Hypergeometric functions: IBP reduction and series expansions

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- Introduced by **Gelfand**, **Kapranov** and **Zelevinsky**.
- Generalize classical hypergeometric functions using:
  - **Toric geometry**
  - **$\mathcal{A}$-hypergeometric systems**
  - **Euler–Mellin integrals**
- Defined via an integer matrix $\mathcal{A}$ and parameter vector $\vec{c} \in \mathbb{C}^{n+k}$.

$$\Phi_{\mathcal{A},\vec{c}}(\alpha, \beta; u) = \int_\gamma f_1(t)^{\alpha_1} \cdots f_k(t)^{\alpha_k} t^{\vec{\beta}} \frac{dt_1 \wedge \cdots \wedge dt_n}{t_1 \cdots t_n}$$

together with $\vec{\beta} = (\beta_1, \ldots \beta_n) \in \mathbb{C}^n$ and $t^{\vec{\beta}} = t_1^{\beta_1} \ldots t_n^{\beta_n}$.

Each $f_j(t)$ is a **Laurent** polynomial with support $S_j \subset \mathbb{Z}^n$

$$f_j(t) = \sum_{\vec{m} \in S_j} u_{j,\vec{m}} t^{\vec{m}}$$

and integration is over a suitable cycle γ

GKZ Hypergeometric functions and D-modules

$\mathcal{A}$-matrix construction:

- For each polynomial $f_j(t)$ enumerate exponents of monomials $\vec{m} \in S_j$:

$$S_j = \{\vec{m}_{j,1}, \dots, \vec{m}_{j,r_j}\}, \quad r_j = |S_j|, \ N \equiv \sum_{j=1}^{k} r_j$$

- $\mathcal{A}$-matrix of size $(n+k) \times N$ is given by

$$\mathcal{A} = \begin{pmatrix} \vec{m}_{1,1} & \cdots & \vec{m}_{1,r_1} & \cdots & \vec{m}_{k,r_k} \\ \vec{e}_1 & \cdots & \vec{e}_1 & \cdots & \vec{e}_k \end{pmatrix} \in \mathbb{Z}^{(n+k)\times N}$$

Bottom k rows is the selector matrix with $\vec{e}_j \in \mathbb{Z}^k$ being standard basis vector (1 in position j and 0 elsewhere)

Example (2 variables and 2 polynomials):

Suppose:

- $f_1(t) = u_{1,1} + u_{1,2}t_1 + u_{1,3}t_2$,
- $f_2(t) = u_{2,1} + u_{2,2}t_1^2 t_2$

Then:

- $S_1 = \{(0,0), (1,0), (0,1)\}$,
- $S_2 = \{(0,0), (2,1)\}$

GKZ Hypergeometric functions and D-modules

and $\mathcal{A}$-matrix is

$$\mathcal{A} = \begin{pmatrix} 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

4 rows: 2 variables + 2 polynomials

5 columns: $5 u_{i,j}$ coefficients

Determination of $\vec{c} \in \mathbb{C}^{n+k}$ vector:

- Fixing an index $i \in \{1, \ldots, n\}$ and considering transformation $u_{j,\vec{m}} \to t^{m_i} u_{j,\vec{m}}$ and variable change $t_i' = t t_i$ we get ($t \in \mathbb{C}^\times$):

$$\Big(\sum_{j,\vec{m}} m_i \theta_{j,\vec{m}}\Big) \Phi_{\mathcal{A},\vec{c}}(\vec{\alpha}, \vec{\beta}; \vec{u}) = \beta_i \Phi_{\mathcal{A},\vec{c}}(\vec{\alpha}, \vec{\beta}; \vec{u})$$

  where $\theta_{j,\vec{m}} = u_{j,\vec{m}} \frac{\partial}{\partial u_{j,\vec{m}}}$ is Euler partial derivative.

- Scaling all coefficients $u_{l,\vec{m}}$ of $f_l(t)$ by s we get

$$\Big(\sum_{\vec{m} \in S_l} \theta_{l,\vec{m}}\Big) \Phi_{\mathcal{A},\vec{c}}(\vec{\alpha}, \vec{\beta}; \vec{u}) = \alpha_l \Phi_{\mathcal{A},\vec{c}}(\vec{\alpha}, \vec{\beta}; \vec{u})$$

- Thus

$$\vec{c} = (\beta_1, \ldots, \beta_k, \alpha_1, \ldots, \alpha_n).$$

GKZ Hypergeometric functions and D-modules

The complete GKZ $\mathcal{A}$-hypergeometric system is given by the system of equations:

- $k+n$ first-order Euler equations

$$\vec{a}_1 u_1 \frac{\partial \Phi}{\partial u_1} + \ldots + \vec{a}_N \frac{\partial \Phi}{\partial u_N} = \vec{c}\Phi$$

- infinite set of equations of order N or less

$$\prod_{l_i<0} \left(\frac{\partial}{\partial u_i}\right)^{-l_i} \Phi = \prod_{l_i>0} \left(\frac{\partial}{\partial u_i}\right)^{l_i} \Phi$$

where

$$\mathbb{L} = \{\vec{l} = (l_1, \ldots, l_N) \in \mathbb{Z}^N \,|\, \mathcal{A} \cdot \vec{l} = 0\}$$

and

$$\mathcal{A} = (\vec{a_1}, \ldots, \vec{a_N}).$$

General holonomic D-modules

Other ways to obtain differential equations for **Euler-Melin** integrals:

- In the context of **Feynman** integrals it is the solution of momentum space IBP relations with, for example, **Laporta** algorithm
- IBP relations can be also obtained in parametric representation with parametric annihilators of integrand expression. For example, for

$$I_{\nu_1,\ldots,\nu_n} = \frac{\Gamma(d/2)}{\Gamma(d/2-\omega)} \prod_{i=1}^{N} \int_0^\infty \frac{z_i^{\nu_1-1} dz_i}{\Gamma(\nu_i)} \mathcal{G}^{-d/2}$$

and IBP ideal in parametric space is given by $(1 \le i \le N)$:

$$\mathcal{G}\partial_i + \frac{d}{2}(\partial_i \mathcal{G}) \in \mathsf{Ann}(G^{-d/2})\,.$$

Integral representation gives us an isomorphism betwen **Weyl** algebra in parametric space and shift algebra in integral indices $\nu_1, \ldots \nu_n$.

General holonomic D-modules

- The shift algebra relations can be also obtained by considering equations

$$\partial^a Q_i \equiv 0 \mod RI, i = 1, \dots, r$$

  where a is multi-index and Q_i form a basis of ideal I for **Weyl** algebra R in coefficients of monomials of $\mathcal{G}$-polynomial or parameters they depend on. It is the annihilator of **Euler-Mellin** integrals. For example, D-ideal formed by GKZ $\mathcal{A}$-hypergeometric system. Last, but not least, this way we could obtain **Pfaffian** system associated to ideal I

- The solution of shift relations for example with **Gauss** elimination and **Laporta** algorithm give us both a possibility to reduces indices of **Euler-Mellin** integrals and general holomonic differential systems for the latter.

General holonomic D-modules

D-modules M are modules over **Weyl** algebra A_n, which is free algebra over $\mathbb{C}$

$$A_n := \mathbb{C}[x_1, \ldots, x_n]\langle \partial_1, \ldots, \partial_n \rangle$$

modulo relations

$$[\partial_i, x_j] = \partial_i x_j - x_j \partial_i, \quad i, j = 1, \ldots, n$$

Rational **Weyl** algebra:

$$R_n := \mathbb{C}(x_1, \ldots, x_n)\langle \partial_1, \ldots, \partial_n \rangle$$

D-modules are defined through the ideal of differential operators $P_1, \ldots, P_r$

$$I = \langle P_1, \ldots, P_n \rangle, \quad P_i(x, \partial) = \sum_{\alpha,\beta} c_{\alpha,\beta} x^\alpha \partial^\beta$$

such that for any $Q \in D_n$

$$Q P_j f(x_1, \ldots, x_n) = 0, \quad f(x) \in M$$

where $f(x)$ are holonomic solutions.

In general, the solution space is defined as

$$Sol_M(I) := \{m \in M | P \circ m = 0, P \in I\}$$

There is an insomorphism

$$Hom_{D_n}(D_n / D_n I, M) \cong Sol_M(I)$$

allowing to study obstructions for local solutions to become global with $Ext^i_{D_n}$ groups

The $\mathbb{C}$-vector space of solutions for holonomic ideal I outside singular locus of I has finite dimension:

$$\text{rank}(I) = \dim_{\mathbb{C}(x_1, \ldots, x_n)}(R_n / R_n I)$$

General holonomic D-modules

To find series solutions for D-modules we need to introduce the notion of weight vectors:

$$(v,w) = (v_1,\dots,v_n,w_1,\dots,w_n) \in \mathbb{Z}^{2n}, \quad v_j + w_j \geq 0$$

for monomial orders: $\mathsf{ord}_{(v,w)}(x^\alpha \partial^\beta) = \sum_{i=1}^n v_i\alpha_i + w_i\beta_i$.

The initial form $\mathsf{in}_{(v,w)}(P)$ of operator $P \in D_n$ is the part with maximal (v,w)-weight. Then

- The characteristic ideal of D-ideal I is

$$\mathsf{in}_{(0,1)}(I) = \langle \mathsf{in}_{(0,1)}(P) | P \in I \rangle \subset \mathbb{C}[x_1,\dots,x_n][\xi_1,\dots,\xi_n]$$

- The characteristic variety of ideal I is

$$\mathsf{Char}(I) = V(\mathsf{in}_{(0,1)}(I)) = \{(x,\xi) | p(x,\xi) = 0, p \in \mathsf{in}_{(0,1)}(I)\}$$

  Ideal is holonomic if $\mathsf{dim}\ \mathsf{Char}(I) = n$

- The singular locus $\mathsf{Sing}(I)$ is the vanishing set of the ideal

$$(\mathsf{in}_{(0,1)}(I) : (\xi_1,\dots,\xi_n)^{(\infty)}) \cap \mathbb{C}[x_1,\dots,x_n]$$

  solved with saturation + elimination

General holonomic D-modules

Grobner deformations for **Frobenius** series expansions are considered with weights $(-w, w)$, $w \in \mathbb{R}^n$ and initial ideals

$$\mathrm{in}_w(I) = \mathrm{in}_{-w,w}(I)$$

The indicial ideal of I is $\mathbb{C}[\theta_1, \ldots, \theta_n]$-ideal

$$\mathrm{ind}_w(I) = R_n \cdot \mathrm{in}_{-w,w}(I) \cap \mathbb{C}[\theta_1, \ldots, \theta_n], \quad \theta_i = x_i \partial_i .$$

Zeroes of $\mathrm{ind}_w(I)$ in $\mathbb{C}^n$ are exponents of I, such that starting monomials of solutions to I are of the form $x^A \log(x)^B$ with $A \in V(\mathrm{ind}_w(I))$.

From starting monomials algorithm of **Saito**, **Strumfels** and **Takayama** construct **Frobenius** series solutions

$$\Phi_k(x) = x^A \cdot \sum_{\substack{0 \le p \cdot w \le k, p \in C_{\mathbb{Z}}^* \\ 0 \le b_j < \mathrm{rank}(I)}} c_{p,b} x^p \log^b(x) \,,$$

so $\Phi_k \in$ Nilson ring $N_w(I)$ with respect to weight w.

The convergence regions are determined by maximum cones in Grobner fan. The latter are regions in weight space where $\mathrm{in}_w(I)$ does not change

Toric geometry and combinatorics of polyhedral fans

The toric geometry consideration of GKZ systems starts with primary polytope in $\mathbb{R}^d$ space defined by $\mathcal{A}$-matrix:

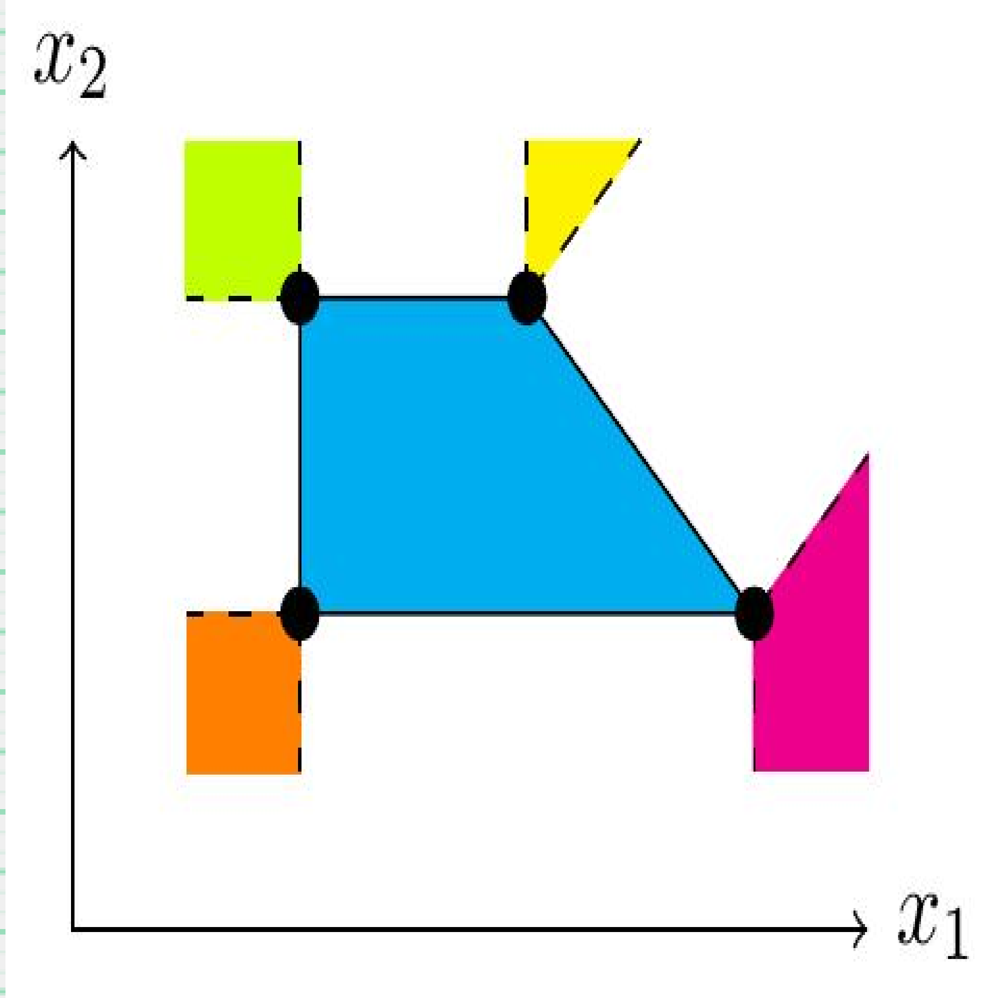
$$P = \triangle_{\mathcal{A}} = \mathsf{conv}(A) = \mathsf{conv}(a_1, \ldots a_n) \subset \mathbb{R}^d \,,$$

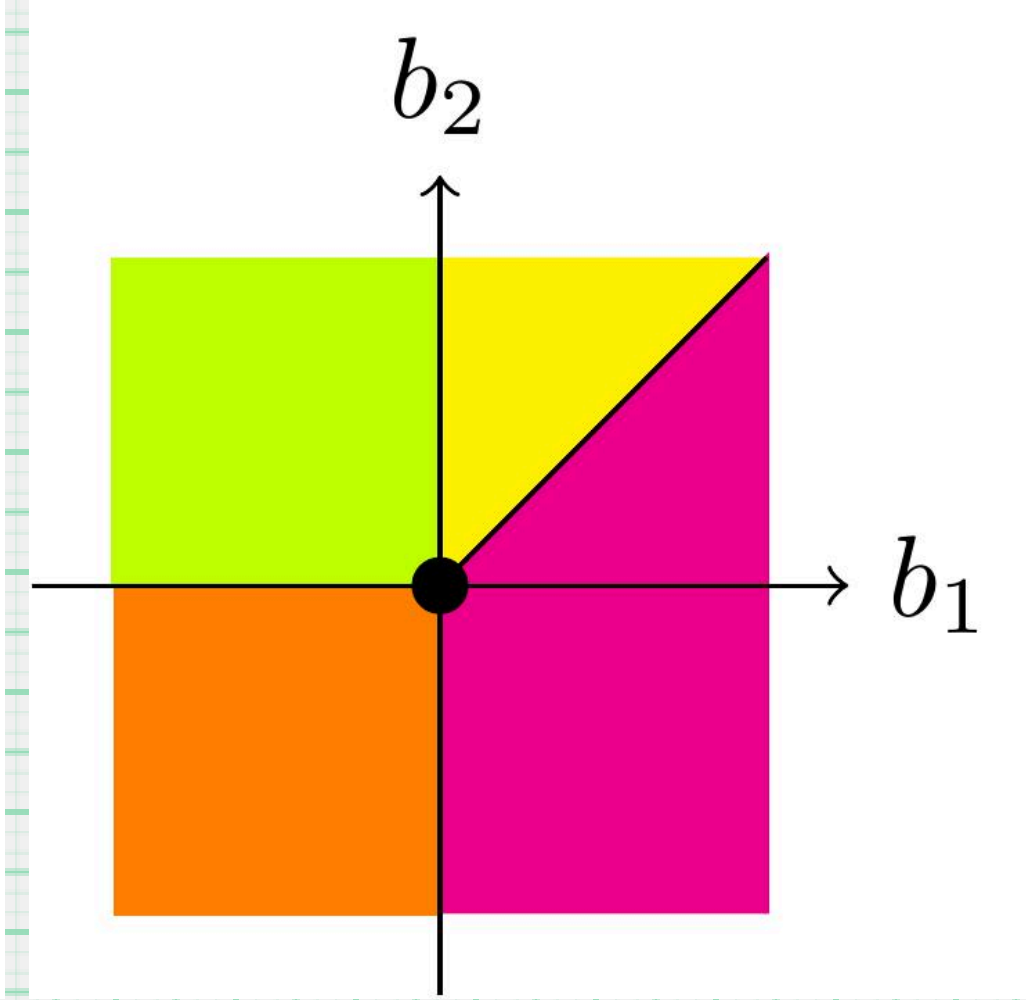
convex hull of $\mathcal{A}$-matrix columns from which we build its normal fan. The latter is the collection of normal cones

$$N_F(P) = \{b \in (\mathbb{R}^d)^* | F \subseteq \{x \in P | b \cdot x = \mathsf{max}_{y \in P}(b \cdot y)\}\}$$

Normal cones live in the dual $(\mathbb{R}^d)^*$ space and the toric variety described by this fan is in general singular. Using height functions to produce regular triangulations of the primary polytope (lifted to $\mathbb{R}^{d+1}$ triangulation is convex) induces the resolution of singularities of the original toric variety by subdividing singular cones (affine charts of toric variety) to make them regular.

Toric geometry and combinatorics of polyhedral fans





Toric geometry and combinatorics of polyhedral fans

Next, for any regular triangulation $T = \{\sigma_1, \sigma_2, \ldots\}$ of P we define a point $q_T \in \mathbb{R}^n$ with coordinates $q_{1T}, \ldots q_{nT}$:

$$q_{jT} = \sum_{\sigma \in T, a_j \in \text{vertices}(\sigma)} \text{vol}_0(\sigma)\,, \quad \text{vol}_0(\sigma) = d!\text{vol}(\sigma)$$

The secondary polytope $\Sigma(\mathcal{A})$ is defined as convex hull

$$\Sigma(\mathcal{A}) = \text{conv}(S) \subset \mathbb{R}^n$$

of the set of points $S = \{q_{T_1}, q_{T_2}, \ldots\} \in \mathbb{R}^n$. The secondary fan (moduli space) of GKZ system is build as the normal fan of the latter.

Now, the Γ-series solutions of GKZ-system are build from the lattice $\mathbb{L}$ and vector $\vec{\gamma} \in \mathbb{C}^n$

$$\Phi_{\mathbb{L},\vec{\gamma}}(x_1, \ldots, x_n) = \sum_{(l_1,\ldots,l_n)\in\mathbb{L}} \prod_{j=1}^{n} \frac{x_j^{l_j+\gamma_j}}{\Gamma(l_j + \gamma_j + 1)}\,,$$

where $\mathcal{A} \cdot \mathbb{L} = 0$ and $\vec{c} = \mathcal{A} \cdot \vec{\gamma}$.

Toric geometry and combinatorics of polyhedral fans

If $\vec{c}$ is generic and $\triangle_{\mathcal{A}}$ admits unimodular triangulation, then the dimension of the solution space is $\mathsf{vol}_0(\triangle_{\mathcal{A}})$.

For a maximal cone C of the secondary fan $N(\Sigma(\mathcal{A}))$ we take a corresponding triangulation and construct T_C as a list of subsets of indices $(1, \ldots, n)$, such that each subset denotes the indices of the vertices of maximal simplicies in this triangulation. Then we look for vectors $\vec{\gamma} \in \mathbb{C}^n$ such that

$$\vec{c} = a \cdot \vec{\gamma}$$
$$\exists J \in T_C, \text{such that } \gamma_i \in \mathbb{Z}_{\leq 0} \text{ for} j \notin J$$

If the number of such solutions is less then $\mathsf{vol}_0(\triangle_{\mathcal{A}})$ the vector $\vec{c}$ is resonant. In the latter case to find logarithmic solutions we use differentiation with rescpect to its components.

Frobenius solutions of Pfaffian systems

These techniques were already partially implemented in public *Mathematica* package **PrecisionLauricella**.

Consider the evaluation of

$$F_1\left(\frac{1}{2}; 1, \epsilon; \frac{3}{2}; \frac{4}{3}, \frac{7}{4}\right)$$

Choosing function basis as

$$J = \left\{F_1, \, x\frac{\partial}{\partial x}F_1, \, y\frac{\partial}{\partial y}F_1\right\}$$

the corresponding Pfaffian system takes the form

$$dJ = (M_x\, dx + M_y\, dy)\, J\,,$$

$$M_x = \begin{pmatrix} 0 & \frac{1}{x} & 0 \\ \frac{1}{2-2x} & \frac{-3x^2+(3-2\varepsilon)xy+x+(2\varepsilon-1)y}{2(x-1)x(x-y)} & \frac{y-1}{(x-1)(x-y)} \\ 0 & \frac{\varepsilon y}{x^2-xy} & \frac{1}{y-x} \end{pmatrix},$$

and

$$M_y = \begin{pmatrix} 0 & 0 & \frac{1}{y} \\ 0 & \frac{\varepsilon}{x-y} & \frac{x}{y(y-x)} \\ \frac{\varepsilon}{2-2y} & -\frac{\varepsilon(x-1)}{(y-1)(x-y)} & \frac{x+y}{2xy-2y^2} - \frac{\varepsilon}{y-1} \end{pmatrix}$$

Selecting the path $x = \frac{t}{3}$ and $y = \frac{7t}{16}$, the M_t differential system in the variable t is given by the matrix

$$M_t = \begin{pmatrix} 0 & \frac{1}{t} & \frac{1}{t} \\ \frac{1}{6-2t} & -\frac{3(t-1)}{2(t-3)t} & \frac{1}{3-t} \\ \frac{7\varepsilon}{32-14t} & \frac{7\varepsilon}{16-7t} & \frac{16-7(2\varepsilon+1)t}{2t(7t-16)} \end{pmatrix}$$

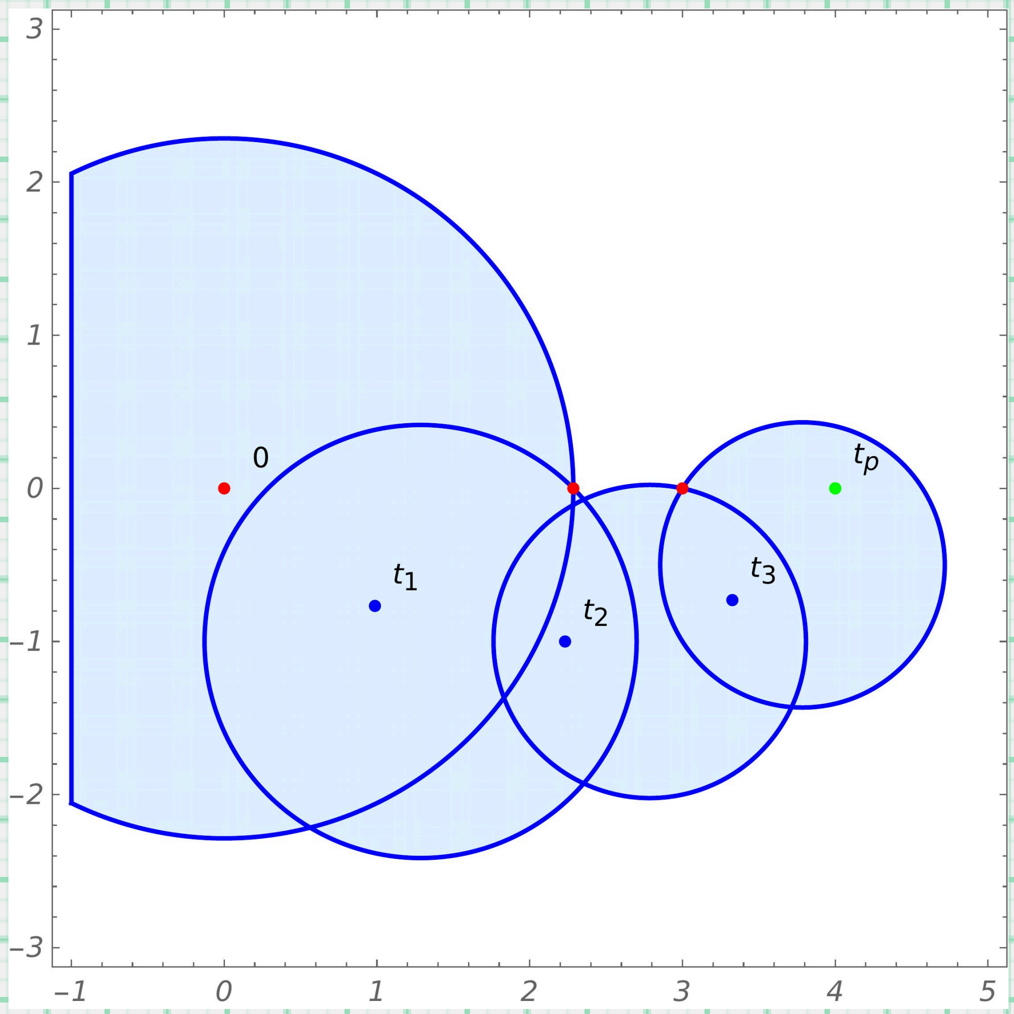
Frobenius solutions of Pfaffian systems

Analytical continuation

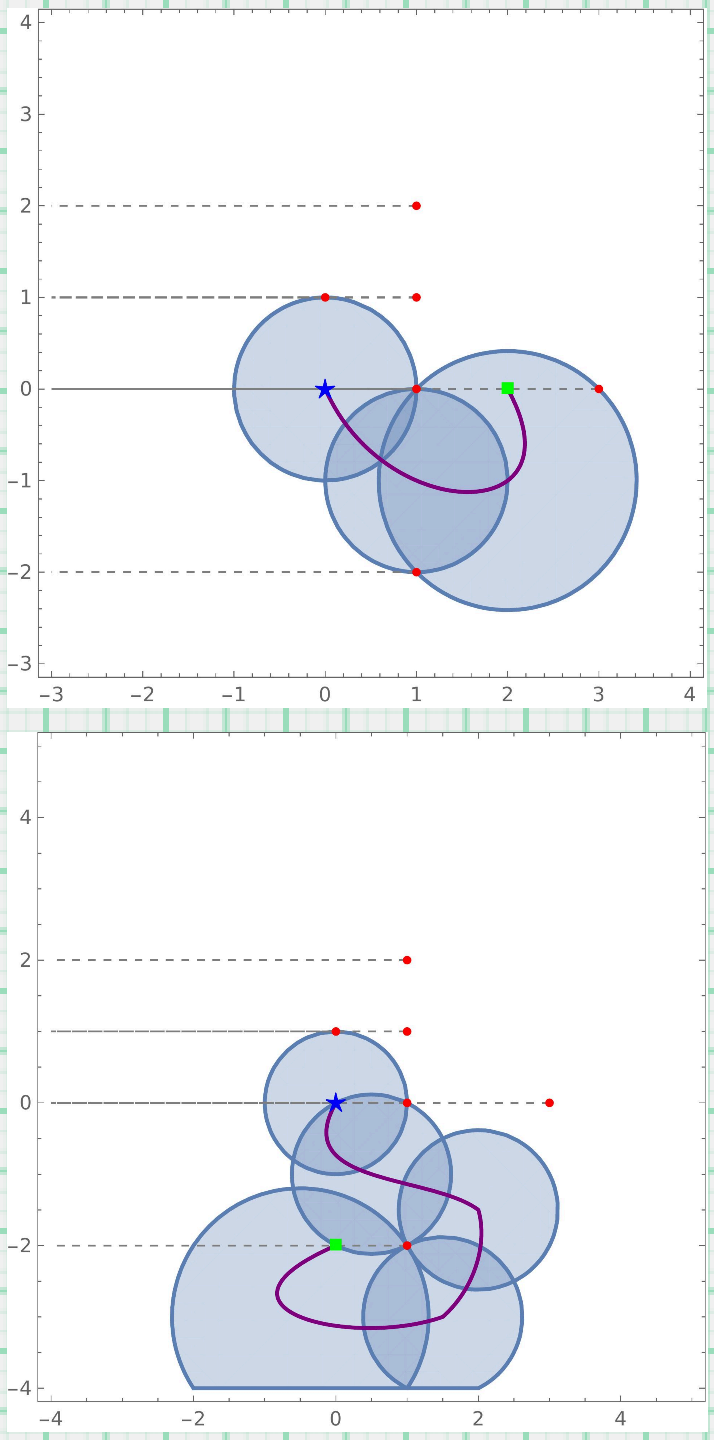
$$F_1\left(\frac{1}{2};1,\epsilon;\frac{3}{2};\frac{4}{3},\frac{7}{4}\right) = U^{\frac{53}{14}-\frac{i}{2}}(t_p)$$
$$\times\left(U^{\frac{53}{14}-\frac{i}{2}}(t_3)\right)^{-1} U^{\frac{39}{14}-i}(t_3)$$
$$\times\left(U^{\frac{39}{14}-i}(t_2)\right)^{-1} U^{\frac{9}{7}-i}(t_2)$$
$$\times\left(U^{\frac{9}{7}-i}(t_1)\right)^{-1} U^{0}(t_1)\left(U^{0}(0)\right)^{-1} b\,,$$

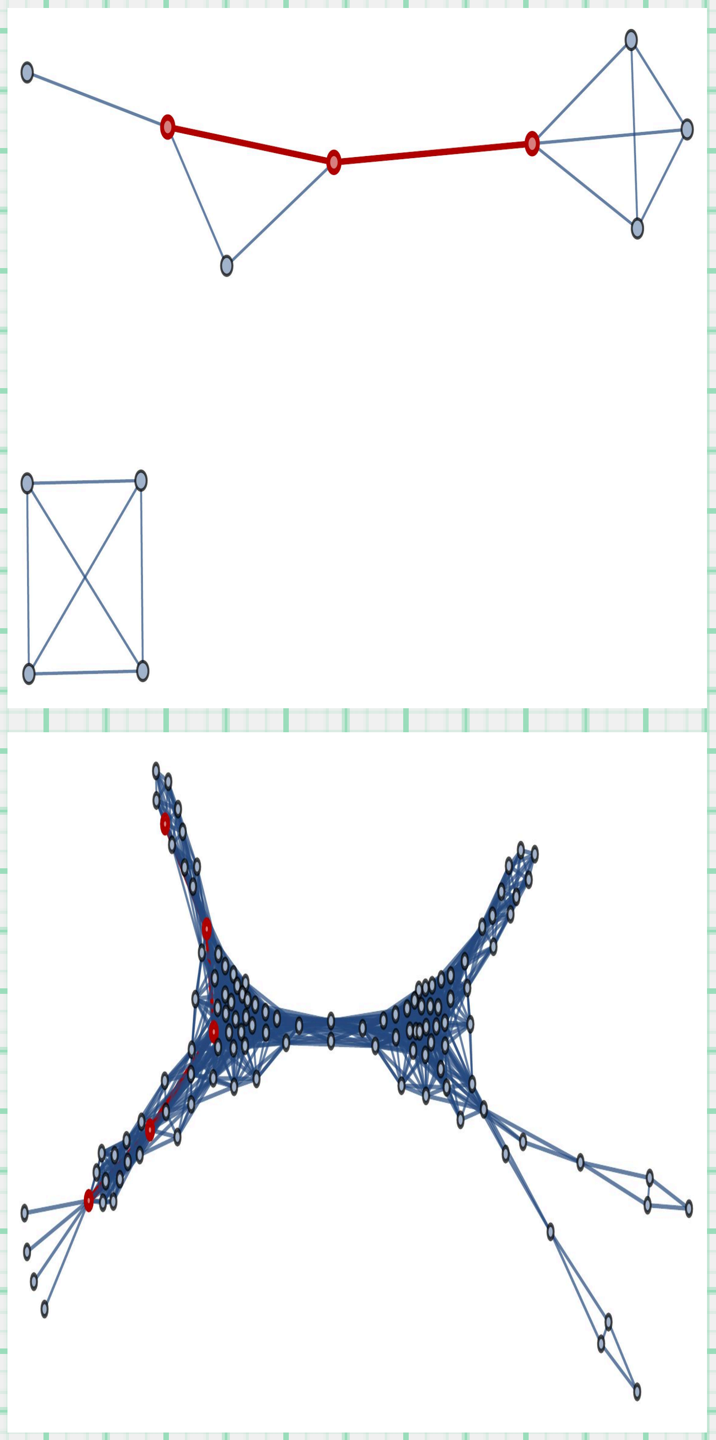
and final result

$$F_1\left(\frac{1}{2};1,\epsilon;\frac{3}{2};\frac{4}{3},\frac{7}{4}\right)$$
$$=(1.1405189944-1.3603495231\,i)$$
$$-(1.9381695438+1.5059564172\,i)\varepsilon$$
$$-(1.6764200809-2.0776109157\,i)\varepsilon^2$$
$$+(1.6422823823+1.4396930521\,i)\varepsilon^3$$
$$+\mathcal{O}(\varepsilon^4)\,.$$



Frobenius solutions of Pfaffian systems





Conclusion and future directions

- Development of general purpose expert system for the solution of holomonic D-modules, in particular GKZ hypergeometric systems.
- Development of expert system for the reduction of indices of general Euler-Mellin integrals
- Development of expert system for training neural networks representing solutions of holonomic D-modules
- Further study of different combinatoric techniques provided by toric geometry with applications to, in particular, variation of mixed Hodge stuctures, calculation of intersection numbers, twisted cohomology, monodromy groups, resolution of singularities and asymptotic expansions of Euler-Mellin integrals