$4d$ Superconformal indices, $3d$ partition functions and many-body systems

V. P. Spiridonov

BLTP JINR and NRU HSE

Superconformal indices of $4d$, $\mathcal{N}=1$ supersymmetric gauge field theories:

- Space-time symmetry group $SU(2,2|1)$: Lorentz group, $J_i, \overline{J}_i$; supertranslations, P_μ, Q_α, $\overline{Q}_{\dot\alpha}$; special superconformal transformations, $K_\mu, S_\alpha, \overline{S}_{\dot\alpha}$; dilatations, H; $U(1)_R$-symmetry, R.
- Local gauge group G and global flavor group F (generators F_k).
- Take $Q := \overline{Q}_1$ and $Q^\dagger := \overline{S}_1 \Rightarrow$

$$QQ^\dagger + Q^\dagger Q = 2\mathcal{H}, \qquad \mathcal{H} = H - 2\overline{J}_3 - 3R/2$$

  The superconformal index (Romelsberger, 2005; KMMR, 2005):

$$I(p,q,y_k) = \mathrm{Tr}\Big((-1)^{\mathcal{F}} p^{\mathcal{R}/2+J_3} q^{\mathcal{R}/2-J_3} \prod_k y_k^{F_k}\, e^{-\beta\mathcal{H}}\Big), \quad \mathcal{R} = H - \tfrac{1}{2}R.$$

- $p, q, y_k, \beta =$ group parameters for maximal Cartan generators preserving taken SUSY. Gets contributions only from BPS-states

$$Q|\psi\rangle = Q^\dagger|\psi\rangle = \mathcal{H}|\psi\rangle = 0, \qquad Q^2 = (Q^\dagger)^2 = 0,$$

  $\Rightarrow$ does not depend on $\beta \Rightarrow$ a topological object, coinciding with SUSY partition function of theories on $S^3 \times S^1$ in external fields.

Computation $\Rightarrow$ the matrix integral:

$$I(y;p,q) \;=\; \int_G d\mu(z)\, \exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\mathrm{ind}\big(p^n,q^n,z^n,y^n\big)\right)$$

with the Haar measure $d\mu(z)$ and single particle states index

$$\mathrm{ind}(p,q,z,y) = \frac{2pq-p-q}{(1-p)(1-q)}\chi_{adj_G}(z)$$
$$+\sum_j \frac{(pq)^{R_j/2}\chi_{r_F,j}(y)\chi_{r_G,j}(z)-(pq)^{1-R_j/2}\chi_{\bar{r}_F,j}(y)\chi_{\bar{r}_G,j}(z)}{(1-p)(1-q)}.$$

$\chi_{R_F,j}(y)$, $\chi_{R_G,j}(z)$ = characters of the group representations, R_j = R-charges of chiral fields.

Example: $G=SU(2), F=SU(8)$, (f,f) chiral superfield with $R=1/2$ (Dolan, Osborn, 2008)

$$I_E = V(t_1,\ldots,t_8;p,q) = \frac{(p;p)_\infty(q;q)_\infty}{4\pi i}\int_{\mathbb{T}}\frac{\prod_{j=1}^8\Gamma(t_jx^{\pm1};p,q)}{\Gamma(x^{\pm2};p,q)}\frac{dx}{x},$$

$t_j=y_j(pq)^{R/2}\Rightarrow\prod_{j=1}^8 t_j=(pq)^2$, $(z;p)_\infty=\prod_{j=0}^\infty(1-zp^j)$,

$$\Gamma(z;p,q)=\prod_{j,k=0}^{\infty}\frac{1-z^{-1}p^{j+1}q^{k+1}}{1-zp^jq^k},\quad |q|,|p|<1.$$

This is the elliptic hypergeometric function V.S., 2003

$W(E_7)$-symmetry $\Rightarrow$ Seiberg duality.

The elliptic hypergeometric equation

$$\frac{\prod_{j=1}^{8}\theta(\varepsilon_j z;p)}{\theta(z^2;p)\theta(qz^2;p)}(\psi(qz)-\psi(z))+\frac{\prod_{j=1}^{8}\theta(\varepsilon_j z^{-1};p)}{\theta(z^{-2};p))\theta(qz^{-2};p)}(\psi(q^{-1}z)-\psi(z))$$

$$+\prod_{k=1}^{6}\theta(\frac{\varepsilon_k\varepsilon_8}{q};p)\psi(z)=0, \qquad \prod_{k=1}^{8}\varepsilon_k=p^2q^2, \quad \varepsilon_8=q\varepsilon_7,$$

where $\theta(z)=(z;p)_\infty(pz^{-1};p)_\infty$ and $\psi(z)\propto V(\ldots,cz,c/z,\ldots;p,q),\ c=\sqrt{\varepsilon_6\varepsilon_8}/p^2$.

The most general known solvable (discrete) Schrödinger equation V.S., 2004
= a particular eigenvalue equation for the one-particle Hamiltonian
of integrable many-body van Diejen model (generalized Ruijsenaars model)

NB: Quantum mechanical wave function $\sim$ QFT partition function

$4d \to 3d$ **reduction:** Dolan, Vartanov, V.S., 2011

$$\Gamma(e^{-2\pi v u}; e^{-2\pi v\omega_1}, e^{-2\pi v\omega_2}) \underset{v\to 0^+}{=} e^{-\pi\frac{2u-\omega_1-\omega_2}{12 v\omega_1\omega_2}} \gamma^{(2)}(u;\omega_1,\omega_2).$$ Ruijsenaars, 1997

Faddeev's (1994) modular dilogarithm, or hyperbolic gamma function

$$\gamma^{(2)}(u;\omega) = e^{-\frac{\pi i}{2}B_{2,2}(u;\omega)} \frac{(\tilde{\mathbf{q}} e^{2\pi i \frac{u}{\omega_1}}; \tilde{\mathbf{q}})_\infty}{(e^{2\pi i\frac{u}{\omega_2}}; \mathbf{q})_\infty}, \qquad \mathbf{q} = e^{2\pi i \frac{\omega_1}{\omega_2}}, \quad \tilde{\mathbf{q}} = e^{-2\pi i\frac{\omega_2}{\omega_1}},$$

$$B_{2,2}(u;\omega) = \frac{1}{\omega_1\omega_2}\left((u - \frac{\omega_1+\omega_2}{2})^2 - \frac{\omega_1^2+\omega_2^2}{12}\right),$$

Then $$V(e^{-2\pi v g_k}; e^{-2\pi v\omega_1}, e^{-2\pi v\omega_2}) \underset{v\to 0^+}{=} e^{\frac{\pi}{4v}\left(\frac{1}{\omega_1}+\frac{1}{\omega_2}\right)} I_h(\underline{g}),$$ Rains, 2006

$$I_h(\underline{g}) = \int_{-i\infty}^{i\infty} \frac{\prod_{j=1}^8 \gamma^{(2)}(g_j \pm z;\omega_1,\omega_2)}{\gamma^{(2)}(\pm 2z;\omega_1,\omega_2)} \frac{dz}{2i\sqrt{\omega_1\omega_2}},$$

SUSY partition function of a particular $3d$ field theory on the squashed S_b^3.

A particular $3d$ $\mathcal{N}=2$ quiver gauge field theory on duality wall of $4d$ $\mathcal{N}=4$ theory Hosomichi, Lee, Park, 2010

$$G = U(1)\times U(2)\times \ldots \times U(N-1), \qquad F = U(N)$$

with the explicit partition function

$$\Psi_{\underline{\lambda_n}}(\underline{x_n};g)=\Lambda_n(\lambda_n)\Psi_{\underline{\lambda_{n-1}}}(\underline{x_{n-1}};g),\qquad \underline{x_n}=(x_1,\dots,x_n),\quad \Psi_{\lambda_1}(x_1)=e^{\frac{2\pi i}{\omega_1\omega_2}\lambda_1 x_1},$$

$$(\Lambda_n(\lambda_n)f)(\underline{x_n})=\frac{\gamma^{(2)}(g)^{n-1}}{(n-1)!}\int_{\mathbb{R}^{n-1}}\Lambda(\underline{x_n},\underline{y_{n-1}};\lambda)\,f(\underline{y_{n-1}})\prod_{k=1}^{n-1}\frac{dy_k}{\sqrt{\omega_1\omega_2}},$$

$$\Lambda(\underline{x_n},\underline{y_{n-1}};\lambda)=e^{\frac{2\pi i}{\omega_1\omega_2}\lambda(\sum_{i=1}^n x_i-\sum_{i=1}^{n-1}y_i)}\prod_{i=1}^{n}\prod_{j=1}^{n-1}K(x_i-y_j)\prod_{\substack{i,j=1\\ i\neq j}}^{n-1}\mu(y_i-y_j),$$

$$K(x)=\gamma^{(2)}(\tfrac{1}{2}g^*\pm ix),\qquad \mu(x)=\frac{\gamma^{(2)}(g^*\pm ix)}{\gamma^{(2)}(\pm ix)},\quad g^*=\omega_1-\omega_2-g.$$

Origin: K - bifundamental chiral fields, g - mass term, μ - vector supermultiplet, λ_j - FI terms.

Surprize: the eigenfunction of hyperbolic Ruijsenaars Hamiltonian (Hallnas, Ruijsenaars, 2014)

$$H\Psi_{\underline{\lambda_n}}(\underline{x_n};g)=\big(\sum_{j=1}^{n}e^{\frac{2\pi}{\omega_2}\lambda_j}\big)\Psi_{\underline{\lambda_n}}(\underline{x_n};g),\quad H=\sum_{j=1}^{n}\prod_{k=1,\neq j}^{n}\frac{\operatorname{sh}\frac{\pi}{\omega_2}(x_j-x_k-ig)}{\operatorname{sh}\frac{\pi}{\omega_2}(x_j-x_k)}e^{-i\omega_1\partial_{x_i}}.$$

Full solution: Belousov, Derkachov, Kharchev, Khoroshkin, 2023

Quantum integrable many-body systems

Calogero model (1969-1971)

$$H\left(\underline{p},\underline{x}\right)=\frac{1}{2}\sum_{n=1}^{N}\left(p_n^2+\omega^2x_n^2\right)+g^2\sum_{m,n=1;m\neq n}^{N}\frac{1}{\left(x_n-x_m\right)^2},\quad [x_n,p_m]=i\hbar\delta_{nm}$$

Trigonometric(hyperbolic) and elliptic extensions: Sutherland, Inozemtsev, etc.

Relativistic generalization: Ruijsenaars, van Diejen

Two particle hyperbolic Ruijsenaars Hamiltonians in the center of mass

$$H_h=\sum_{\varepsilon=\pm1}\frac{\operatorname{sh}\frac{\pi}{\omega_2}(x-i\varepsilon g)}{\operatorname{sh}\frac{\pi}{\omega_2}x}\,e^{\varepsilon\omega_1 p},\qquad H_h'=\sum_{\varepsilon=\pm1}\frac{\operatorname{sh}\frac{\pi}{\omega_1}(x-i\varepsilon g)}{\operatorname{sh}\frac{\pi}{\omega_1}x}\,e^{\varepsilon\omega_2 p},\quad p=-i\frac{d}{dx}$$

Hyperbolic wave function:

$$F_\lambda^g(x)=\gamma^{(2)}(g;\omega)\int_{\mathbb{R}}e^{\frac{2\pi i}{\omega_1\omega_2}\lambda z}\gamma^{(2)}\left(\tfrac{1}{2}g^*\pm\tfrac{1}{2}ix\pm iz;\omega\right)\frac{dz}{\sqrt{\omega_1\omega_2}}$$

Eigenvalue problems

$$H_hF_\lambda^g(x)=2\operatorname{ch}\frac{\pi\lambda}{\omega_2}F_\lambda^g(x),\qquad H_h'F_\lambda^g(x)=2\operatorname{ch}\frac{\pi\lambda}{\omega_1}F_\lambda^g(x).$$

Standard degenerations: 1) $x \to \omega_1 x$, $g = b\omega_1$, $\omega_1 \to 0$, 2) x fixed, $g = b\omega_1$, $\omega_1 \to 0$,

$$1)\ H_h \to \frac{x - ib}{x} e^{p} + \frac{x + ib}{x} e^{-p}, \qquad 2)\ H_h \to -\partial_x^2 - \pi^2 \frac{b(b-1)}{\operatorname{sh}^2 \pi x}.$$

New complex degenerations (Sarkissian, V.S., 2022; Belousov, Sarkissian, V.S., 2025)

3) rational relativistic: $\delta \to 0$, $N \in \mathbb{Z}$, $N \to \infty$, $N\delta \to \alpha \in \mathbb{R}$ in

$$\omega_1 = i + \delta, \quad \omega_2 = -i + \delta, \quad x = n + u\delta, \quad g = i(r + h\delta), \quad \lambda = N + \beta.$$

$$H_{cr}(b) = \sum_{\varepsilon = \pm 1} \frac{z \pm b}{z} \, e^{\varepsilon \partial_z}, \quad H'_{cr}(b') = \sum_{\varepsilon = \pm 1} \frac{z' \pm b'}{z'} \, e^{\varepsilon \partial_{z'}}, \qquad e^{\partial_z} := e^{\partial_n} e^{-i\partial_u}, \quad e^{\partial_{z'}} := e^{-\partial_n} e^{-i\partial_u},$$

$$z = \frac{n + iu}{2}, \quad b = \frac{r + ih}{2}, \quad z - z' = n, \quad b - b' = r, \quad n, r \in \mathbb{Z}.$$

4) non-relativistic of Sutherland type: $\delta \to 0$, $x = N + \beta$, $N \in \mathbb{Z}$, $N \to \infty$, $N\delta \to \alpha$,

$$\gamma = \alpha + i\beta, \quad g = i(r + h\delta), \quad H_h \to -\partial_\gamma^2 - \pi^2 \frac{b(b-1)}{\operatorname{sh}^2 \pi\gamma}, \quad H'_h \to -\partial_{\bar\gamma}^2 - \pi^2 \frac{b'(b'-1)}{\operatorname{sh}^2 \pi\bar\gamma},$$

A degeneration of the hyperbolic gamma function — Sarkissian, V.S., 2020

$$\gamma^{(2)}(i\sqrt{\omega_1\omega_2}(n + u\delta); \omega) \underset{\delta \to 0^+}{=} e^{\frac{\pi i}{2} n^2} (4\pi\delta)^{iu-1} \mathbf{\Gamma}(u, n), \qquad \sqrt{\frac{\omega_1}{\omega_2}} = i + \delta,$$

where $\mathbf{\Gamma}(u, n)$ is the complex gamma function

$$\mathbf{\Gamma}(u, n) = \frac{\Gamma(\frac{n+iu}{2})}{\Gamma(1 + \frac{n-iu}{2})}, \quad n \in \mathbb{Z}.$$

Limiting wave function $\propto$

$$F^{r,h}_{\alpha,\beta}(u,n) = \frac{1}{4\pi\mathbf{\Gamma}(h,r)} \sum_{k\in\mathbb{Z}+\epsilon} \int_{-\infty}^{\infty} e^{-2\pi i(\beta k+\alpha y)}\, \mathbf{\Gamma}(\tfrac{1}{2}h \pm \tfrac{1}{2}u \pm y, \tfrac{1}{2}r \pm \tfrac{1}{2}n \pm k)\, dy,$$

where $\epsilon \in \{0, \frac{1}{2}\}$, satisfies eigenvalue problems

$$H_{cr}(1-b)\, F^{r,h}_{\alpha\beta}(u,n) = -2\,\mathrm{ch}\,\pi\gamma\, F^{r,h}_{\alpha\beta}(u,n), \qquad H'_{cr}(1-b')\, F^{r,h}_{\alpha\beta}(u,n) = -2\,\mathrm{ch}\,\pi\bar{\gamma}\, F^{r,h}_{\alpha\beta}(u,n).$$

and their duals

$$H_c\, F^{r,h}_{\alpha,\beta}(u,n) = -(\pi z)^2 F^{r,h}_{\alpha,\beta}(u,n), \qquad H'_c\, F^{r,h}_{\alpha,\beta}(u,n) = -(\pi z')^2 F^{r,h}_{\alpha,\beta}(u,n),$$

$$H_c = -\partial^2_\gamma - 2\pi b\coth(\pi\gamma)\,\partial_\gamma - (\pi b)^2, \qquad H'_c = -\partial^2_{\bar{\gamma}} - 2\pi b'\coth(\pi\bar{\gamma})\,\partial_{\bar{\gamma}} - (\pi b')^2.$$

5) complementary rational relativistic: $\omega_1 = 1 + i\delta, \quad \omega_2 = 1 - i\delta, \quad \delta \to 0,$

$$x = i(-k + u\delta), \quad g = \ell - h\delta, \quad H_h \to \sum_{\varepsilon=\pm 1} \frac{z \pm b}{z}\, e^{\varepsilon\partial_z}, \quad H'_h \to \sum_{\varepsilon=\pm 1} \frac{z' \pm b'}{z'}\, e^{-\varepsilon\partial_{z'}}$$

6) complimentary non-relativistic: $\delta \to 0, \quad N \in \mathbb{Z},\ N \to \infty, \quad N\delta \to \alpha, \quad x = i(N+\beta),$

$$g = \ell + h\delta, \quad y = \alpha + i\beta, \quad H_h \to -\partial^2_{\bar{y}} - \pi^2 \frac{b'(b'+1)}{\mathrm{sh}^2\,\pi\bar{y}}, \quad H'_h \to -\partial^2_y - \pi^2 \frac{b(b-1)}{\mathrm{sh}^2\,\pi y}, \quad b' = -\bar{b}.$$