Two-loop chiral effective potential in general $\mathcal{N} = 1$ supersymmetric Yang-Mills model

(based on 2508.xxxxx)

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SUSY Effective potential

- ⋆ Non-renormalization theorem: all loop corrections to the effective action in such a theories are expressed by integrals over the whole superspace, but not over its chiral subspace
- ⋆

$$\int d^8z\ u(\Phi)\left(-\frac{D^2}{4\Box}\right)v(\Phi) = \int d^6z\ u(\Phi)v(\Phi), \quad d^8z = d^4xd^2\theta d^2\bar{\theta}, \quad d^6z = d^4xd^2\theta$$

Possible only for massless diagrams

- ⋆ Effective potential [I.L. Buchbinder, S. Kuzenko and Z. Yarevskaya, '94]

$$\Gamma[\Phi] = \int d^8z\ (\mathbf{K}+\mathbf{A}) + \left(\int d^6z\ \mathbf{W} + \text{h.c.}\right)$$

$\mathbf{K}(\Phi,\bar{\Phi})$ is the Kähler effective potential,
$\mathbf{A}(\Phi, D\Phi, \bar{\Phi}, \bar{D}\bar{\Phi})$ is the effective superpotential of auxiliary fields,
$\mathbf{W}(\Phi)$ is the chiral effective superpotential.

- ⋆ Corrections to the chiral effective potential

$$\mathbf{W} = \sum_{L=1}^{\infty} \hbar^L \mathbf{W}^{(L)}$$

- ⋆ $n_{D^2} + 1 = n_{\bar{D}^2}$ [I.L. Buchbinder, S.M. Kuzenko, A.Yu. Petrov '94]

Setup: $\mathcal{N}=1$ super-Yang-Mills model

⋆ Action

$$\mathcal{S}_c = \int d^4x\, \mathcal{L} = \text{tr}\int d^8z\ \bar{\Phi} e^{-2gV}\Phi + \frac{1}{2}\text{tr}\int d^6z\ \mathcal{W}^2 +$$
$$+ \int d^8z\ (\bar{\Psi}_1 e^{-gV}\Psi_1 - \bar{\Psi}_2 e^{gV}\Psi_2) + \lambda\,\text{tr}\int d^6z\ \Psi_1\Phi\Psi_2 + \text{h.c.}$$

where $W_\alpha = -\frac{1}{8}\bar{D}^2(e^{-2gV}D_\alpha e^{2gV})$
V – vector field (**adj.** repr. of $SU(N)$)
Φ – chiral superfield (**adj.** repr. of $SU(N)$)
Ψ_1 (Ψ_2) – chiral superfields (**(anti)fund.** repr. of $SU(N)$)
g – coupling constant

⋆ Gauge-fixing term

$$\mathcal{S}_{GF} = -\frac{1}{16\xi}\text{tr}\int d^8z\ D^2V\bar{D}^2V, \quad \xi = 1$$

⋆ Faddeev-Popov ghost term

$$\mathcal{S}_{FP} = \text{tr}\int d^8z\ \left[\bar{c}'c - c'\bar{c} + \frac{1}{2}(c' + \bar{c}')[V, c+\bar{c}] + \cdots\right]$$

Superfield background splitting

- ⋆ Effective action can be written as a sum of classical action and quantum corrections under the shifting [I.L. Buchbinder and S.M. Kuzenko, '94]

$$\Phi \to \Phi + \sqrt{\hbar}\phi, \;\; \Psi_I \to \Psi_I + \sqrt{\hbar}\psi_I,$$

- ⋆ $S^{(2)}$ defines the superpropagators
- ⋆ Condition imposed on effective action $\partial_a \Phi = \partial_a \Psi_I = \partial_a \bar{\Phi} = \partial_a \bar{\Psi}_I = 0$
- ⋆ The superfield effective potential can be written as a series in the number of loops

$$\Gamma[\Phi, \bar{\Phi}] = \sum_{L=1}^{\infty} \hbar^L \Gamma^{(L)}[\Phi, \Psi_I | \bar{\Phi}, \bar{\Psi}_I]$$

- ⋆ Chiral potential

$$\mathbf{W}[\Phi, \Psi_I] = \sum_{L=1}^{\infty} \hbar^L \mathbf{W}^{(L)}[\Phi, \Psi_I]$$

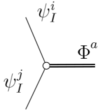
Feynman rules

 $= \langle \phi_a \bar{\phi}_b \rangle = -\frac{\bar{D}_1^2 D_2^2}{16\Box}\delta_{ab}\delta^8\left(z_1 - z_2\right)$

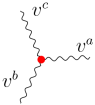
 $= \langle \psi_i \bar{\psi}_j \rangle = -\frac{\bar{D}_1^2 D_2^2}{16\Box}\delta_{ij}\delta^8\left(z_1 - z_2\right)$

 $= \langle v_a v_b \rangle = \left(-\frac{D^\alpha \bar{D}^2 D_\alpha}{8\Box^2} + \xi \frac{\{D^2, \bar{D}^2\}}{16\Box^2}\right)\delta_{ab}\delta^8\left(z_1 - z_2\right) \overset{\xi=1}{=} \frac{1}{16\Box}\delta_{ab}\delta^8\left(z_1 - z_2\right)$

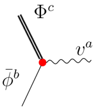
 $= \langle c'_a c_b \rangle = \langle c'_b c_a \rangle = \frac{1}{16\Box}\delta_{ab}\delta^8(z_1 - z_2)$



$-ig(t_a)_{ij}\Phi^a$



$-gf_{abc}$



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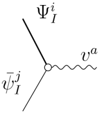
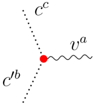


$-ig(t_a)_{ij}\Psi_I^i$



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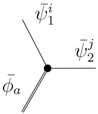


$-ig(t_a)_{ij}$

$SU(N)$ group generators $[t_a, t_b] = i(T^c)_{ab}t_c$, $(t^a)^i_j(t^a)^j_k = C_F\delta^i_k$, $T_{abc}T^{c'}_{ab} = C_A\delta^{c'}_c$, $\mathrm{tr}[t_a t_b] = T_A\delta_{ab}$

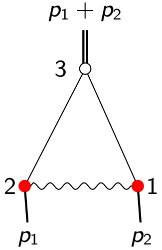
$C_F = \frac{N^2-1}{2N}$, $C_A = N$

$T_A = 1/2$ is the Dynkin index of fundamental representation of gauge group $SU(N)$

One-loop chiral integral

$$\int d^8z\; u(\Phi)\left(-\frac{D^2}{4\Box}\right)v(\Phi) = \int d^6z\; u(\Phi)v(\Phi)$$

$$D^2\bar{D}^2D^2 = 16\Box D^2$$
$$d^8z = d^4x d^2\theta d^2\bar{\theta}$$
$$d^6z = d^4x d^2\theta$$



$$\mathbf{W}^{(1)} = \lim_{p_1,p_2\to 0}\frac{\lambda g^2}{4}(2C_F - C_A)\int\prod_{l=1}^{3}d^8z_l\;\Phi(z_1)\Psi_1(z_2)\Psi_2(z_3)$$
$$\left\{\frac{1}{\Box_2}\delta_{1,2}\frac{D_1^2\bar{D}_3}{16\Box_1}\delta_{1,3}\frac{D_2^2}{4}\delta_{2,3}\frac{1}{\Box_2}\right\}$$

In assumption that $\Psi_1(y_1,\theta)\Psi_1(y_2,\theta)\Phi(x,\theta)\simeq[\Psi_1\Phi\Psi_2](x,\theta)$
Integration results (in agreement with [P.C. West '91])

$$\mathbf{W}^{(1)} = \frac{\hbar}{(4\pi)^2}\frac{g^2}{4}(2C_F - C_A)\Upsilon^{(1)}W_{tree}, \quad W_{tree} = \lambda\,\mathrm{tr}\int d^6z\;\Psi_1\Phi\Psi_2$$

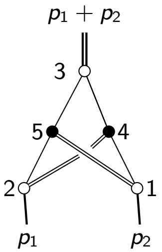
where $\Upsilon^{(1)}$ is the reduced Davydychev-Usyukina triangle integral [Usyukina,Davydychev'92]:

$$\Upsilon^{(1)} = \lim_{p_1,p_2\to 0}\int d^4q\frac{(p_1+p_2)^2}{q^2(q-p_1)^2(q_1+p_2)^2} = \int_0^1 d\tau\frac{2\log(\tau)}{\tau^2-\tau+1}.$$

Finite two-loop chiral superfield graph

[West '91] [I. Jack, D.R.T. Jones and P.C. West '91]

[I.L. Buchbinder, S.M. Kuzenko and A.Y. Petrov '94]



$$\mathbf{W}^{(2)} = \lim_{p_1, p_2 \to 0} \frac{g^4}{8C_A^2}(C_A^2 + 1) \int \prod_{l=1}^{5} d^8 z_l \; \lambda \Psi_1(z_3)\Phi(z_4)\Psi_2(z_5)$$

$$\left\{ \frac{1}{\Box_1}\delta_{1,3} \frac{D_2^2 \bar{D}_3^2}{16\Box_2}\delta_{3,2} \frac{1}{16\Box_2}\delta_{2,4} \frac{D_1^2 \bar{D}_4^2}{16\Box_1}\delta_{1,4} \frac{D_1^2 \bar{D}_5^2}{16\Box_1}\delta_{1,5} \frac{D_2^2}{4\Box_2}\delta_{2,5} \right\}.$$

Master-integral in this case is given by

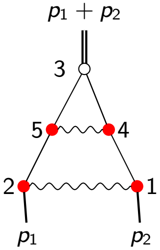
$$I^{(2)} = \lim_{p_{1,2} \to 0} \int \frac{d^4 q_1}{(4\pi)^4} \frac{d^4 q_2}{(4\pi)^4} \frac{q_1^2 p_1^2 + q_2^2 p_2^2 - 2p_1 p_2 (q_1 q_2)}{q_1^2 q_2^2 (q_1 + q_2)^2 (q_1 - p_1)^2 (q_2 - p_2)^2 (q_1 + q_2 - p_1 - p_2)^2}.$$

At the end

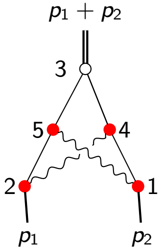
$$\mathbf{W}^{(2)} = \frac{\hbar^2}{(4\pi)^4} \frac{3}{4C_A^2}(C_A^2 + 1)|\lambda|^4 \zeta(3) \times W_{tree},$$

where $\zeta(n)$ is Riemann zeta-function.

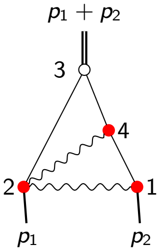
Finite two-loop supergraphs with gauge interaction



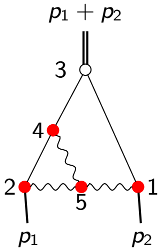
$$W^{(2)}_{fin} = \frac{3g^4}{8N^2}\zeta(3) \times W_{tree}$$



$$W^{(2)}_{fin} = -\frac{3g^4}{4N^2}(N^2+1)\zeta(3) \times W_{tree}$$

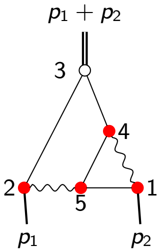


$$W^{(2)}_{fin} = \frac{3g^4}{4N^2}\zeta(3) \times W_{tree}$$

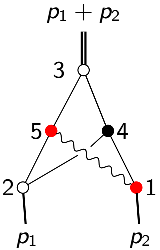


$$W^{(2)}_{fin} = \frac{g^4}{8}\Upsilon^{(2)} \times W_{tree},$$

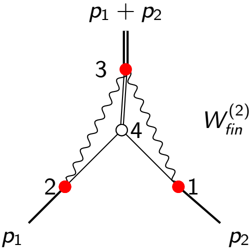
$$\Upsilon^{(2)} = \int_0^1 d\tau \frac{2\log^3(\tau)}{\tau^2 - \tau + 1}$$



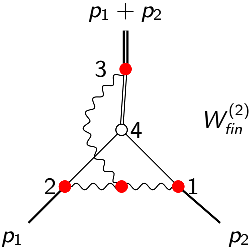
$$W_{fin}^{(2)} = -\frac{3g^4}{8N^2}\zeta(3) \times W_{tree}$$



$$W_{fin}^{(2)} = 0$$



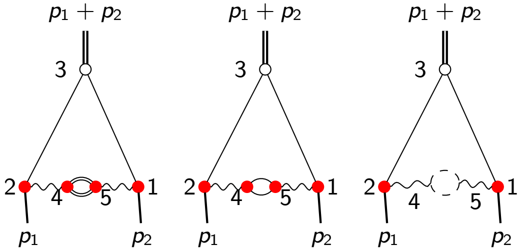
$$W_{fin}^{(2)} = 0$$

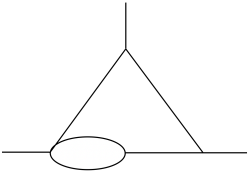


$$W_{fin}^{(2)} = 0$$

$$\mathbf{W}_{fin}^{(2)} = \left(\frac{1}{8}g^4\Upsilon^{(2)} + \frac{3}{4}\left(\frac{N^2+1}{N^2}|\lambda|^4 - g^4\right)\zeta(3)\right) \times W_{tree}$$

Divergent two-loop supergraphs with gauge interaction

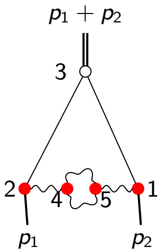




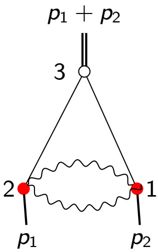
$$\mathbf{W}^{(2)}_{div,1} = \frac{1}{2} g^4 \left(2NT_A - C_A\right)\left(2C_F - C_A\right) \times J^{(1)}_{1,1} \Upsilon^{(1)} \times W_{tree},$$

where $J^{(1)}_{1,1}(k)$ is the one-loop integral

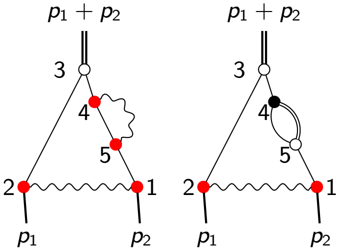
$$J^{(1)}_{1,1}(k) = \left(\frac{1}{\epsilon} + 2 + O(\epsilon^1)\right)(k^2/\mu^2)^{-\epsilon}.$$



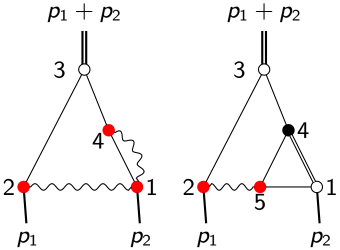
$\mathbf{W}^{(2)}_{div} = 0$



$\mathbf{W}^{(2)}_{div} = 0$



and



$$\mathbf{W}^{(2)}_{div,2} = \frac{1}{8} g^2 (2C_F - C_A) \left(|\lambda|^2 - g^2 \right) \times J^{(1)}_{1,1} \, \Upsilon^{(1)} \times W_{tree}, \quad W_{tree} = \lambda \, \mathrm{tr} \int d^6 z \; \Psi_1 \Phi \Psi_2$$

where $J^{(1)}_{1,1}$ is the one-loop two-point divergent subgraph.

Counterterms & Final result

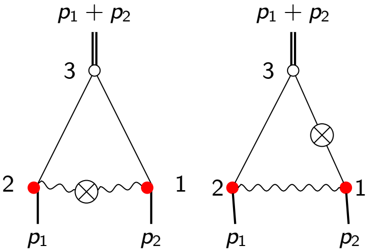


Figure: All divergent diagrams are equivalent to the triangle scalar master integral with one-loop subgraph inserted in the internal propagator.

Thus, the final result for the divergent part of the chiral effective superpotential on the two-loop level can be expressed as follows

$$\mathbf{W}^{(2)}_{div} = \Big(g^4\,\{2NT_A - C_A\} + g^2(|\lambda^2| - g^2)\Big) \times$$

$$\times\ \frac{1}{4}(2C_F - C_A)J^{(1)}_{1,1}\Upsilon^{(1)} \times W_{tree}\,,$$

$$W_{tree} = \lambda\,\mathrm{tr}\int d^6z\ \Psi_1\Phi\Psi_2$$

Final Result for $\mathcal{N} = 2$ SYM theory

Beta function of $\mathcal{N} = 2$ SYM theory

$$\beta(g) = \frac{2g^3}{(4\pi)^2}(2NT_A - C_A).$$

The finiteness of the $\mathcal{N} = 2$ SYM model with an arbitrary gauge group can be determined by the following condition [I.G. Koh and S. Rajpoot,'94]

$$\sum_i m_i T_A(R_i) = C_A$$

where R_i is some representation of the gauge group and m_i is a number of hypermupltiplets.
The final result for the two-loop finite chiral effective superpotential

$$\mathbf{W}^{(2)}_{fin,\mathcal{N}=2} = g^4 \left(\frac{1}{8}\Upsilon^{(2)} + \frac{3}{4N^2}\zeta(3)\right) \times W_{tree}$$

Large N limit in chiral effective superpotential for finite $\mathcal{N}=2$ SYM

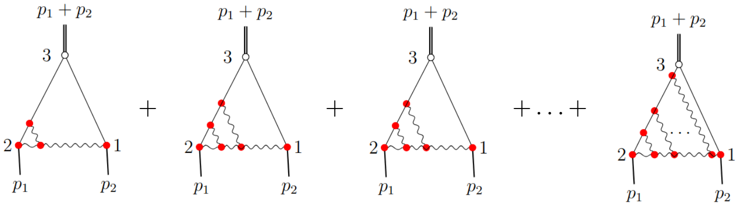


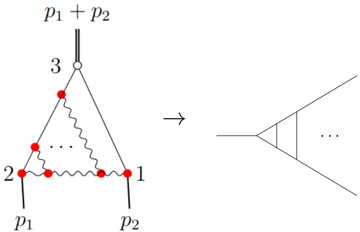
Figure: Sum of triangle-type diagrams series.

$$\mathbf{W}'^{(m)} = \lim_{p_1,p_2\to 0}(-1)^{m+1}\frac{g^{2m}}{2^{m+2}}N^{m-2}\int\prod_{l=1}^{2m+1}d^8z_l\ \lambda\Psi_1(z_1)\Phi(z_2)\Psi_2(z_3)\times$$

$$\times\left\{\frac{1}{\Box_5}\delta_{5,1}\frac{D_1^2\bar{D}_3^2}{16\Box_1}\delta_{1,3}\frac{D_4^2}{4\Box_3}\delta_{3,4}\frac{\bar{D}_4^2D_6^2}{16\Box_4}\delta_{4,6}\dots\right.$$

$$\dots\frac{\bar{D}_{2m-2}^2D_{2m}^2}{\Box_{2m-2}}\delta_{2m-2,2m}\frac{\bar{D}_{2m}^2D_2^2}{\Box_{2m}}\delta_{2m,2}\frac{1}{\Box_{2m+1}}\delta_{2m+1,2}\dots$$

$$\left.\frac{1}{\Box_{2m}}\delta_{2m+1,2m}\dots\frac{1}{\Box_4}\delta_{5,4}\right\}.$$



The result for this diagram in general form after evaluating D-algebra and re-expressing the integral through the Υ-functions can be written as:

$$\mathbf{W}'^{(m)} = (-1)^{m-1} \frac{g^{2m}}{2^{m+2}} N^{m-2} \Upsilon^{(m)} \times W_{tree}.$$

The formal sums over these contributions are given ($y = g^2 N/2$)

$$\mathbf{W}'^{lead} = \frac{y}{2N^2} \int_0^1 d\tau \frac{\log(\tau)\ (1-\tau)}{(1 + y\log^2(\tau))\ (1+\tau^3)} \times W_{tree} = \Upsilon^{tot} \times W_{tree},$$

and after integration one has the closed result

$$\Upsilon^{tot} = \frac{1}{4N^2} \sum_{m=1}^{\infty} ((\pi - 2\mathrm{Si}(x))\sin(x) - 2\mathrm{Ci}(x)\cos(x)) U_m(1/2),$$

with $x = \frac{m+1}{\sqrt{y}}$, Si and Ci are integral sine and cosine functions and $U_n(x)$ is Chebyshev polynomials.

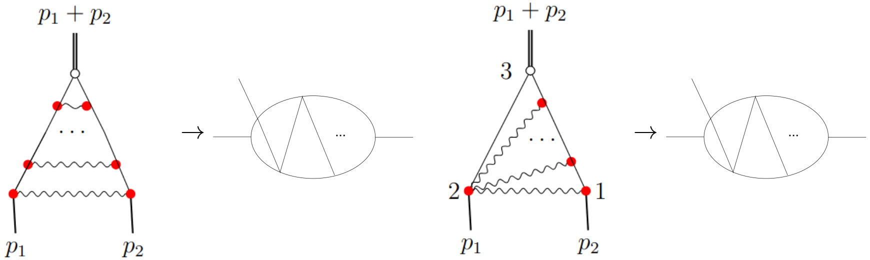


Figure: Examples of reduction of higher loop supergraphs to scalar conformal zig-zag master-integrals.

$$Z(L+1) = 4C_L \sum_{p=1}^{\infty} \frac{(-1)^{(p-1)(L+1)}}{p^{2(L+1)-3}} = \begin{cases} 4\, C_L\, \zeta(2L-1) \ \text{ for } \ L = 2N+1\,, \\ 4\, C_L\, (1-2^{2(1-L)})\, \zeta(2L-1) \ \text{ for } \ L = 2N\,, \end{cases}$$

[D.J. Broadhurst and D. Kreimer '95], [S. Derkachov, A.P. Isaev and L. Shumilov 2022], [S.E. Derkachov, A.P. Isaev and L.A. Shumilov 2023]

where L is a number of loops, $C_L = \frac{1}{(L+1)}\binom{2L}{L}$ is the Catalan number. Thus the whole finite subleading correction to chiral effective superpotential can be expressed as

$$\mathbf{W}^{sub} \sim g^{2L}\ c_L/N^L \times Z(L+1) \times W_{tree}\,,$$

Results & Prospects

✔ Performed one-, two-loop calculations of the chiral effective potential in the $\mathcal{N}=1$ supersymmetric gauge theory with chiral matter

- One-loop chiral quantum corrections turn to be finite
- At 2-loop level there are only a few types of two-loop supergraph topologies leading to a chiral effective superpotential
- The divergent two-loop diagrams include the one-loop two-point corrections to the vector and the chiral superfield Green function
- The divergent contributions can be reduced by introducing appropriate counterterms at the diagram level or by renormalizing the fields in the original Lagrangian

✔ In the considered finite $\mathcal{N}=2$ $SU(N)$ non-Abelian Yang-Mills model, summation of specific individual sequences of diagrams have been shown in the large N limit

⋆ Find the connections of chiral contributions with conformal zig-zags and the possibility of obtaining exact values for chiral superpotentials in various models.

Thank you for attention!