Yang-Baxter structure of extended space

based on the work with Kirill Gubarev

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Statements

- (Almost abelian) bi-vector deformations are equivalent to coordinate transformations in doubled space
- It is possible to construct uni-vector deformations generating solutions in Einstein–Maxwell dilaton theory.

Why deformations?

Under gauge/gravity duality families of CFT's correspond to families of supergravity backgrounds.

Bi-vector deformations

- Type II supergravity fields: $G_{mn}, B_{mn}, \varphi, C_{(p)}$
- A general bi-vector Yang-Baxter deformation
  [Bakhmatov, Colgain, Sheikh-Jabbari, Yavatanoo (2018)]

$$
\begin{aligned}
&\bullet \quad (G+B)^{-1} = g^{-1} + \beta && \text{no initial flux} \\
&\bullet \quad (G+B)^{-1} = (g+b)^{-1} + \beta && \text{with a flux of } b_{mn}
\end{aligned}
\tag{1}
$$

- Sufficient conditions to have a solution

$$
\begin{aligned}
[k_a, k_b] &= f_{ab}{}^c k_c && \text{(Killing vector algebra)} \\
\beta^{mn} &= k_a{}^m k_b{}^n r^{ab} && \text{(bi-Killing anzats);} \\
r^{b_1[a_1} r^{|b_2|a_2} f_{b_1 b_2}{}^{a_3]} &= 0 && \text{(classical YB equation);} \\
r^{b_1 b_2} f_{b_1 b_2}{}^a k_a{}^m &= I^m = 0 && \text{(unimodularity condition);}
\end{aligned}
\tag{2}
$$

Origin of deformations

From dualities:

- String on $\mathbb{T}^d$ is invariant under (global) $\mathrm{O}(d,d)$

$$\Downarrow$$

bi-vector deformations: $O_\beta = \exp\left(\beta^{mn}(x)T_{mn}\right) \in O(d,d)$,

Geometric:

- Uni-vector deformations can be found in the standard KK reductions of GR
- Rank of the poly-vector is related to structure of internal space
- Deformations are coordinate transformations

KK reduction of the standard GR

The standard General Relativity in $D = 4$

$$S_{GL(4)} = \int d^4x\sqrt{-G}R[G] \tag{3}$$

$$ds^2 = e^{\phi}g_{mn}dx^m dx^n + e^{-\phi}\left(dz + \mathcal{A}_m dx^m\right)^2$$

The Einstein–Maxwell dilaton theory in $D = 3$:

$$S_{EMd} = \int d^3x\sqrt{-g}\Big(R[g] - \tfrac{1}{2}\partial_m\phi\,\partial^m\varphi - \tfrac{1}{4}e^{-2\phi}\mathcal{F}^{mn}\mathcal{F}_{mn}\Big), \tag{4}$$

has hidden symmetries:

1. global GL(4) (analogue of the global $O(d, d)$);
2. local diffeos modulo the “section condition” $\partial_z = 0$.

Uni-vector deformations

Uni-vector deformations $\text{bas}\,\mathfrak{gl}(4) = \{T_m{}^4, T_m{}^n, T_4{}^n\}$:

$$
\begin{aligned}
O_\alpha &= \exp\left(\alpha^m T_m{}^4\right) = \begin{bmatrix} 1 & 0 \\ \alpha^m & 1 \end{bmatrix}, \\
G'_{MN}(x) &= O_M{}^K O_N{}^L G_{KL}(x)
\end{aligned}
\tag{5}
$$

Non-linear transformations of the fields of the Einstein–Maxwell dilaton theory

$$
\begin{aligned}
e^{-\tilde\varphi} &= e^{\varphi}\alpha_k\alpha^k + e^{-\varphi}(1 + \mathcal{A}_k\alpha^k)^2\,, \\
\tilde{\mathcal{A}}_m &= e^{\tilde\varphi}(e^{\varphi}\alpha_m + e^{-\varphi}\mathcal{A}_m(1 + \mathcal{A}_k\alpha^k))\,, \\
\tilde g_{mn} &= e^{-\tilde\varphi}(e^{\varphi}g_{mn} + e^{-\varphi}\mathcal{A}_m\mathcal{A}_n) - e^{-2\tilde\varphi}\tilde{\mathcal{A}}_m\tilde{\mathcal{A}}_n\,.
\end{aligned}
\tag{6}
$$

Generates solutions if $\alpha^m = \alpha^m(x)$ is a $D = 3$ Killing vector:

$$
\mathcal{L}_\alpha\,\varphi = 0, \quad \mathcal{L}_\alpha\,g_{mn} = 0, \quad \mathcal{L}_\alpha\,\mathcal{A}_m = 0, \tag{7}
$$

.

Example I

Flat D=3 space: $ds^2 = -dt^2 + dr^2 + r^2 d\theta^2$, $\varphi = 0$, $\mathcal{A}_m = 0$

Deformation along $\alpha = \eta\, \partial_\theta$ gives:

$$\begin{aligned} ds^2 &= (1+\eta^2 r^2)\big(-dt^2 + dr^2\big) + r^2 d\theta^2, \\ \tilde{\mathcal{A}} &= \frac{\eta r^2}{\eta^2 r^2 + 1} d\theta, \\ \tilde{\varphi} &= -\ln\left(1+\eta^2 r^2\right). \end{aligned} \tag{8}$$

The transformation is non-trivial:

$$\tilde{R} = \frac{2\eta^2\left(\eta^2 r^2 - 1\right)}{(\eta^2 r^2 + 1)^3}, \quad \tilde{\mathcal{F}} = \frac{2\eta r}{(\eta^2 r^2 + 1)^2} dr \wedge d\theta. \tag{9}$$

Example II

Schwarzschild Black hole: $f(r) = 1 + \frac{r_g}{r}$

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad \varphi = 0, \quad \mathcal{A}_m = 0,$$

Deformation along $\alpha = c\,\partial_\phi$ gives:

$$\begin{aligned}
ds^2 &= \sqrt{1 + c^2 r^2 \sin^2\theta}\left[-f(r)dt^2 + f(r)^{-1}dr^2 + r^2\left(d\theta^2 + \frac{\sin^2\theta}{1 + c^2 r^2 \sin^2\theta} d\phi^2\right)\right], \\
\mathcal{A} &= \frac{c\, r^2 \sin^2\theta}{1 + c^2 r^2 \sin^2\theta} d\phi, \\
\tilde{\varphi} &= -\frac{\sqrt{3}}{2} \ln\left(1 + c^2 r^2 \sin^2\theta\right),
\end{aligned} \tag{10}$$

This is not equivalent to the Gibbons–Maeda solution.

Origin of the uni-vector symmetry

- Uni-vector deformations provide a solution generating technique for the Einstein–Maxwell dilaton theory

- These are nothing but coordinate transformation

$$x'^M = e^{\xi(x)} x^M, \quad \xi(x) = z\,\alpha^m(x)\partial_m \tag{11}$$

in the parent theory if $L_\alpha = 0$.

$$G'_{MN}(x') = \frac{\partial x^K}{\partial x'^M}\frac{\partial x^L}{\partial x'^N} G_{KL}(x) \tag{12}$$

- No algebraic condition arises

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Are Yang–Baxter bi-vector deformations also a coordinate transformation?

Double Field Theory

	D-dim Einstein–Maxwell dilaton	10-dim Supergravity
fields:	$g_{mn},\ \mathcal{A}_m,\ \phi$	$g_{mn},\ b_{mn},\ \phi$
parent:	GR in $D+1$	Double Field Theory
coordinates:	$x^M = (x^m, z)$	$x^M = (x^m, \tilde{x}_m)$
section condition:	$\partial_z = 0$	$\tilde{\partial}^m = 0$
hidden symmetry:	$GL(D+1)$	$O(10,10)$
deform. param.:	α^m	$\beta^{mn} = r^{i_1 i_2} k_{i_1}{}^m k_{i_2}{}^n$
conditions:	$L_\alpha = 0$	$L_{k_i} = 0$ CYBE + unimodularity

Reduction of DFT

Double Field Theory in $D = 10 + 10$

$$S_{GL(4)} = \int d^{20}X e^{-2d}\mathcal{R}[G] \tag{13}$$

$$ds^2 = (g_{mn} - B_{mk}B_n{}^k)dx^m dx^n + B_m{}^n dx^m d\tilde{x}_n + g^{mn} d\tilde{x}_m d\tilde{x}_n$$

NS-NS sector of the $D = 10$ SUGRA:

$$S_{sugra} = \int d^{10}x\sqrt{-g}e^{-2\phi}\left(R[g] - 4\partial_m\phi\,\partial^m\varphi - \frac{1}{12}H^{mnk}H_{mnk}\right), \tag{14}$$

has hidden symmetries:

1. global $O(d, d)$;
2. local generalized diffeos modulo the section condition $\tilde{\partial}^m = 0$.

Bi-vector deformations

Under the breaking $O(10,10) \to GL(10)$ the generators split as

$$\text{bas}\,\mathfrak{o}(d,d) = \{T_\alpha\} = \{T_{[mn]}, T_m{}^n, T^{[mn]}\}, \quad m = 1, \dots, 10, \tag{15}$$

The deformation matrix:

$$\begin{aligned} \mathcal{O}_\beta &= \exp\left[\beta^{mn} T_{mn}\right] = \begin{bmatrix} 1 & 0 \\ \beta & 1 \end{bmatrix}, \\ G'_{MN}(x) &= O_M{}^K O_N{}^L G_{KL}(x) \end{aligned} \tag{16}$$

In the Bi-Killing ansatz $\beta^{mn} = r^{ab} k_a{}^m k_b{}^n$ the deformation generates solutions if

$$\begin{aligned} r^{b_1[a_1} r^{|b_2|a_2} f_{b_1 b_2}{}^{a_3]} &= 0 && \text{(classical YB equation);} \\ r^{b_1 b_2} f_{b_1 b_2}{}^a k_a{}^m &= I^m = 0 && \text{(unimodularity condition)} \end{aligned} \tag{17}$$

We want to see that this is a coordinate transformation in the doubled space

The Yang-Baxter condition

Coordinate transformation

$$X'^M = e^{\xi} X^M, \quad \xi = \beta^{mn} \tilde{x}_m \partial_n. \tag{18}$$

Closure of these into themselves requires

$$[\delta_{\xi_{\tilde{x}}}, \delta_{\xi_{\tilde{y}}}] = \delta_{[\xi_{\tilde{x}}, \xi_{\tilde{y}}]} = \delta_{\xi_{\tilde{z}}} \quad \Longleftarrow \quad \beta^{l[m} \partial_l \beta^{nk]} = 0 \tag{19}$$

$$\boxed{\beta^{mn} = r^{ab} k_a{}^m k_b{}^n}$$

Classical Yang-Baxter equation:

$$r^{b_1[a_1} r^{|b_2|a_2} f_{b_1 b_2}{}^{a_3]} = 0 \tag{20}$$

Coordinate transformation in DFT

Transformation matrix for tensors must be in $O(10,10)$ [Hohm,Zwiebach (2012)]

$$G'(X')_{MN} = \mathcal{F}_M{}^K \mathcal{F}_N{}^L \mathcal{H}_{KL}(X),$$
$$\mathcal{F}_M{}^N = \frac{1}{2}\left(\frac{\partial X^P}{\partial X'^M}\frac{\partial X'_P}{\partial X_N} + \frac{\partial X'_M}{\partial X^P}\frac{\partial X^N}{\partial X'_P}\right). \tag{21}$$

Given the section condition $\eta^{MN}\partial_M \bullet \partial_N \bullet$ this implies the correct rule

$$\eta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\partial'_M = F_M{}^N \partial_N = \frac{\partial X^N}{\partial X^M}\partial_N \tag{22}$$

For $\xi = \beta^{mn}\tilde{x}_m\partial_n$ this should give (does not quite work)

$$\begin{aligned}
\mathcal{H}_{mn}(x)' &= \mathcal{H}_{mn}(x), \\
\mathcal{H}_m{}^n(x)' &= \mathcal{H}_m{}^n(x) + \beta^{nk}\mathcal{H}_{mk}, \\
\mathcal{H}^{mn}(x)' &= \mathcal{H}^{mn}(x) + 2\beta^{(m|k|}\mathcal{H}_k{}^{n)} + \beta^{mk}\beta^{nl}\mathcal{H}_{kl}.
\end{aligned} \tag{23}$$

Almost abelian bi-vector deformations

All non-abelian unimodular rank-4 r-matrices of $\mathfrak{so}(2,4)$ were classified in [Borsato, Wulff (2016)]

Take a subclass of them:

$$\beta = p_1 \wedge p_2 + q \wedge j, \tag{24}$$

where the only non-vanishing commutators are

$$[j, p_i] = \varepsilon_i q. \tag{25}$$

We show, that the corresponding bi-vector deformations are equivalent to

1. the coordinate transformation $\xi = \beta^{mn}\tilde{x}_m \partial_n$
2. a further TsT transformation.

At the linear level and for general TsT transformations the same has been observed in [Sakamoto, Sakatani, Yoshida (2017)]

Conclusions

1. Bi-vector Yang-Baxter deformations generalize up and down
2. Uni-vector and almost-abelian bi-vector Yang-Baxter deformations are equivalent to coordinates transformation **in the parent theory**
3. Classical Yang–Baxter equation follows from the consistency of the algebra of such transformations

Uses and further work:

- What is the origin of unimodularity in this language?
- Generalize to Yang–Baxter deformation of general form.
- Prove that all (almost-abelian) YB deformations preserve integrability
- Generalize to tri-vector deformations (need tensorial transformation law).

Thank you!

Deformations open a way to the world of new knowledge