi^2} (-3(C_2)^2 + 2N_f C_2 T(R) + 4N_f C(R) T(R)) - \frac{\alpha^2}{\pi^3} (N_f)^2 T(R) \dim R (\ln a_1 + 1 + \frac{A}{2} + \tilde{b}_{21} - d_1) - \frac{1}{2\pi^3} N_f T(R) \times (\alpha + \alpha_s C(R))^2 + \frac{\alpha \alpha_s}{2\pi^3} N_f C_2 T(R) - \frac{3\alpha_s^2}{8\pi^3} (C_2)^3 (1 + 3b_{21} - 3b_{11}) + \frac{\alpha_s^2}{4\pi^3} N_f (C_2)^2 \times T(R) (1 + 3b_{21} - 3b_{11} + 3b_{22} - 3b_{12}) + \frac{3\alpha_s^2}{2\pi^3} N_f C_2 C(R) T(R) (\ln a_\varphi + \frac{4}{3} + \frac{A}{2} + b_{23} - b_{11}) - \frac{\alpha_s^2}{2\pi^3} (N_f)^2 C_2 T(R)^2 (b_{22} - b_{12}) - \frac{\alpha_s^2}{\pi^3} (N_f)^2 C(R) T(R)^2 (\ln a_G + 1 + \frac{A}{2} + b_{23} - b_{12}) + O(\alpha_s^3, \alpha_s^2 \alpha, \alpha_s \alpha^2, \alpha^3).$$

They satisfy the exact relation if $\tilde{b}_{21} = \tilde{d}_2$, $b_{21} = b_{11}$, $b_{22} = b_{12}$, $b_{23} = d_2$. In particular, this relation is valid in the HD+MSL scheme.

The three-loop results for the $\overline{\text{DR}}$ scheme

The three-loop β -functions in the $\overline{\text{DR}}$ scheme are given by

$$\left. \frac{\beta(\alpha_s, \alpha)}{\alpha^2} \right|_{\overline{\text{DR}}} = \frac{N_f \dim R}{\pi} \left\{ 1 + \frac{\alpha}{\pi} + \frac{\alpha_s}{\pi} C(R) - \frac{1}{2\pi^2} (\alpha + \alpha_s C(R))^2 - \frac{3\alpha^2}{4\pi^2} N_f \dim R \right. \\ \left. + \frac{9\alpha_s^2}{8\pi^2} C_2 C(R) - \frac{3\alpha_s^2}{4\pi^2} N_f C(R) T(R) + O(\alpha_s^3, \alpha_s^2 \alpha, \alpha_s \alpha^2, \alpha^3) \right\};$$

$$\left. \frac{\beta_s(\alpha_s, \alpha)}{\alpha_s^2} \right|_{\overline{\text{DR}}} = -\frac{1}{2\pi} (3C_2 - 2N_f T(R)) + \frac{\alpha}{\pi^2} N_f T(R) + \frac{\alpha_s}{4\pi^2} (-3(C_2)^2 + 2N_f C_2 T(R) \\ + 4N_f C(R) T(R)) - \frac{3\alpha^2}{4\pi^3} (N_f)^2 T(R) \dim R - \frac{1}{2\pi^3} N_f T(R) (\alpha + \alpha_s C(R))^2 + \frac{\alpha \alpha_s}{2\pi^3} N_f C_2 \\ \times T(R) - \frac{21\alpha_s^2}{32\pi^3} (C_2)^3 + \frac{5\alpha_s^2}{8\pi^3} N_f (C_2)^2 T(R) + \frac{13\alpha_s^2}{8\pi^3} N_f C_2 C(R) T(R) - \frac{\alpha_s^2}{8\pi^3} (N_f)^2 C_2 T(R)^2 \\ - \frac{3\alpha_s^2}{4\pi^3} (N_f)^2 C(R) T(R)^2 + O(\alpha_s^3, \alpha_s^2 \alpha, \alpha_s \alpha^2, \alpha^3).$$

In this case the relation between the β -functions is not satisfied starting from the three-loop approximation,

$$\left(1 - \frac{\alpha_s C_2}{2\pi} \right) \frac{\beta_s(\alpha_s, \alpha)}{\alpha_s^2} + \frac{3C_2}{2\pi} - \frac{T(R)}{\dim R} \cdot \frac{\beta(\alpha_s, \alpha)}{\alpha^2} = -\frac{9\alpha_s^2}{32\pi^3} (C_2)^3 \\ + \frac{3\alpha_s^2}{8\pi^3} (C_2)^2 N_f T(R) + O(\alpha_s^3, \alpha_s^2 \alpha, \alpha_s \alpha^2, \alpha^3) \neq O(\alpha_s^3, \alpha_s^2 \alpha, \alpha_s \alpha^2, \alpha^3).$$

The minimal scheme for $\mathcal{N} = 1$ SQCD+SQED

The explicit dependence of the β -functions on the renormalization parameters allows constructing a subtraction scheme in which the relation between the β -function is valid and RGFs are as simple as possible. In this case

$$b_{12} = b_{11} + \ln \frac{a_G}{a_\varphi}; \quad d_2 = b_{11} - \ln a_\varphi - 1 - \frac{A}{2}; \quad \tilde{d}_2 = d_1 - \ln a_1 - 1 - \frac{A}{2}.$$

In this case the expressions for the β -functions take the simplest form

$$\frac{\beta(\alpha_s, \alpha)}{\alpha^2} = \frac{N_f \dim R}{\pi} \left\{ 1 + \frac{\alpha}{\pi} + \frac{\alpha_s}{\pi} C(R) - \frac{1}{2\pi^2} (\alpha + \alpha_s C(R))^2 + O(\alpha_s^3, \alpha_s^2 \alpha, \alpha_s \alpha^2, \alpha^3) \right\};$$

$$\begin{aligned} \frac{\beta_s(\alpha_s, \alpha)}{\alpha_s^2} = & -\frac{1}{2\pi} (3C_2 - 2N_f T(R)) + \left(1 + \frac{\alpha_s C_2}{2\pi}\right) \left[\frac{\alpha}{\pi^2} N_f T(R) + \frac{\alpha_s}{4\pi^2} (-3(C_2)^2 + 2N_f \right. \\ & \left. \times C_2 T(R) + 4N_f C(R) T(R)) \right] - \frac{1}{2\pi^3} N_f T(R) (\alpha + \alpha_s C(R))^2 + O(\alpha_s^3, \alpha_s^2 \alpha, \alpha_s \alpha^2, \alpha^3). \end{aligned}$$

This scheme is analogous to the minimal scheme for $\mathcal{N} = 1$ SQED with N_f flavors, in which only terms without N_f survive in the anomalous dimension and

$$\frac{\beta(\alpha)}{\alpha^2} = \frac{N_f}{\pi} (1 - \gamma(\alpha)) = \frac{N_f}{\pi} \left(1 + \frac{\alpha}{\pi} - \frac{\alpha^2}{2\pi^2} + \frac{\alpha^3}{2\pi^3} + O(\alpha^4) \right),$$

I. Shirokov and K.S., JHEP 04 (2022), 108.

The Minimal Supersymmetric Standard Model (MSSM)

The analogous research for MSSM has been done in

D. Rystsov, K.S., Phys. Rev. D **111** (2025) no.1, 016012.

The MSSM is the simplest supersymmetric extension of the Standard Model. It is a gauge theory with the group $SU_3 \times SU_2 \times U_1$ and softly broken supersymmetry. Quarks, leptons, and Higgs fields are components of the chiral matter superfields:

Superfield	SU_3	SU_2	$U_1 (Y)$	Superfield	SU_3	SU_2	$U_1 (Y)$
$3 \times Q$	$\bar{3}$	2	$-1/6$	$3 \times N$	1	1	0
$3 \times U$	3	1	$2/3$	$3 \times E$	1	1	-1
$3 \times D$	3	1	$-1/3$	H_d	1	2	$1/2$
$3 \times L$	1	2	$1/2$	H_u	1	2	$-1/2$

where for the superfields which include left quarks and leptons we use the brief notations

$$Q = \begin{pmatrix} \tilde{U} \\ \tilde{D} \end{pmatrix}; \quad L = \begin{pmatrix} \tilde{N} \\ \tilde{E} \end{pmatrix}.$$

The MSSM couplings and superpotential

The MSSM contains **three gauge couplings**

$$\alpha_3 = \frac{e_3^2}{4\pi}; \quad \alpha_2 = \frac{e_2^2}{4\pi}; \quad \alpha_1 = \frac{5}{3} \cdot \frac{e_1^2}{4\pi}$$

corresponding to the subgroups $SU(3)$, $SU(2)$, and $U(1)$, respectively. (The factor $5/3$ in the coupling constant α_1 is introduced in order that the unification of couplings has the form $\alpha_1 = \alpha_2 = \alpha_3$.) There are also **dimensionless Yukawa couplings** $(Y_U)_{IJ}$, $(Y_D)_{IJ}$, and $(Y_E)_{IJ}$ (which are 3×3 matrices) inside **the superpotential**

$$\begin{aligned} W = & (Y_U)_{IJ} \begin{pmatrix} \tilde{U} & \tilde{D} \end{pmatrix}_I^a \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} H_{u1} \\ H_{u2} \end{pmatrix} U_{aJ} \\ & + (Y_D)_{IJ} \begin{pmatrix} \tilde{U} & \tilde{D} \end{pmatrix}_I^a \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} H_{d1} \\ H_{d2} \end{pmatrix} D_{aJ} \\ & + (Y_E)_{IJ} \begin{pmatrix} \tilde{N} & \tilde{E} \end{pmatrix}_I \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} H_{d1} \\ H_{d2} \end{pmatrix} E_J \\ & + \mu (H_{u1} \ H_{u2}) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} H_{d1} \\ H_{d2} \end{pmatrix}. \end{aligned}$$

Moreover, the superpotential includes a term with **the parameter μ** , which has **the dimension of mass**.

The NSVZ equations for the MSSM

The renormalization group running of the gauge couplings in the MSSM can be described exactly in all loops with the help of **the NSVZ β -functions**

M. A. Shifman, Int. J. Mod. Phys. A **11** (1996), 5761,

$$\begin{aligned}\frac{\beta_1}{\alpha_1^2} &= -\frac{3}{5} \cdot \frac{1}{2\pi} \left[-11 + \text{tr} \left(\frac{1}{6} \gamma_Q + \frac{4}{3} \gamma_U + \frac{1}{3} \gamma_D + \frac{1}{2} \gamma_L + \gamma_E \right) + \frac{1}{2} \gamma_{H_u} + \frac{1}{2} \gamma_{H_d} \right]; \\ \frac{\beta_2}{\alpha_2^2} &= -\frac{1}{2\pi(1 - \alpha_2/\pi)} \left[-1 + \text{tr} \left(\frac{3}{2} \gamma_Q + \frac{1}{2} \gamma_L \right) + \frac{1}{2} \gamma_{H_u} + \frac{1}{2} \gamma_{H_d} \right]; \\ \frac{\beta_3}{\alpha_3^2} &= -\frac{1}{2\pi(1 - 3\alpha_3/2\pi)} \left[3 + \text{tr} \left(\gamma_Q + \frac{1}{2} \gamma_U + \frac{1}{2} \gamma_D \right) \right].\end{aligned}$$

They relate three gauge β -functions of the theory to the anomalous dimensions of the chiral matter superfields. The renormalization group functions (RGFs) are defined by the equations

$$\beta_i(\alpha, Y) = \left. \frac{d\alpha_i}{d \ln \mu} \right|_{\alpha_0, Y_0 = \text{const}}; \quad \gamma_i(\alpha, Y) = \left. \frac{d \ln Z_i}{d \ln \mu} \right|_{\alpha_0, Y_0 = \text{const}},$$

where the subscript 0 denotes the bare values.

The exact equations describing the renormalization of the MSSM Yukawa couplings

RGFs describing the renormalization of the Yukawa couplings and of the parameter μ can also be related to the anomalous dimensions of the matter superfields due to the nonrenormalization of the superpotential

M. T. Grisaru, W. Siegel, M. Rocek, Nucl. Phys. B **159** (1979), 429.

$$\begin{aligned}\frac{dY_U}{d\ln\mu} &= \frac{1}{2} \left(\gamma_{H_u} Y_U + (\gamma_Q)^T Y_U + Y_U \gamma_U \right); \\ \frac{dY_D}{d\ln\mu} &= \frac{1}{2} \left(\gamma_{H_d} Y_D + (\gamma_Q)^T Y_D + Y_D \gamma_D \right); \\ \frac{dY_E}{d\ln\mu} &= \frac{1}{2} \left(\gamma_{H_d} Y_E + (\gamma_L)^T Y_E + Y_E \gamma_E \right); \\ \frac{d\mu}{d\ln\mu} &= \frac{1}{2} \left(\gamma_{H_u} + \gamma_{H_d} \right) \mu.\end{aligned}$$

It is important that these equations are valid in the HD+MSL scheme because in this scheme all renormalization constants contain only powers of $\ln \Lambda/\mu$, where Λ is the dimensionful regularization parameter.

The equations for the determinants of the Yukawa matrices

The renormalization group equations for the Yukawa couplings can be multiplied by the corresponding inverse matrices. After that, it is possible to calculate traces of the resulting equations using the formula

$$\mathrm{tr}\left[M^{-1} \frac{dM}{d\ln\mu}\right] = \frac{d}{d\ln\mu} \mathrm{tr} \ln M = \frac{d}{d\ln\mu} \ln \det M.$$

Then (taking into account that the indices numerating generations range from 1 to 3), we see that **the equations describing how the determinants of the Yukawa matrices** depend on the renormalization point μ are written as

$$\begin{aligned}\gamma_{\det Y_U} &\equiv \frac{d \ln \det Y_U}{d \ln \mu} = \mathrm{tr}\left[(Y_U)^{-1} \frac{dY_U}{d \ln \mu}\right] = \frac{1}{2} \left(3\gamma_{H_u} + \mathrm{tr}(\gamma_Q + \gamma_U)\right); \\ \gamma_{\det Y_D} &\equiv \frac{d \ln \det Y_D}{d \ln \mu} = \mathrm{tr}\left[(Y_D)^{-1} \frac{dY_D}{d \ln \mu}\right] = \frac{1}{2} \left(3\gamma_{H_d} + \mathrm{tr}(\gamma_Q + \gamma_D)\right); \\ \gamma_{\det Y_E} &\equiv \frac{d \ln \det Y_E}{d \ln \mu} = \mathrm{tr}\left[(Y_E)^{-1} \frac{dY_E}{d \ln \mu}\right] = \frac{1}{2} \left(3\gamma_{H_d} + \mathrm{tr}(\gamma_L + \gamma_E)\right).\end{aligned}$$

They can be solved together with the NSVZ equations and the equation describing the renormalization of the parameter μ .

The renormalization group equations for the (rigid part of the) MSSM

Collecting the above equations we obtain the system of differential equations describing the renormalization of the MSSM parameters exactly in all orders

$$\frac{d}{d \ln \mu} \left(\frac{5}{3} \cdot \frac{2\pi}{\alpha_1} \right) = -11 + \text{tr} \left(\frac{1}{6} \gamma_Q + \frac{4}{3} \gamma_U + \frac{1}{3} \gamma_D + \frac{1}{2} \gamma_L + \gamma_E \right) + \frac{1}{2} \gamma_{H_u} + \frac{1}{2} \gamma_{H_d};$$

$$\frac{d}{d \ln \mu} \left(\frac{2\pi}{\alpha_2} + 2 \ln \alpha_2 \right) = -1 + \text{tr} \left(\frac{3}{2} \gamma_Q + \frac{1}{2} \gamma_L \right) + \frac{1}{2} \gamma_{H_u} + \frac{1}{2} \gamma_{H_d};$$

$$\frac{d}{d \ln \mu} \left(\frac{2\pi}{\alpha_3} + 3 \ln \alpha_3 \right) = 3 + \text{tr} \left(\gamma_Q + \frac{1}{2} \gamma_U + \frac{1}{2} \gamma_D \right);$$

$$\frac{d \ln \det Y_U}{d \ln \mu} = \frac{1}{2} \left(3 \gamma_{H_u} + \text{tr}(\gamma_Q + \gamma_U) \right);$$

$$\frac{d \ln \det Y_D}{d \ln \mu} = \frac{1}{2} \left(3 \gamma_{H_d} + \text{tr}(\gamma_Q + \gamma_D) \right);$$

$$\frac{d \ln \det Y_E}{d \ln \mu} = \frac{1}{2} \left(3 \gamma_{H_d} + \text{tr}(\gamma_L + \gamma_E) \right);$$

$$\frac{d \ln \mu}{d \ln \mu} = \frac{1}{2} \left(\gamma_{H_u} + \gamma_{H_d} \right).$$

The anomalous dimensions of the chiral matter superfields and μ can be eliminated, thereby obtaining a differential equation which contains only derivatives of the gauge and Yukawa couplings.

Eliminating the anomalous dimensions of the matter superfields

First, we eliminate $\text{tr}(\gamma_L)$, $\text{tr}(\gamma_E)$, and $\gamma_{H_u} + \gamma_{H_d}$. The resulting equations contain γ_Q , γ_D , and γ_U only in the combination $\text{tr}(2\gamma_Q + \gamma_U + \gamma_D)$,

$$\frac{d}{d \ln \mu} \left(\frac{2\pi}{\alpha_3} + 3 \ln \alpha_3 \right) = 3 + \frac{1}{2} \text{tr}(2\gamma_Q + \gamma_U + \gamma_D);$$

$$\frac{d}{d \ln \mu} \left(\frac{2\pi}{\alpha_2} + 2 \ln \alpha_2 + \frac{5}{3} \cdot \frac{2\pi}{\alpha_1} - 2 \ln \mu - 2 \ln \det Y_E + 2 \ln \det Y_D \right) = -12 + \frac{4}{3} \text{tr}(2\gamma_Q + \gamma_U + \gamma_D);$$

$$\frac{d}{d \ln \mu} \left(\ln \det Y_D + \ln \det Y_U - 3 \ln \mu \right) = \frac{1}{2} \text{tr}(2\gamma_Q + \gamma_U + \gamma_D).$$

This allows either eliminating the one-loop constants or eliminating the parameter μ . **The resulting equations take the form**

$$\begin{aligned} \frac{d}{d \ln \mu} \left(\frac{2\pi}{\alpha_3} + 3 \ln \alpha_3 + \frac{\pi}{2\alpha_2} + \frac{1}{2} \ln \alpha_2 + \frac{5\pi}{6\alpha_1} \right. \\ \left. - \frac{1}{2} \ln \det Y_E - \frac{7}{6} \ln \det Y_D - \frac{5}{3} \ln \det Y_U + \frac{9}{2} \ln \mu \right) = 0; \end{aligned}$$

$$\begin{aligned} \frac{d}{d \ln \mu} \left(\frac{2\pi}{\alpha_3} + 3 \ln \alpha_3 - \frac{\pi}{\alpha_2} - \ln \alpha_2 - \frac{5\pi}{3\alpha_1} \right. \\ \left. + \ln \det Y_E - \frac{2}{3} \ln \det Y_D + \frac{1}{3} \ln \det Y_U - 9 \ln \mu \right) = 0, \end{aligned}$$

respectively.

The all-loop RGIs for the MSSM

Integrating the first equation we obtain the expression RGI_1 , which does not explicitly depend on the scale μ , but contains the parameter μ . Integrating the second equation gives the expression RGI_2 independent of μ , but containing the scale μ ,

$$\text{RGI}_1 = \frac{\mu^{9/2} (\alpha_3)^3 (\alpha_2)^{1/2}}{(\det Y_E)^{1/2} (\det Y_U)^{5/3} (\det Y_D)^{7/6}} \exp \left(\frac{2\pi}{\alpha_3} + \frac{\pi}{2\alpha_2} + \frac{5\pi}{6\alpha_1} \right);$$
$$\text{RGI}_2 = \frac{(\alpha_3)^3 \det Y_E (\det Y_U)^{1/3}}{\mu^9 \alpha_2 (\det Y_D)^{2/3}} \exp \left(\frac{2\pi}{\alpha_3} - \frac{\pi}{\alpha_2} - \frac{5\pi}{3\alpha_1} \right).$$

Instead of the renormalization group invariants (RGI_1 , RGI_2) it is possible to use the equivalent set (RGI_3 , RGI_4), where the expressions

$$\text{RGI}_3 \equiv \left(\frac{\text{RGI}_1}{\text{RGI}_2} \right)^{2/3} = \frac{\mu^3 \mu^6 \alpha_2}{(\det Y_E) (\det Y_U)^{4/3} (\det Y_D)^{1/3}} \exp \left(\frac{\pi}{\alpha_2} + \frac{5\pi}{3\alpha_1} \right);$$
$$\text{RGI}_4 \equiv (\text{RGI}_1)^{2/3} (\text{RGI}_2)^{1/3} = \frac{\mu^3 (\alpha_3)^3}{\mu^3 \det Y_U \det Y_D} \exp \left(\frac{2\pi}{\alpha_3} \right)$$

also have a rather simple form.

The three-loop check

Differentiating $\ln(\text{RGI}_3)$ and $\ln(\text{RGI}_4)$ with respect to $\ln \mu$ we obtain the equations

$$0 = \left(\frac{1}{\alpha_2} - \frac{\pi}{\alpha_2^2} \right) \beta_2 - \frac{5\pi}{3\alpha_1^2} \beta_1 + 6 + 3\gamma_\mu - \gamma_{\det Y_E} - \frac{4}{3} \gamma_{\det Y_U} - \frac{1}{3} \gamma_{\det Y_D};$$
$$0 = \left(\frac{3}{\alpha_3} - \frac{2\pi}{\alpha_3^2} \right) \beta_3 - 3 + 3\gamma_\mu - \gamma_{\det Y_U} - \gamma_{\det Y_D}.$$

The scheme dependence of these equations becomes essential starting from the order $O(\alpha^2, \alpha Y^2, Y^4)$ corresponding to the three-loop approximation for the β -functions and to the two-loop approximation for the anomalous dimensions.

The three-loop β -functions for the MSSM in the HD+MSL scheme have been calculated in

O. Haneychuk, V. Shirokova, K.S., JHEP **09** (2022), 189.

and in the $\overline{\text{DR}}$ scheme in

I. Jack, D. R. T. Jones, A. F. Kord, Annals Phys. **316** (2005), 213.

In the HD+MSL scheme the above equations are satisfied in the three-loop approximation independently of the regularization parameters.

The three-loop verification, the $\overline{\text{DR}}$ scheme

However, in the $\overline{\text{DR}}$ scheme these equations are not valid starting from the three-loop approximation,

$$\begin{aligned}
 & \left[\left(\frac{1}{\alpha_2} - \frac{\pi}{\alpha_2^2} \right) \beta_2 - \frac{5\pi}{3\alpha_1^2} \beta_1 + 6 + 3\gamma_\mu - \gamma_{\det Y_E} - \frac{4}{3} \gamma_{\det Y_U} - \frac{1}{3} \gamma_{\det Y_D} \right]_{\overline{\text{DR}}} \\
 &= \frac{1}{2\pi^2} \left(\frac{1243\alpha_1^2}{400} + \frac{17\alpha_2^2}{16} - 5\alpha_3^2 \right) + \frac{1}{16\pi^3} \text{tr}(Y_U Y_U^+) \left(\frac{143\alpha_1}{180} + \frac{11\alpha_2}{4} + \frac{44\alpha_3}{9} \right) \\
 &+ \frac{1}{16\pi^3} \text{tr}(Y_D Y_D^+) \left(\frac{14\alpha_1}{45} + 2\alpha_2 + \frac{32\alpha_3}{9} \right) + \frac{1}{16\pi^3} \text{tr}(Y_E Y_E^+) \left(\frac{9\alpha_1}{10} + \frac{3\alpha_2}{2} \right) \\
 &- \frac{1}{(16\pi^2)^2} \left[11 \text{tr}((Y_U Y_U^+)^2) + 8 \text{tr}((Y_D Y_D^+)^2) + 6 \text{tr}((Y_E Y_E^+)^2) + \frac{19}{3} \text{tr}(Y_D Y_D^+ Y_U Y_U^+) \right. \\
 &\left. + 11 (\text{tr}(Y_U Y_U^+))^2 + 8 (\text{tr}(Y_D Y_D^+))^2 + 2 (\text{tr}(Y_E Y_E^+))^2 + \frac{26}{3} \text{tr}(Y_E Y_E^+) \text{tr}(Y_D Y_D^+) \right] \\
 &+ O(\alpha^3, \alpha^2 Y^2, \alpha Y^4, Y^6) \neq O(\alpha^3, \alpha^2 Y^2, \alpha Y^4, Y^6); \\
 & \left[\left(\frac{3}{\alpha_3} - \frac{2\pi}{\alpha_3^2} \right) \beta_3 - 3 + 3\gamma_\mu - \gamma_{\det Y_U} - \gamma_{\det Y_D} \right]_{\overline{\text{DR}}} = \frac{1}{2\pi^2} \left(\frac{363\alpha_1^2}{400} + \frac{9\alpha_2^2}{16} - \frac{21\alpha_3^2}{8} \right) \\
 &+ \frac{1}{16\pi^3} \text{tr}(Y_U Y_U^+) \left(\frac{13\alpha_1}{30} + \frac{3\alpha_2}{2} + \frac{8\alpha_3}{3} \right) + \frac{1}{16\pi^3} \text{tr}(Y_D Y_D^+) \left(\frac{7\alpha_1}{30} + \frac{3\alpha_2}{2} + \frac{8\alpha_3}{3} \right) \\
 &- \frac{1}{(16\pi^2)^2} \left[6 \text{tr}((Y_U Y_U^+)^2) + 6 \text{tr}((Y_D Y_D^+)^2) + 6 (\text{tr}(Y_U Y_U^+))^2 + 6 (\text{tr}(Y_D Y_D^+))^2 \right. \\
 &\left. + 2 \text{tr}(Y_E Y_E^+) \text{tr}(Y_D Y_D^+) + 4 \text{tr}(Y_D Y_D^+ Y_U Y_U^+) \right] + O(\alpha^3, \alpha^2 Y^2, \alpha Y^4, Y^6) \\
 &\neq O(\alpha^3, \alpha^2 Y^2, \alpha Y^4, Y^6).
 \end{aligned}$$

The $6D, \mathcal{N} = (1, 0)$ higher derivative theory

Usual supersymmetric theories in higher dimensions are not renormalizable. However, in this case it is possible to investigate theories with higher derivatives.

Following the paper

I. L. Buchbinder, A. S. Budekhina, E. A. Ivanov and K.S.,
Phys. Rev. D **111** (2025) no.12, 125014.

we consider the theory similar to the one proposed in

E. A. Ivanov, A. V. Smilga and B. M. Zupnik, Nucl. Phys. B **726** (2005), 131,

which in the $6D, \mathcal{N} = (1, 0)$ harmonic superspace

A. Galperin, E. Ivanov, S. Kalitzin, V. Ogievetsky and E. Sokatchev, Class. Quant. Grav. **1** (1984), 469; A. S. Galperin, E. A. Ivanov, V. I. Ogievetsky and E. S. Sokatchev, "Harmonic superspace", Cambridge, UK: Univ. Pr. (2001) 306 p.

is described by the action

$$\begin{aligned} S = & \pm \frac{1}{2e_0^2} \text{tr} \int d\zeta^{(-4)} (F^{++})^2 - \frac{2}{e_0^2} \text{tr} \int d\zeta^{(-4)} \widetilde{q^+} \nabla^{++} q^+ \\ & = \text{tr} \int d^6x \left\{ \pm \frac{1}{e_0^2} (\mathcal{D}_\mu F^{\mu\nu})^2 + \dots \right\}. \end{aligned}$$

The NSVZ-like exact (?) β -function

The theory could contain quadratic and logarithmical divergences, but the quadratic divergences cancel each other in the one-loop approximation (and presumably in all loops) due to the presence of the hypermultiplet in the adjoint representation. Moreover, due to the presence of the hypermultiplet, the theory is not anomalous

A. V. Smilga, Phys. Lett. B **647** (2007), 298

and seems to be renormalizable. The hypermultiplet and ghosts do not receive divergent quantum corrections and only the gauge coupling is renormalized. Surprisingly, it seems that this theory possesses the exact NSVZ-like β -function

$$\beta(\alpha_0) = \mp \frac{\alpha_0^2 C_2}{2\pi^2 \left(1 \mp \alpha_0 C_2 / 8\pi^2\right)}.$$

Similarly to the pure $4D$, $\mathcal{N} = 1$ SYM theory, it is possible to integrate the renormalization group equation and obtain the expression that does not receive quantum corrections in any order of the perturbation theory,

$$\left(\frac{\alpha}{\mu^4}\right)^{C_2} \exp\left(\pm \frac{8\pi^2}{\alpha}\right) = \text{RGI}.$$

- The running of gauge and Yukawa couplings in various supersymmetric theories obeys some equations, which are **exact in all orders** of the perturbation theory.
- Due to these equations, **various couplings** in supersymmetric theories **do not run independently**.
- Some of the exact relations between running couplings in supersymmetric theories can be written in the form of **RGIs**, which, by definition, are **scale independent in all orders**.
- In particular, for $\mathcal{N} = 1$ SQCD+SQED it is possible to construct **an RGI from two gauge couplings**.
- For **MSSM** there are **two independent RGIs** involving gauge and Yukawa couplings.
- The exact equations relating RGFs are valid only for certain **renormalization prescriptions** which include **the HD+MSL scheme** and, in general, do not include $\overline{DR}$ scheme.
- Presumably, it is also possible to write **the NSVZ-like equations** and construct the corresponding RGIs for certain **higher-derivative supersymmetric theories in higher dimensions**.

Thank you for the attention!