Exploring multiparticle amplitudes by means of Exact Landau method

Sergei Demidov, Bulat Farkhtdinov , Dmitry Levkov

The International Conference
"Advances in Quantum Field Theory 2025"

arXiv: `2212.03268`, `2111.04760` *& work in progress*

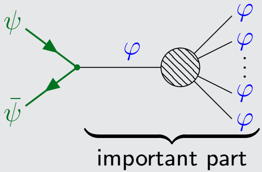
Peculiarities of multiparticle production in QFT

Real scalar field $\varphi(t,\, \boldsymbol{x})$ theory in $d = 3 + 1$ dimensions

$S_{\varphi^4} = \frac{1}{2}\int dt\, d^3\boldsymbol{x} \left[(\partial_\mu \varphi)^2 - \varphi^2 - \frac{\lambda}{2}\varphi^4 \right]$, $m = 1$. Weak coupling: $\lambda \ll 1$.

Creation of arbitrary number of particles n at threshold

$\mathcal{A}_{1\to n} \sim$



$= \langle n,\, \boldsymbol{p}{=}0\, |\hat{\mathcal{S}}\, \hat{\varphi}(0)\, |0\rangle =$

Brown '92; Voloshin'92

Libanov et al '94

$$= \underbrace{n!\left(\frac{\lambda}{16}\right)^{\frac{n-1}{2}}}_{\mathcal{A}^{\text{tree}}} \left[\underbrace{\ 1\ }_{\text{tree}} + \underbrace{\lambda B(n^2 - 4n + 3)}_{\text{1 loop} \sim \lambda n^2} + \underbrace{\lambda^2 \frac{B^2}{2}(n^4 + \dots)}_{\text{2 loops} \sim (\lambda n^2)^2} + \dots \right], B \in \mathbb{C}$$

Non-perturbative at $n \gg \lambda^{-1/2}$ due to large number of diagrams

Peculiarities of amplitudes $1 \to n$ in QM

Quantum particle at position $\varphi(t)$ in $d = 0+1$ dimensions

$$S_{\varphi^4} = \frac{1}{2}\int dt \left[(\partial_t\varphi)^2 - \varphi^2 - \frac{\lambda(t)}{2}\varphi^4\right], \; \omega = 1. \; \lambda = \lambda(t) \text{ for scattering.}$$

Weak coupling and IR regularization: $\lambda(t) = \lambda_0 \, \mathrm{e}^{-2\epsilon t}; \; \lambda_0, \epsilon \ll 1.$

High-multiplicity adiabatic transition

$$\mathcal{A}_{1\to n} = \langle n|\hat{\mathcal{S}}\,\hat{\varphi}(0)\,|0\rangle \underset{\substack{\uparrow \\ \text{adiabatic at } \epsilon \to 0\text{ !}}}{=} \mathrm{e}^{-\frac{i}{\epsilon}\int\limits_0^\infty dt(E_n - E_0 - n)} \langle n|\hat{\varphi}\,|0\rangle\Big|_{t=0}$$

Jaeckel, Schenk, 2018

$$E_n - E_0 - n = \frac{\lambda_0}{8}(3n^2 + \dots) - \frac{\lambda_0^2}{64}(17n^3 + \dots) + \dots$$

$$\langle n|\hat{\varphi}|0\rangle = \mathcal{A}^{\text{tree}}\left[1 - \frac{\lambda_0}{32}(17n^2 + 5n - 12) + \frac{\lambda_0^2}{2048}(289n^4 + \dots) + \dots\right]$$

$$\mathcal{A}^{\text{tree}} = \sqrt{\frac{n!}{2}}\left(\frac{\lambda_0}{8}\right)^{(n-1)/2}$$

Qualitatively the same blow-up at $n \gg \lambda^{-1/2}$ as in QFT!

Exponentiation of $1 \to n$ amplitudes

Conjecture

$$\mathcal{A}_{1\to n} \overset{!}{=} \mathcal{A}^{\text{tree}}\ \exp\left[\frac{1}{\lambda}F_{-1}(\lambda n) + F_0(\lambda n) + \lambda F_1(\lambda n) + \ldots\right]$$

resummation of all loops!

Libanov et al '94

Relation to perturbation theory

- Expansions in both $\lambda \ll 1$ and $\lambda n \ll 1$ = perturbation theory
- 't Hooft parameter $\lambda n \gtrsim 1$ stands for non-perturbative regime

Results from explicit resummation

- QFT : $F_{-1} = B(\lambda n)^2 + \mathcal{O}(\lambda n)^3$
- QM : $F_{-1} = -(\frac{3i}{16\epsilon} + \frac{17}{32})(\lambda_0 n)^2 + (\frac{17i}{256\epsilon} + \frac{125}{256})(\lambda_0 n)^3 + \mathcal{O}(\lambda_0 n)^4$

$$F_0 = -(\frac{3i}{16\epsilon} + \frac{5}{32})\,\lambda_0 n + (\frac{51i}{512\epsilon} + \frac{99}{512})\,(\lambda_0 n)^2 + \mathcal{O}(\lambda_0 n)^3$$

$$F_1 = \frac{3}{8} + \mathcal{O}(\lambda_0 n)$$

Libanov et al '94; Jaeckel, Schenk 2018

Expressions have a semiclassical form

Main goals

- Construct an exact method which can systematically calculate F_{-1}, F_0, $\dots$ at λ, $\lambda n \ll 1$;
- Proof exponentiation conjecture at least in QM;
- Construct a method for calculation of F_{-1}, F_0, $\dots$ at $\lambda n \gtrsim 1$.

Resulting Landau method expression

Amplitudes $\mathcal{A}_{1\to n}$ are computed

1. Using the same theory with a classical δ-functional source

$$S_j = \frac{1}{2}\int d^dx \left[(\partial_\mu\varphi)^2 - \varphi^2 - \frac{\lambda(t)}{2}\varphi^4\right] + ij\varphi(0),\ \lambda(t) = \lambda_0\,\mathrm{e}^{-2\epsilon t},\ \epsilon \to 0;$$

2. On a classical singular background

$$\hat\varphi(x) = \underbrace{\phi_B(t-t_*)}_{\text{classical \& singular at } \boxed{t=t_*}} + \delta\hat\varphi(t,\boldsymbol{x}),$$

Brown solution: $\phi_B = \dfrac{i\sqrt{2}}{\sqrt{\lambda}\,\sin(t-t_*+i\epsilon t)}$.

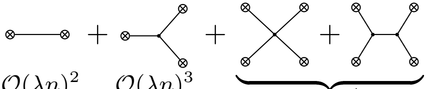
Then a perturbatively exact relation is valid:

$$\boxed{\mathcal{A}_{1\to n} = -i\mathcal{A}^{\text{tree}}\cdot\sqrt{\frac{\lambda_0}{2}}\;\underbrace{\lim_{\phi_0\to\infty}\phi_0^2}_{\text{limit}}\;\underbrace{\int\frac{dj\,dt_*}{2\pi}\,\mathrm{e}^{j\phi_0+it_*n}}_{\text{Laplace transform}}\;\underbrace{\mathcal{A}^{(j,\,\phi_B)}_{\text{vac}\to\text{vac}}}_{\text{with source}}}$$

Feynman graphs for $\lambda n \ll 1$ expansion

Our formula recreates: $$\boxed{\mathcal{A}_{1\to n} = \mathcal{A}^{\text{tree}} \cdot \exp\Big[\underbrace{\frac{1}{\lambda}F_{-1} + F_0 + \lambda F_1 + \dots}_{\text{connected graphs}}\Big]}$$

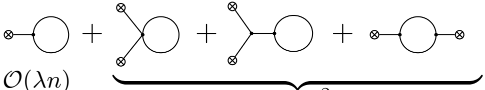
$F_{-1} = \underbrace{\Delta F_{-1}}_{\text{extra int-ls}} +$



$+ \cdots =$ tree graphs

$(\otimes = j)$

$F_0 = \underbrace{\Delta F_0}_{\text{extra int-ls}} +$



$+ \cdots =$ 1-loop, etc

Series for $F_l(\lambda n)$: $l + 1$ — number of loops

- Extra integrals via saddle-point method: $\boxed{j_s \sim \lambda n/\phi_0^2}$ $\boxed{t_{*,s} \sim \phi_0^{-1}}$
- $\phi_0 \to \infty \Rightarrow j \to 0$, singular ϕ_B $(t_* \to 0)$ *cf. Son '95*
- ϕ_0 cancels in diagrams, λn remains

Expansion in j produces series for $F_l(\lambda n)$ at $\lambda n \ll 1$

Semiclassical nature of amplitudes

Recall conjecture:

$$\boxed{\mathcal{A}_{1\to n} = \mathcal{A}^{\text{tree}} \cdot \exp\left[\underbrace{\frac{1}{\lambda}F_{-1} + F_0 + \lambda F_1 + \dots}_{\text{connected graphs}}\right]}$$

Non-perturbative limit $\lambda \to 0$, $\lambda n = \mathcal{O}(1)$ ideas

- $F_{-1}(\lambda n) = \sum$ tree graphs — continues to $\lim\limits_{j\to 0} F_0[\phi_{\text{cl}}]$

  Rubakov–Son-Tinyakov conjecture

  $\phi_{\text{cl}}(t,\, \boldsymbol{x}) \in \mathbb{C}$ — classical solution at $j \neq 0$ ← D.T. Son's method;
- $F_0(\lambda n) = \sum$ 1-loop graphs — continues to $\lim\limits_{j\to 0} \ln\det[\text{fluctuations}]\dots$

Rubakov et al '92, Son '95

Semiclassical methods at finite λn, $j \neq 0$
reconstruct full answer at $j \to 0$ also for $\lambda n \gtrsim 1$

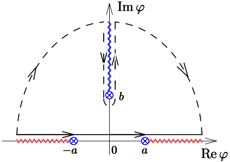
Exact Landau method

Setup

- $\langle n|\hat{\varphi}|0\rangle = \int d\varphi \varphi \Psi_n \Psi_0$ is calculated using WKB wave functions;
- $\boxed{\lambda \propto \hbar}$ is the semiclassical parameter: $\varphi \to \varphi/\sqrt{\lambda} \Rightarrow S_{\varphi^4} \to S_{\varphi^4}/\lambda$;
- $\varphi = \infty$ is a singular point of the potential.

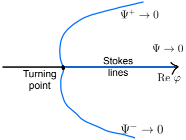
Steps for Exact Landau method

1. Decompose $\Psi = \Psi^+ + \Psi^-$, $\Psi^\pm$ — solutions of Schrödinger equation;
2. Use Borel-summable expressions for Ψ, $\Psi^\pm$ from exact WKB;
3. Deform the contour to ∞: $\langle n|\hat{\varphi}|0\rangle = \sum \int_{\mathcal{C}^\pm_\infty} d\varphi\, \varphi\, \Psi^\pm_n\, \Psi_0$;
4. Integrals become residual on $\mathcal{C}^\pm_\infty$: $\boxed{\Psi^\pm_n \Psi_0 \to \text{const}/\varphi^2 + \text{exp. small}}$.



Perturbatively exact expression:

$$\boxed{\langle n|\hat{\varphi}|0\rangle = 2\pi \sum_{\pm} \lim_{\varphi = \pm i\infty} \varphi^2\, \Psi^\pm_n\, \Psi_0}$$



WKB details

WKB terms

- Momentum $p^2(\varphi) = 2\lambda E - \varphi^2 - \varphi^4/2$; $\mathcal{P}^2 = p^2 + \lambda^2\sqrt{\mathcal{P}}\left(\frac{1}{\sqrt{\mathcal{P}}}\right)''$;
- Basis functions
$$\xi^{\pm}_{\varphi_*,\infty}(\lambda,\varphi) = \frac{\exp\left(\pm i\cdot\lambda^{-1}\int_{\varphi_*}^{\varphi} p(\varphi')d\varphi' \pm i\cdot\lambda^{-1}\int_{\infty}^{\varphi}[\mathcal{P}(\lambda,\varphi') - p(\varphi')]d\varphi'\right)}{\sqrt{\mathcal{P}(\lambda,\varphi)}};$$
- General solution of Schrödinger equation: $\Psi = C_+\xi^+ + C_-\xi^-$.

Parameter dependence

- Quantization condition (Γ_R encycles classically allowed region): $\oint_{\Gamma_R}\mathcal{P}(\varphi)d\varphi \sim 2\pi(\lambda n + \lambda/2) \Rightarrow E = E(\lambda,\,\lambda n)$;
- Normalization condition: $\int_{-\infty}^{+\infty} d\varphi\Psi^2 = 1 \Rightarrow C = C(\lambda,\,\lambda n)$;
- $\langle n|\hat{\varphi}|0\rangle = 2\pi\sum_{\pm}\lim_{\varphi=\pm i\infty}\varphi^2\,\Psi_n^{\pm}\,\Psi_0$ is perturbative at $\lambda n \ll 1$ and exponentially small at $\lambda n \gg 1$. $\lambda \ll 1$ in both cases.

Landau method from path integral in QFT and QM

- **Separate** $\varphi(\boldsymbol{x} = t = 0)$:

$$\mathcal{A}_{1\to n} \equiv \langle n|\hat{\mathcal{S}}\,\hat{\varphi}(0)\,|0\rangle = \int \mathcal{D}\varphi(x)\;\varphi(0)\;\mathrm{e}^{iS[\varphi]}\;\langle n|\varphi(t_f)\rangle$$

$$\overbrace{1 = \int d\phi_0\,\delta(\phi_0 - \varphi(0)) = \int \frac{d\phi_0\,dj}{2\pi i}\;\mathrm{e}^{j(\phi_0-\varphi(0))}}$$

$$= N\int d\phi_0\phi_0\;\int dj\mathrm{e}^{j\phi_0}\int \mathcal{D}\varphi(x)\;\mathrm{e}^{-j\varphi(0)+iS[\varphi]}\;\langle n|\varphi(t_f)\rangle;$$

- **Replace** $\int d\phi_0\phi_0 \to \lim\limits_{\phi_0\to\infty}\phi_0^2$;

- **Simplify final state**: $|n\rangle = (\hat{a}^\dagger_{\boldsymbol{p}=0})^n|0\rangle = N\int dt_*\;\mathrm{e}^{it_*n}\,\underbrace{\exp(\mathrm{e}^{-it_*}\hat{a}^\dagger)|0\rangle}_{\text{coherent}}$

- **Use background solution to cancel new BCs**

$$\hat{\varphi}(x) = \underbrace{\phi_B(t-t_*)}_{\text{satisfies BCs}} \;+\; \underbrace{\delta\hat{\varphi}(x)}_{\text{vacuum BCs}}$$

Resulting expression and relation to WKB

Path integral:

$$\boxed{\mathcal{A}_{1\to n} = N \underbrace{\lim_{\phi_0\to\infty} \phi_0^2}_{\text{limit}} \underbrace{\int dj\,dt_*\, \mathrm{e}^{j\phi_0 + i t_* n}}_{\text{Laplace transform}} \underbrace{\int \mathcal{D}\delta\varphi(x)\, \mathrm{e}^{j\varphi(0)+i\tilde{S}[\delta\varphi;\phi_B]}}_{\mathcal{A}^{(j,\,\varphi_B)}_{\text{vac}\to\text{vac}}}}$$

$\varphi = \phi_B + \delta\varphi$, $\qquad \phi_B(t-t_*) = \frac{i\sqrt{2}}{\sqrt{\lambda}\,\sin(t-t_*+i\epsilon t)}$ — Brown solution.

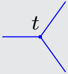
WKB:

$$\boxed{\mathcal{A}_{1\to n} = 2\pi\, \mathrm{e}^{-\frac{i}{\epsilon}\int\limits_0^\infty dt(E_n - E_0 - n)} \sum_\pm \lim_{\varphi=\pm i\infty} \varphi^2\, \Psi_n^\pm\, \Psi_0\Big|_{t=0}}$$

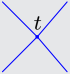
Checking the expression

Quantize the theory: $\varphi(t) = \phi_B + \delta\varphi$

$= G_B(t,\, t'),$



$= -6i\int dt\, \phi_B \sqrt{\lambda},$

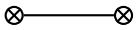
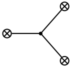


$= -6i\int dt\, \lambda,$

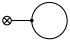
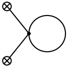
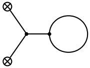
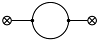
$= -j\Big|_{t=0}$

$$j_s\sqrt{\lambda_0} = -\frac{\lambda_0 n\sqrt{2}}{\phi_0^2\lambda_0} + \mathcal{O}(\lambda_0 n)^2, \qquad t_{*,s} = -\frac{i\sqrt{2}}{\phi_0\sqrt{\lambda_0}} + \mathcal{O}(\lambda_0 n)$$

$$\boxed{\mathcal{A}_{1\to n} = \mathcal{A}^{\text{tree}} \cdot \mathrm{e}^{\frac{1}{\lambda}F_{-1}(\lambda n) + F_0(\lambda n) + \lambda F_1(\lambda n) + \dots}}$$

$F_{-1} = it_*\lambda_0 n + j\phi_0\lambda_0 +$

$+$

$+ \mathcal{O}(\lambda_0 n)^4$

$$= -\left(\frac{3i}{16\epsilon} + \frac{17}{32}\right)(\lambda_0 n)^2 + \left(\frac{17i}{256\epsilon} + \frac{125}{256}\right)(\lambda_0 n)^3 + \mathcal{O}(\lambda_0 n)^4$$

$F_0 = (j,\, t_*$ prefactors$) +$

$+$

$+$

$+$

$+ \mathcal{O}(\lambda_0 n)^3$

$$= -\left(\frac{3i}{16\epsilon} + \frac{5}{32}\right)\lambda_0 n + \left(\frac{51i}{512\epsilon} + \frac{99}{512}\right)(\lambda_0 n)^2 + \mathcal{O}(\lambda_0 n)^3$$

$F_1 = (j,\, t_*$ contributions$) + O(\lambda_0 n) = \frac{3}{8} + O(\lambda_0 n)$

Coincides with explicit resummation! *(Jaeckel, Schenk '18)*

Conclusions

Recall the main goals

- Construct an exact method which can systematically calculate F_{-1}, F_0, $\dots$ at λ, $\lambda n \ll 1$ — done;
- Proof exponentiation conjecture at least in QM — done;
- Construct a method for calculation of F_{-1}, F_0, $\dots$ at $\lambda n \gtrsim 1$ — D.T. Son's method of singular solutions is a good candidate, there are hints from WKB.

What is next?

- Obtain higher order corrections in QFT?
- Apply to broken φ^4 theory?
- Take exponentially small corrections into account?

Thank you for your attention!