# Minimal string theory and topological recursion

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# “Non-critical” string theories

Worldsheet CFT:
ghosts BC-system + Liouville CFT + (conformal) matter;
$c_L + c_M + (-26) = 0$

$$\int_{\overline{\mathcal{M}}_{g,n}} Z_{\mathsf{BC}} \left\langle \prod_{i=1}^{n} V_{1-\Delta_i}(z_i) \right\rangle_{\mathsf{L}} \left\langle \prod_{i=1}^{n} \Phi_{\Delta_i}(z_i) \right\rangle_{\mathsf{M}} \Leftarrow$$

perturbative
“tachyon” amplitudes

moduli space
of curves

Liouville primaries

Matter primaries

Matter theory:

- “timelike Liouville CFT” — “Virasoro minimal string”
  [Eberhardt. Collier
  Muhlmann, Rodriguez ’23]
- Liouville CFT, $c_L = c_M^* \in 13 + i\,\mathbb{R}$ — “complex Liouville string”
  [Eberhardt. Collier
  Muhlmann, Rodriguez ’24]
- Virasoro $(p, q)$ minimal model — “minimal string” ← this talk!

# “Non-critical” string theories

- Known analytic/numeric answers for amplitudes are very simple. **Reason:** all mentioned theories are (conjecturally) dual to “matrix models”.
- Minimal string: the oldest known example [Douglas, V. Kazakov, Daul, Brezin, ...], but (perhaps) most subtle — dictionary of the duality is confusing, “operator mixing”, no arguments from analyticity
- **Goal:** reformulate/simplify the correspondence to
  - improve understanding of the duality
  - understand relations to other examples
  - facilitate computations

# “Old” dictionary

- “Matrix model”: $\langle\mathcal{O}\rangle = \int [dH]\, e^{-N\,\mathrm{tr}\,V(H)}\mathcal{O}(H)$, $H - N \times N$ hermitian
  (or 2-matrix model: tr $[V_1(H_1) + V_2(H_2) + H_1 H_2]$)
  $V(H) = \sum c_k H^k$; $c_k^{(p)}$ – “$p$-critical points”; “double-scaling limit”
  partition function $\mathcal{F} = \sum\limits_{g=0}^{\infty} \mathcal{F}_g N^{2-2g}$ — genus expansion
- in special coordinates $t_k$ near $p$-critical point $\mathcal{F}$ is a special tau-function of (reduced) KdV hierarchy (KP hierarchy for 2-matrix model)

# “Old” dictionary

- tachyon amplitudes in $(2, 2p+1)$ minimal string theory $A_n^g(k_1 \dots k_n)$ [$\mathcal{T}_{1,k} = V_{1,-k}\Phi_{1,k}$ with matter primaries enumerated by $k = 1 \dots p$] are identified with (singular part of)

$$\left.\frac{\partial^n \mathcal{F}_g}{\partial\tau_{k_1} \dots \partial\tau_{k_n}}\right|_{t_k = t_k^{(0)}} \longleftarrow \text{“conformal background”}$$

- $t$ and $\tau$ are related via “resonance transformations”

$$\boxed{t_k = (2p+1) u_0^{k+1} \sum_{n=1}^{\lfloor \frac{k+1}{2} \rfloor} \sum_{\substack{m_1 \dots m_n \geq 1 \\ \sum(m_l+1) = k+1}} \frac{\tau_{m_1} \dots \tau_{m_n}}{n!} \frac{(2p-2k+2n-3)!!}{(2p-2k-1)!!}}$$

[Moore '91
Belavin, Zamolodchikov '08]

needed to satisfy minimal model “fusion rules”

# Topological recursion

Computing $\mathcal{F}_g(t)$ is difficult...
In matrix model it is easier to compute resolvent correlators

$$\left\langle \mathrm{tr}\,\frac{1}{E_1 - H} \ldots \mathrm{tr}\,\frac{1}{E_n - H} \right\rangle_{\text{connected}} = \sum_{g=0}^{\infty} R_{g,n} N^{2-2g-n} \tag{1}$$

$R_{g,n}$ obey "loop equations" that can be translated into **"topological recursion"**

# Topological recursion

Spectral curve: $(x(z), y(z)) \subset \mathbb{C}^2$

- $\begin{cases} x = 2u_0\, T_2(z) \\ y = 2u_0^{p+1/2}\, T_{2p+1}(z) \end{cases} \Rightarrow \omega_{0,1} = y\, dx$

  — encodes the potential $V(H)$, or "conformal background" $t_k^0$ for $(2, 2p+1)$ minimal string [Seiberg, Shih, '01] ($T$ is a Chebyshev polynomial)
- bidifferential $B \equiv \omega_{0,2} = \frac{dz_1 dz_2}{(z_1 - z_2)^2}$ — universal in "one-cut matrix models"

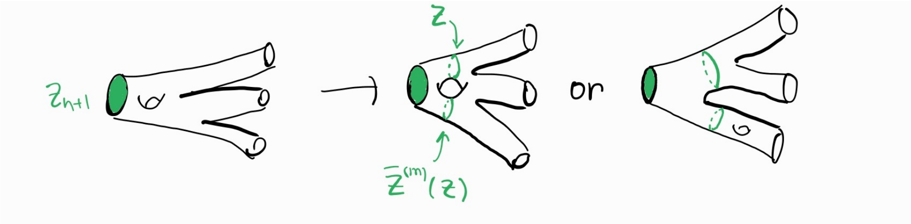
$\left.\begin{matrix} \omega_{0,1} \\ \omega_{0,2} \end{matrix}\right\} \Longrightarrow$ $n$-differentials $\omega_{g,n} = W_{g,n}\, dz_1 \ldots dz_m$; $W_{g,n} \sim R_{g,n}$

# Topological recursion

for each $\zeta_m : dx(\zeta_m) = 0$

- $\overline{z}^{(m)}(z)$ — Galois involution
- $\left.\begin{matrix}\omega_{0,1}\\ \omega_{0,2}\end{matrix}\right\} \Rightarrow K^{(m)}(z_0, z) = \dfrac{\int\limits_{\overline{z}^{(m)}}^{z} B(z_0,\xi)d\xi}{2(y(z)-y(\overline{z}^{(m)}))dx}$ — recursion kernel

$$\omega_{g,n+1}(z_1,\dots,z_n,z_{n+1}) = \sum_m \operatorname*{Res}_{z=\zeta_m} K^{(m)}(z_{n+1},z)\Big(\omega_{g-1,n+2}(z_1,\dots z_n,z,\overline{z}^{(m)})+$$
$$+\sum_{g_1=0}^{g}\ \sum^{\prime}_{J_1+J_2=\{z_1\dots z_n\}} \omega_{g_1,|J|+1}(J_1,z)\omega_{g-g_1,|J_2|+1}(J_2,\overline{z}^{(m)})\Big)$$



# Question

- Simple (universal!) formulas exist that compute $\frac{\partial^n \mathcal{F}_g}{\partial t_{k_1} \ldots \partial t_{k_n}}\Big|_{t_k = t_k^{(0)}}$:

$$\frac{\partial^n \mathcal{F}_g}{\partial t_{k_1} \ldots \partial t_{k_n}} \left|_{t_k = t_k^0} = \operatorname*{Res}_{z_i = \infty} \left( \omega_{g,n}(z_1, \ldots, z_n) \prod_{i=1}^{n} \frac{x^{p-k_i+1/2}(z_i)}{2p - 2k_i + 1} \right)\right.$$

- How to incorporate "resonance transformations" ($\frac{\partial}{\partial t} \to \frac{\partial}{\partial \tau}$) and discard the regular part?

# Surprising observation

On several nontrivial examples we noticed that it is encoded in a so-called "$x - y$ swap"

$$(1) \qquad \begin{tikzcd}
x = 2u_0\,T_2(z) \arrow[dr] & \check{x} = 2u_0^{p+1/2}\,T_{2p+1}(z) \\
y = 2u_0^{p+1/2}\,T_{2p+1}(z) \arrow[ur] & \check{y} = 2u_0\,T_2(z)
\end{tikzcd} \qquad (2)$$

In two-matrix model (1) and (2) compute resolvents associated to the first and second matrix: $\left\langle \mathrm{tr}\, \frac{1}{E_1 - H_1} \ldots \right\rangle$ and $\left\langle \mathrm{tr}\, \frac{1}{E_1 - H_2} \ldots \right\rangle$ — nontrivial transformation!

# Statement

$$\check{A}^g_n(k_1,\dots,k_n) = \operatorname*{Res}_{z_1=\infty} \dots \operatorname*{Res}_{z_n=\infty} \left( \check{\omega}_{g,n}(z_1,\dots,z_n) \prod_{i=1}^{n} \frac{T_{2(p-k_i)+1}(z_i)}{2(p-k_i)+1} \right)$$

coincides with singular part of $\left.\frac{\partial^n \mathcal{F}_g}{\partial\tau_{k_1}\dots\partial\tau_{k_n}}\right|_{t_k=t_k^{(0)}}$ and should compute tachyon correlators.

# Simple examples

Here and in what follows $b^2 \equiv \frac{2}{2p+1}$.
Calculation yields

$$\check{A}_3^0(\vec{k}) = \underset{\substack{\text{normalization}\\\text{(not important)}}}{\frac{1}{2(2p+1)}} \cdot \underline{b^2 \sum_{m=1}^{2p} \frac{\prod_{i=0}^{2} \sin \frac{2\pi m k_i}{(2p+1)}}{\sin \pi m b^2}}$$

From the worldsheet $A_3^0 = \mathcal{N}^{(0)}_{k_1 k_2 k_3}$ — fusion number/dimension of spaces of 3pt conformal blocks in minimal model sector ($0$ or $1$).
The underlined factor is equal to $\mathcal{N}^{(0)}_{k_1 k_2 k_3}$ — s.c. Verlinde formula!

More generally, $\mathcal{N}^{(0)}_{i_1 \ldots i_k} = (-1)^{\sum\limits_l (i_l - 1)} b^2 \sum\limits_{m=1}^{2p} \frac{\prod\limits_{l=1}^{k} \sin \pi m i_l b^2}{(\sin \pi m b^2)^{k-2}}$.

# Simple examples

Next is

$$\check{A}^0_4(\vec{k}) = \frac{V^b_{0,4}(\vec{k})}{2\pi^2}\mathcal{N}^{(0)}_{k_1\ldots k_4} + \sum_{i=2,3,4}\sum_{m=1}^{2p}\left(\frac{1}{2}\mathcal{N}^{(0)}_{k_1 k_i m}\mathcal{N}^{(0)}_{m\bullet\circ}\right)\mathbf{G}(m)$$

- $\mathbf{G}(m) = 4B_2\left(\frac{b^2}{2}|p+1/2-m|\right)$, $B$ — Bernoulli polynomial
- $V^b_{g,n}$ — "quantum volumes" of [Eberhardt. Collier Muhlmann, Rodriguez '23], amplitudes in VMS. They are polynomials in $(p+\frac{1}{2}-k_i)$; e.g. $\frac{V^b_{0,4}}{2\pi^2} = 1 + b^4 - b^4\sum(p+\frac{1}{2}-k_i)^2$

. $\check{A}^0_4$, although superficially different, agrees with previously known results in "matrix models" [Belavin Zamolodchikov '08] and worldsheet [Belavin Zamolodchikov '05] “resonance transformations”:

$$Z_{k_1k_2k_3k_4} = -F_\theta(-2) + \sum_{i=1}^{4} F_\theta(k_i - 1) - F_\theta(k_{12|34}) - F_\theta(k_{13|24}) - F_\theta(k_{14|23}) \tag{2.17}$$

where $k_{ij|lm}$ and the function $F_\theta$ are defined as

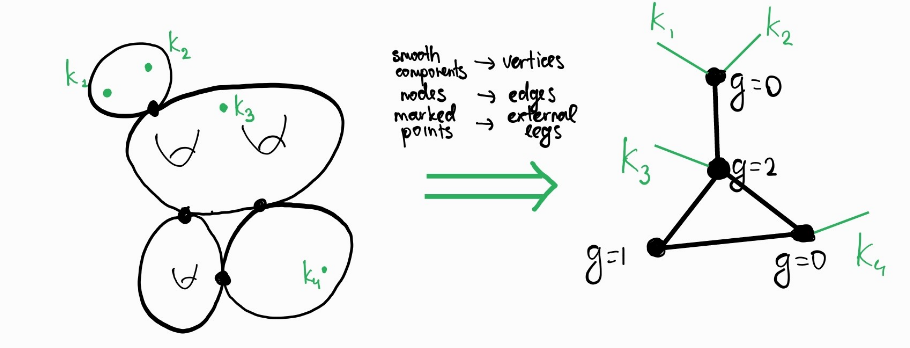
$$k_{ij|lm} = \min(k_i + k_j, k_l + k_m); \quad F_\theta(k) = \frac{1}{2}(p - k - 1)(p - k - 2)\theta(p - 2 - k) \tag{2.18}$$

worldsheet:

$$C_{k_1k_2k_3k_4} = (k_1 + 1)(p + k_1 + 3/2) - \sum_{i=2}^{4} \sum_{s=-k_1:2}^{k_1} \left|p - k_i - s - \frac{1}{2}\right|. \tag{2.26}$$

# General answer

Given in terms of sums over “stable graphs” (they play the role of “Feynman diagrams”).
They enumerate possible degenerations of punctured Riemann surfaces:



# General answer

The simplest way to write the answer:

$$\check{A}^g_n(\vec{k}) = \overset{\substack{\text{stable graphs}\\ \text{of genus } g\\ \text{with } n \text{ legs}}}{\sum_{\Gamma} \frac{1}{|\mathsf{Aut}(\Gamma)|}} \underset{\substack{\text{integer number}\\ \text{for every edge}}}{\sum_{\vec{k_e}\in\mathbb{Z}}} \overset{\substack{\vec{k}_v \text{ — parameters of edges}\\ \text{and legs entering } v;\\ \text{product over vertices}}}{\prod_{v} \left(\mathcal{N}^{(g_v)}_{\vec{k}_v} \frac{V^b_{g_v,n_v}(\vec{k}_v)}{(2\pi^2)^{3g_v+n_v-3}}\right)} \times$$

$$\times \underset{\text{product over edges}}{\prod_{e} \left(b^2|p+\frac{1}{2}-k_e|\right)}$$

($\sum_{k_e}$ diverges; interpreted in $\zeta$-regularization produces Bernoulli polynomials)

# Features of the answer

This formula:

- manifestly obeys “fusion rules”
- agrees with previous computations (no general formula known before!)
- naturally generalizes to $(p,q)$ minimal string (not well understood in the usual approach)
- similar to the answer obtained for “complex Liouville string” theory

# Interesting byproducts of this reformulation

- $p \to \infty$ — JT gravity limit; gives an alternative way to compute "Weil-Petersson volumes for surfaces with conical defects" [Eberhatdt Turiaci '23] from $x - y$ swapped "Mirzakhani spectral curve"
- a conjecture for correlators with "ground ring" operators in minimal string — to be studied further [AA V. Belavin, WIP]

Thank you for your attention!