Disparity in sound speeds: implications for unitarity and effective potential in quantum field theory

Based on arXiv:2505.07794, in collaboration with D. Ageev

Supported by Russian Science Foundation Grant No. 24-72-00121

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International Conference "Advances in Quantum Field Theory 2025"

JINR, BLTP, Dubna

1 Outline

1 Motivation

- Probability conservation, embodied in the unitarity of the S-matrix, has long been one of the most powerful consistency checks on quantum field theories (QFTs).
- Unitarity relation and partial wave expansions are fundamental tools for ensuring probability.
- While most classic studies assume relativistic dispersion with a universal ($c = 1$) speed of light $\rightarrow$ many contemporary frameworks (from multi-field inflation to Lorentz-violating extensions of the Standard Model) request scalar degrees of freedom that propagate with different sound speeds.

1 Motivation

- Thus, we consider a theory which contains massive scalar fields with different sound speeds.
- We derive unitarity relations for partial wave amplitudes of $2 \to 2$ scattering, with explicit formulas for contributions of two-particle intermediate states.
- Making use of these relations, we obtain unitarity bounds.
- These bounds can be used for estimating the strong coupling scale of a pertinent EFT.
- Moreover, we derive one-loop Coleman–Weinberg potential for small splitting $\delta = u_2 - u_1$ ($u_2 = c_s + \delta$ and $u_1 = c_s$) and RG flow modification: all quartic β-functions scale $\propto (c_s - \frac{3}{2}\delta)$, revealing an "accidental" fixed line $\delta = \frac{2}{3}c_s$.

2 Action and states

- The quadratic action reads

$$S = \sum_i S_{\phi_i} \ , \quad S_{\phi_i} = \int d^4x \left(\frac{1}{2}\dot{\phi_i}^2 - \frac{1}{2}u_i^2(\vec{\nabla}\phi_i)^2 - \frac{1}{2}m_i^2\phi_i^2 \right) ,$$

$$E_{p_i}^2 = u_i^2 p_i^2 + m_i^2 \ .$$

- Now, we consider an initial state:

$$|\psi, \beta\rangle = \sqrt{2E_{p_1}}\sqrt{2E_{p_2}}\, a_{p_1}^\dagger a_{p_2}^\dagger \,|0\rangle \ ,$$

and the same form we have for the final state $|\psi', \beta'\rangle$.

- Notation β refers to the types of the two particles, $\beta = \{\phi_i, \phi_j\}$, while notation ψ is a shorthand for the pair of momenta, $\psi = \{\vec{p}_1, \vec{p}_2\}$.

2 Unitarity relation

- Start with the scalar product of $|\psi', \beta'\rangle$ and $|\psi, \beta\rangle$ states

$$\langle\psi', \beta' \,|\psi, \beta\,\rangle = (2\pi)^6 2E_{p_1} 2E_{p_2} \delta^{(3)}(\vec{p}_1\,' - \vec{p}_1)\delta^{(3)}(\vec{p}_2\,' - \vec{p}_2)\delta_{\beta'\beta}\ ,$$

- Unitarity of S-matrix, $SS^\dagger = S^\dagger S = 1$ implies

$$T - T^\dagger = iTT^\dagger = iT^\dagger T\ .$$

- In terms of two-particle state of definite angular momentum one has

$$-i\left(T^{(l)}_{m'\beta';m\beta} - T^{(l)\,*}_{m\beta;m'\beta'}\right) = \int d^4\mathcal{P}'' \sum_{m'',\beta''} \frac{1}{N(\beta'')} T^{(l)}_{m'\beta';m''\beta''}\, T^{(l)\,*}_{m\beta;m''\beta''},$$

$$N(\beta) \equiv 2(2\pi)^5 \frac{\left(u^2_{1\beta}E_2 + u^2_{2\beta}E_1\right)}{p}, \quad p \equiv |\vec{p}\,| = |\vec{p}_1| = |\vec{p}_2|,$$

where $E = E_1 + E_2 = \sqrt{u_1^2 p^2 + m_1^2} + \sqrt{u_2^2 p^2 + m_2^2}$.

2 Unitarity relation

- Then in terms of PWA

$$a_{l,\beta'\beta} = \frac{1}{32\pi}\int d(\cos\gamma)\cdot P_l(\cos\gamma)M_{\beta'\beta},$$

which related to $T^{(l)}_{m'\beta';m\beta}$ as

$$T^{(l)}_{m'\beta';m\beta} = 16\pi\cdot(2\pi)^4\delta^{(4)}(\mathcal{P}^{\mu\,\prime} - \mathcal{P}^{\mu})\delta_{m'm}\; a_{l,\beta'\beta}\ ,$$

we obtain

$$\mathrm{Im}\; a_{l,\alpha\beta} = \sum_{\gamma}\frac{2p_\gamma}{(E_2 u^2_{1\gamma} + E_1 u^2_{2\gamma})}a_{l,\alpha\gamma}a^{*}_{l,\beta\gamma}.$$

- If one takes $u_{1\gamma} = u_{2\gamma} = 1$, then this relation coincides with the standard one.

2 Unitarity relation

- Both cases of distinguishable and identical particles can be generalized as follows:

$$\text{Im } a_{l,\alpha\beta} = \sum_{\gamma} g_\gamma a_{l,\alpha\gamma} a^*_{l,\gamma\beta},$$

where

$$g_\gamma = \frac{2p_\gamma}{(E_2 u^2_{1\gamma} + E_1 u^2_{2\gamma})} \quad \text{distinguishable },$$

$$g_\gamma = \frac{1}{2u^2_\gamma} \frac{p_\gamma}{E_{p_\gamma}} \quad \text{identical }.$$

2 Unitarity bound

- One can introduce the rescaled amplitudes as

$$a_{l,\alpha\beta} = \frac{\tilde{a}_{l,\alpha\beta}}{\sqrt{g_\alpha g_\beta}} ,$$

- Then the optical theorem reads

$$\text{Im } \tilde{a}_{l,\alpha\beta} = \sum_\gamma \tilde{a}_{l,\alpha\gamma} \tilde{a}^*_{l,\gamma\beta}.$$

- The unitarity bound is

$$|\text{Re } \tilde{a}_{l,\alpha\alpha}| \leq \frac{1}{2},$$

what is written for the diagonal part of $\tilde{a}_{l,\alpha\beta}$ and it is right for any eigenvalue of $\tilde{a}_l$.

- Now we explicitly illustrate that the obtained unitarity relation is carried out in the lowest order by the coupling constants.
- To this end, consider

$$S = \int d^4x \Big[\frac{1}{2}\dot{\phi}_1^2 + \frac{1}{2}\dot{\phi}_2^2 - \frac{1}{2}u_1^2(\partial\phi_1)^2 - \frac{1}{2}u_2^2(\partial\phi_2)^2 - \frac{1}{2}m_1^2\phi_1^2 - \frac{1}{2}m_2^2\phi_2^2 + \frac{\lambda_1}{4!}\phi_1^4 + \frac{\lambda_2}{4!}\phi_2^4 + \frac{\lambda_3}{4}\phi_1^2\phi_2^2 \Big].$$

- There are three two-particle states $\alpha = (\phi_1, \phi_1)$, $\beta = (\phi_1, \phi_2)$, and $\gamma = (\phi_2, \phi_2)$ in this theory.
- Tree-level PWAs is given by

$$a_{0,\text{tree}} = \frac{1}{32\pi}\int_{-1}^{1} d(\cos\theta) P_0(\cos\theta) M_{\text{tree}} = \frac{M_{\text{tree}}}{16\pi} = \frac{1}{16\pi}\begin{pmatrix} \lambda_1 & 0 & \lambda_3 \\ 0 & \lambda_3 & 0 \\ \lambda_3 & 0 & \lambda_2 \end{pmatrix}.$$

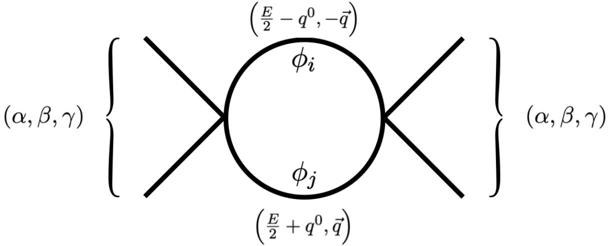

Figure: The 1-loop s-wave diagram. We recall the notations: $\alpha = (\phi_1, \phi_1)$, $\beta = (\phi_1, \phi_2)$, and $\gamma = (\phi_2, \phi_2)$. This picture includes all possible processes: **(1)** $\alpha \to \alpha$ through fields ϕ_2 (so $\phi_i = \phi_j = \phi_2$ in that case and both vertices are λ_3); **(2)** $\alpha \to \alpha$ through fields ϕ_1; **(3)** $\alpha \to \gamma$ through fields ϕ_1; **(4)** $\alpha \to \gamma$ through fields ϕ_2; **(5)** $\gamma \to \gamma$ through fields ϕ_1; **(6)** $\gamma \to \gamma$ through field ϕ_2; **(7)** $\beta \to \beta$ through fields ϕ_1 and ϕ_2.

- On the other hand, we need to evaluate the integrals like:

$$iM^{(1)}_{1-\text{loop}} = \frac{1}{2}(i\lambda_3)^2 \int \frac{d^4q}{(2\pi)^4} \frac{i}{(\frac{l}{2}-q)^2 - m_2^2 + i\epsilon} \frac{i}{(\frac{l}{2}+q)^2 - m_2^2 + i\epsilon}$$

$$= \frac{\lambda_3^2}{2} \int \frac{d^4q}{(2\pi)^4} \frac{1}{(\frac{E}{2}-q^0)^2 - u_2^2\vec{q}^{\,2} - m_2^2 + i\epsilon} \frac{1}{(\frac{E}{2}+q^0)^2 - u_2^2\vec{q}^{\,2} - m_2^2 + i\epsilon},$$

- Using notations $E_{(2)} \equiv \frac{E}{2}$ and $u_2 p_{(2)} \equiv \sqrt{E^2_{(2)} - m_2^2}$ we obtain

$$\text{Im } M^{(1)}_{\text{loop}} = \frac{\lambda_3^2}{16\pi} \frac{p_{(2)}}{E_{(2)}} \frac{1}{2u_2^2}.$$

- The subscript "(2)" reflects that there is the particles related to ϕ_2 field "run" in the loop.

3 Bound for the specific model: example

- Last but not least we present the specific bound for the current model. Let us consider only $\alpha \to \alpha$ process, since the derivations for the rest are totally the same. For chosen $\alpha \to \alpha$ process we immediately arrive to the bound in the form:

$$\left|\frac{\lambda_1}{16\pi}\frac{p_{(1)}}{2u_1^2 E_{(1)}}\right| \leq \frac{1}{2}, \quad l = 0, \quad E_{(1)} = \sqrt{u_1^2 p_{(1)}^2 + m_1^2}.$$

- In terms of energy only

$$\left|\frac{\lambda_1}{16\pi}\frac{\sqrt{E_{(1)}^2 - m_1^2}}{2u_1^3 E_{(1)}}\right| \leq \frac{1}{2}.$$

- To compute the one-loop correction, we consider small fluctuations around the background fields $\phi_1 = \phi_{1c} + \delta\phi_1$ and $\phi_2 = \phi_{2c} + \delta\phi_2$.
- For constant background fields $\phi_1 = \phi_{1c}$ and $\phi_2 = \phi_{2c}$, the kinetic terms vanish, and the classical potential is obtained directly from the potential terms in the Lagrangian:

$$V_{\text{classical}}(\phi_{1c}, \phi_{2c}) = \frac{1}{2} m_1^2 \phi_{1c}^2 + \frac{1}{2} m_2^2 \phi_{2c}^2 + \frac{\lambda_3}{4} \phi_{1c}^2 \phi_{2c}^2 + \frac{\lambda_1}{4!} \phi_{1c}^4 + \frac{\lambda_2}{4!} \phi_{2c}^4.$$

- Expanding the Lagrangian up to quadratic order in the fluctuations

$$\mathcal{L}_{\text{quad}} = \frac{1}{2}(\partial_0 \delta\phi_1)^2 - \frac{u_1^2}{2}(\nabla\delta\phi_1)^2 + \frac{1}{2}(\partial_0 \delta\phi_2)^2 - \frac{u_2^2}{2}(\nabla\delta\phi_2)^2 - \\ - \frac{1}{2} m_{1,\text{eff}}^2 \delta\phi_1^2 - \frac{1}{2} m_{2,\text{eff}}^2 \delta\phi_2^2 - \lambda_3 \phi_{1c}\phi_{2c} \cdot \delta\phi_1 \delta\phi_2,$$

with

$$m_{1,\text{eff}}^2 = m_1^2 + \frac{\lambda_3}{2}\phi_{2c}^2 + \frac{\lambda_1}{2}\phi_{1c}^2, \quad m_{2,\text{eff}}^2 = m_2^2 + \frac{\lambda_3}{2}\phi_{1c}^2 + \frac{\lambda_2}{2}\phi_{2c}^2.$$

- The standard evaluations lead to

$$V_{\text{1-loop}} = \frac{1}{2}\int \frac{d^3\mathbf{k}}{(2\pi)^3}(\omega_+ + \omega_-), \tag{2}$$

where $\mathbf{k}^2 = k_x^2 + k_y^2 + k_z^2$ and

$$\omega_\pm^2(\mathbf{k}) = \frac{(u_1^2 + u_2^2)\mathbf{k}^2 + m_+ \pm \sqrt{\left((u_1^2 - u_2^2)\mathbf{k}^2 + m_-\right)^2 + 4s^2}}{2}, \tag{3}$$

where $s = \lambda_3\phi_{1c}\phi_{2c}$.

- The integration in (2) with (3) cannot be performed analytically, so we make the expansion when the sound speeds difference is relatively small.

4 β-functions and fixed line

- For simplicity considering massless case one obtains that beta functions of chosen theory are given by

$$\beta_{\lambda_1} = \frac{3\left(\lambda_3^2 + \lambda_1^2\right)}{16\pi^2 c_s^4}\left(c_s - 3/2\delta\right),$$

$$\beta_{\lambda_2} = \frac{3\left(\lambda_3^2 + \lambda_2^2\right)}{16\pi^2 c_s^4}\left(c_s - 3/2\delta\right),$$

$$\beta_{\lambda_3} = \frac{\lambda_3\left(\lambda_1 + \lambda_2 + 4\lambda_3\right)}{16\pi^2 c_s^4}\left(c_s - 3/2\delta\right).$$

- So far in our approximation the original beta function scales as $c_s - 3/2\delta$, surprisingly leading us the vanishing of beta function at special $\delta = 2/3c_s$.

5 Outline

5 Results and outlook

- We are going to address the optical theorem and unitarity relation in the case of fully anisotropic dispersion relation, i.e. for theory with the following Lagrangian

$$S = \sum_i S_{\phi_i} ,$$

$$S_{\phi_i} = \int d^4x \Big(\frac{1}{2}\dot{\phi}_i^{\,2} - \frac{1}{2}u_{x,i}^2(\partial_x\phi_i)^2 - \frac{1}{2}u_{y,i}^2(\partial_y\phi_i)^2 - \frac{1}{2}u_{z,i}^2(\partial_y\phi_i)^2 - \frac{1}{2}m_i^2\phi_i^2\Big).$$

- It is possible to rewrite optical theorem as

$$2\; p_{init}(u_1^2E_2 + u_2^2E_1)\sum_X \sigma(A\to X) = \mathrm{Im} M(A\to A),$$

where $|A\rangle$ is a two-particle state. As usual, the optical theorem states that the imaginary part of the forward scattering amplitude is proportional to the total scattering cross section.
- Then the Froissart bound can be derived from the analyticity.


thank you so much