A New Approach to Study the Structure of Renormalization Group Equation Solutions
(in progress)

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Motivation

- ⋆ The Bogolyubov-Parasyuk theorem allows us to obtain an analogue of the RG-equation only due to the locality of the counter-terms [Bork, Kazakov, Kompaniets, Tolkachev, Vlasenko, 2015].
- ⋆ Solution of the equation for a running coupling (for an infinite sum of leading [Kazakov, Iakhibbaev, Tolkachev, 2022] and sub-leading logarithms):

$$\frac{1}{1+\log(x)}, \quad \frac{\log(\log(x)+1)}{(\log(x)+1)^2}. \tag{1}$$

- ⋆ Solution of the two-loop RG-equation for a running coupling contains Lambert function:

$$\frac{1}{1+W(y)}$$

Linear and nonlinear RG-equation

Equation for a running coupling:

$$\frac{\partial}{\partial z}\bar{\alpha} = \beta(\bar{\alpha}) = \beta_0\bar{\alpha}^2 + \beta_1\bar{\alpha}^3 + \beta_2\bar{\alpha}^4 + \beta_3\bar{\alpha}^5 + O(\bar{\alpha}^6) \qquad (2)$$

Linear and nonlinear RG-equation

Equation for a running coupling in a one-loop approximation:

$$\frac{\partial}{\partial z}\bar{\alpha} = \beta_0 \bar{\alpha}^2, \tag{3}$$

solution $A(z)$

Linear and nonlinear RG-equation

Equation for a running coupling in a two-loop approximation:

$$\frac{\partial}{\partial z}\bar{\alpha} = \beta_0\bar{\alpha}^2 + \beta_1\bar{\alpha}^3, \tag{4}$$

solution $B(z)$

You have to integrate

Equation for a running coupling in a two-loop approximation

$$\frac{\partial}{\partial z}\bar{\alpha} = \beta_0\bar{\alpha}^2 + \beta_1\bar{\alpha}^3, \tag{5}$$

$$B''(z) = B(z)A'(z) + A'(z) + A(z)B'(z) + A(z)^2B(z) + A(z)^3, \tag{6}$$

where $B(z)$ is the sum of all subleading singular contributions and the finite contributions are determined by singularities

$$B'_{Log}(z) = \bar{\alpha}'_{BP} = B(z)A'(z) + A'(z) + A(z)B'(z) + A(z)^2B(z) + A(z)^3 \tag{7}$$

Solutions for leading + subleading logs

$$\frac{1}{1+\log(y)} + \frac{\log(\log(y)+1)}{(\log(y)+1)^2},$$

doesn't look like

$$\frac{1}{1+W(x)}$$

There are mostly logarithms

Running coupling

$$\bar{\alpha}(\alpha, z) = \alpha \left(1 + \sum_{k=1}^{\infty} A_k \; \alpha^k z^k + \sum_{k=2}^{\infty} B_k \; \alpha^k z^{k-1} + \sum_{k=3}^{\infty} C_k \; \alpha^k z^{k-2} + \ldots \right) \tag{8}$$

$$z = \log(y) \text{ where } y = Q^2/\mu^2$$

$$\bar{\alpha}(\alpha, z) = \sum_{\mathcal{X}=A,B,C\ldots}^{\infty} \mathcal{X} = A(\alpha, z) + B(\alpha, z) + C(\alpha, z) + \ldots \tag{9}$$

Substituting Eq.(9) in Eq.(2) and keeping only linear contributions (the exception is the leading order) in a certain order we get

$$\frac{\partial}{\partial z} A(z) = \beta_0 A^2(z), \;\; A(0) = 1 \tag{10}$$

Linearized equations

$[\mathcal{X}] \sim [A^n]$

$$
\begin{gathered}
\frac{\partial}{\partial z} B = \beta_1 A^3 + 2\beta_0 BA \\
\frac{\partial}{\partial z} C = \beta_2 A^4 + 3\beta_1 BA^2 + 2\beta_0 CA + \beta_0 B^2 \\
\frac{\partial}{\partial z} D = \beta_3 A^5 + 4\beta_2 BA^3 + 3\beta_1 CA^2 + A\left(3\beta_1 B^2 + 2\beta_0 D\right) + 2\beta_0 BC \\
\frac{\partial}{\partial z} E = \beta_4 A^6 + 5\beta_3 BA^4 + 4\beta_2 CA^+ \\
+3A^2\left(2\beta_2 B^2 + \beta_1 D\right) + 2A\left(3\beta_1 BC + \beta_0 E\right) + \left(\beta_1 B^3 + 2\beta_0 DB + \beta_0 C^2\right) \\
\frac{\partial}{\partial z} F = \beta_5 A^7 + 6\beta_4 BA^5 + 5\beta_3 CA^4 + 2A^3\left(5\beta_3 B^2 + 2\beta_2 D\right) + \\
+3A^2\left(4\beta_2 BC + \beta_1 E\right) + A\left(4\beta_2 B^3 + 6\beta_1 DB + 3\beta_1 C^2 + 2\beta_0 F\right) + \\
+\left(3\beta_1 CB^2 + 2\beta_0 EB + 2\beta_0 CD\right)
\end{gathered}
\tag{11}
$$

$$\cdots$$

Solutions for linearized equations

$$L = \log(1-\beta_0 z), \quad z = \log(y), \quad \xi_N = \beta_N/\beta_0$$

$$B = -A^2\xi_1 L$$

$$C = A^3\left(-L^2\xi_1^2 + L\xi_1^2 + z\beta_0(\xi_1^2 - \xi_2)\right)$$

$$D = A^4\xi_1\left(-L^3\xi_1^2 + \frac{5}{2}L^2\xi_1^2 - \xi_2 L\right) + z\beta_0 A^4\left(2L\xi_1^3 - 2L\xi_2\xi_1 - \xi_2\xi_1 + \xi_3\right) - \\ -\frac{1}{2}z^2\beta_0^2 A^4\left(\xi_1^3 - 2\xi_2\xi_1 + \xi_3\right)$$

$$E = A^5\left(-L^4\xi_1^4 + \frac{13}{3}L^3\xi_1^4 - \frac{3}{2}L^2\left(\xi_1^2 + 2\xi_2\right)\xi_1^2 + L\xi_3\xi_1\right) + \\ +\frac{1}{3}z^3\beta_0^3 A^5\left(\xi_1^4 - 3\xi_2\xi_1^2 + 2\xi_3\xi_1 + \xi_2^2 - \xi_4\right) - \dots)$$

Vertical logarithms (Leading Metalogarithms)

Metalogarithms - summed infinite number of logarithms

$$B = -A^2\xi_1 L$$

$$C = A^3\left(-L^2\xi_1^2 + L\xi_1^2 + z\beta_0(\xi_1^2 - \xi_2)\right)$$

$$D = A^4\xi_1\left(-L^3\xi_1^2 + \frac{5}{2}L^2\xi_1^2 - \xi_2 L\right) + z\beta_0 A^4\left(2L\xi_1^3 - 2L\xi_2\xi_1 - \xi_2\xi_1 + \xi_3\right) - \\ -\frac{1}{2}z^2\beta_0^2 A^4\left(\xi_1^3 - 2\xi_2\xi_1 + \xi_3\right)$$

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Vertical logarithms (Subleading metalogarithms)

Metalogarithms - summed infinite number of logarithms

$$B = -A^2 \xi_1 L$$

$$C = A^3 \left(-L^2 \xi_1^2 + L\xi_1^2 + z\beta_0(\xi_1^2 - \xi_2)\right)$$

$$D = A^4 \xi_1 \left(-L^3 \xi_1^2 + \frac{5}{2} L^2 \xi_1^2 - \xi_2 L\right) + z\beta_0 A^4 \left(2L\xi_1^3 - 2L\xi_2\xi_1 - \xi_2\xi_1 + \xi_3\right) - \\ -\frac{1}{2} z^2 \beta_0^2 A^4 \left(\xi_1^3 - 2\xi_2\xi_1 + \xi_3\right)$$

$$E = A^5 \left(-L^4 \xi_1^4 + \frac{13}{3} L^3 \xi_1^4 - \frac{3}{2} L^2 \left(\xi_1^2 + 2\xi_2\right) \xi_1^2 + L\xi_3\xi_1\right) + \\ + \frac{1}{3} z^3 \beta_0^3 A^5 \left(\xi_1^4 - 3\xi_2\xi_1^2 + 2\xi_3\xi_1 + \xi_2^2 - \xi_4\right) - \ldots)$$

Vertical logarithms (Subsubleading metalogarithms)

Metalogarithms - summed infinite number of logarithms

$$B = -A^2 \xi_1 L$$

$$C = A^3 \left(-L^2 \xi_1^2 + L\xi_1^2 + z\beta_0(\xi_1^2 - \xi_2)\right)$$

$$D = A^4 \xi_1 \left(-L^3 \xi_1^2 + \frac{5}{2} L^2 \xi_1^2 - \xi_2 L\right) + z\beta_0 A^4 \left(2L\xi_1^3 - 2L\xi_2\xi_1 - \xi_2\xi_1 + \xi_3\right) -$$
$$-\frac{1}{2} z^2 \beta_0^2 A^4 \left(\xi_1^3 - 2\xi_2\xi_1 + \xi_3\right)$$

$$E = A^5 \left(-L^4 \xi_1^4 + \frac{13}{3} L^3 \xi_1^4 - \frac{3}{2} L^2 \left(\xi_1^2 + 2\xi_2\right) \xi_1^2 + L\xi_3\xi_1\right) +$$
$$+\frac{1}{3} z^3 \beta_0^3 A^5 \left(\xi_1^4 - 3\xi_2\xi_1^2 + 2\xi_3\xi_1 + \xi_2^2 - \xi_4\right) - \ldots)$$

The Sum of metalogarithms

We can sum the logarithms vertically

$$
\begin{aligned}
\mathbf{L}_0 &\equiv \frac{1}{(1-\beta_0 z)} \equiv A \\
\mathbf{L}_1 &= \mathbf{L}_0 \frac{1}{\left(1+\frac{\xi_1 L}{(1-\beta_0 z)}\right)} \\
\mathbf{L}_2 &= \mathbf{L}_0^2 \frac{\xi_1 \log\left(1+\frac{\xi_1 L}{1-\beta_0 z}\right)}{\left(1+\frac{\xi_1 L}{1-\beta_0 z}\right)^2} \\
\mathbf{L}_3 &= \mathbf{L}_0^3 \frac{\xi_1^2 \left(-\frac{L\xi_1}{\beta_0 z-1} - \log^2\left(\frac{L\xi_1}{\beta_0 z-1}+1\right) + \log\left(\frac{L\xi_1}{\beta_0 z-1}+1\right)\right) + \frac{L\xi_2\xi_1}{\beta_0 z-1}}{\left(1-\frac{L\xi_1}{1-\beta_0 z}\right)^3}
\end{aligned}
\tag{12}
$$

The general rule

We have to replace the argument z to $\frac{B}{\beta_0 A}$.

$$\mathbf{L}_n = A^n \mathcal{X}\left(\frac{B}{\beta_0 A}\right) \tag{13}$$

The general rule

We have to replace the argument z to $\frac{B}{\beta_0 A}$.
Let's try to sum it up one more time.

$$\mathbf{L}_n = A^n \mathcal{X}\left(\frac{B}{\beta_0 A}\right)(z) \tag{14}$$

Let's follow the second metalogarithms

$$\mathbf{L}_n^{(2)} = A^n(z) A^n\left(\frac{B}{\beta_0 A}\right) \mathcal{X}\left(\frac{B\left(\frac{B}{\beta_0 A}\right)}{\beta_0 A\left(\frac{B}{\beta_0 A}\right)}\right) \tag{15}$$

Just make a replacement

$$\mathbf{L}_n = A^n \mathcal{X}\left(\frac{B}{\beta_0 A}\right)(z) \tag{16}$$

Let's follow the second metalogarithms

$$\mathbf{L}_n^{(2)} = A^n(z) A^n \left(\frac{B}{\beta_0 A}\right) \mathcal{X}\left(\frac{B\left(\frac{B}{\beta_0 A}\right)}{\beta_0 A\left(\frac{B}{\beta_0 A}\right)}\right) \tag{17}$$

General rule z → B

$$\mathbf{L}_n^{(k)} \sim \mathcal{X}(B(B(..B)..)) \tag{18}$$

Result

- ⋆ The logarithmic quantum contributions are determined by the linear structure, this allows one to consider the linearized renormalization group equation.
- ⋆ With this approach, it is easy to see the following repeating structure of the solution for logarithms $\mathcal{X}(B(B(..B)..))$.
- ⋆ This can be used to analyze problems related to the Landau pole and UV fixed points in various models of quantum field theory.
- ⋆ It is also possible to study the decomposition of non-elementary functions using this method ($\beta_{N>1} = 0$ to W).

Thank you for attention!