Meson mass spectrum in Ising Field Theory

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AQFT2025

Ising field theory

IFT is defined via the formal action

$$\mathcal{A}_{\mathrm{IFT}} = \mathcal{A}_{c=\frac{1}{2}\ \mathrm{CFT}} + m\int \varepsilon(x)\, d^2x + h\int \sigma(x)\, d^2x,$$

where $\mathcal{A}_{c=\frac{1}{2}\ \mathrm{CFT}}$ is the action of the minimal model $\mathcal{M}_{3,4}$ of 2D CFT and the couplings m and h are

$$m \sim T_c - T, \quad h \sim H.$$

The fields $\varepsilon(x)$ and $\sigma(x)$ possess scaling dimensions

$$(\Delta_\varepsilon, \bar{\Delta}_\varepsilon) = \left(\tfrac{1}{2}, \tfrac{1}{2}\right) \quad (\Delta_\sigma, \bar{\Delta}_\sigma) = \left(\tfrac{1}{16}, \tfrac{1}{16}\right)$$

Thus the theory is characterized by the single scaling parameter

$$\eta = \frac{m}{|h|^{\frac{8}{15}}}.$$

Ising field theory

IFT exhibits a rich structure as a two-dimensional QFT:

1. At zero magnetic field $h = 0$ it reduces to the free theory of massive Majorana fermions of mass $|m|$ (Onsager 1943)
2. At $m = 0$, $h \neq 0$ IFT becomes integrable and has eight stable particles in the spectrum, also known as E_8 particles (Zamolodchikov 1989)
3. Additional integrable structure emerges near the Yang-Lee singularity at imaginary magnetic field (Lee-Yang 1952). In this regime (perturbed $\mathcal{M}_{2,5}$), there is only one stable particle.

No other real or complex values of the parameters are known to lead to integrable QFT.

Ising field theory

According to McCoy-Wu scenario while η changes from $-\infty$ to ∞, the spectrum IFT undergoes evolution from a single particle to an infinite tower of "mesons" formed by confined "quarks".

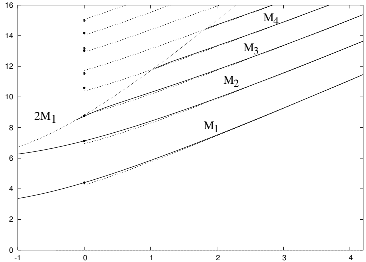


Figure: The mass spectrum $M_n(\eta)$ of IFT particles obtained using the TFFSA (solid lines) and the Bethe-Salpeter equation (dashed lines).

Ising field theory

A non-perturbative approach to analyzing the meson mass spectrum in IFT was developed by Fonseca and Zamolodchikov and is based on the Bethe–Salpeter (BS) equation formulated within the two-quark approximation:

$$\left[m^2 - \frac{M^2}{4\cosh^2\theta}\right]\psi(\theta) =$$
$$= f_0 \fint_{-\infty}^{\infty} \frac{d\theta'}{2\pi}\left[\frac{2\cosh(\theta-\theta')}{\sinh^2(\theta-\theta')} + \frac{\sinh\theta\,\sinh\theta'}{4\cosh^2\theta\,\cosh^2\theta'}\right]\psi(\theta').$$

where $f_0 \sim |m|^{\frac{1}{8}}h$ represents the bare string tension.

Bethe-Salpeter equation

It is convenient to change the variables to rewrite BS equation in the following form (similar to 't Hooft equation for large N_c 2D QCD)

$$\left(\frac{2\alpha}{\pi}+\nu\tanh\frac{\pi\nu}{2}\right)\Psi(\nu)-\frac{1}{16}\frac{\nu}{\cosh\frac{\pi\nu}{2}}\int\limits_{-\infty}^{\infty}d\nu'\frac{\nu'}{\cosh\frac{\pi\nu'}{2}}\Psi(\nu')=$$

$$=\lambda\int\limits_{-\infty}^{\infty}d\nu'\frac{\pi(\nu-\nu')}{2\sinh\frac{\pi(\nu-\nu')}{2}}\Psi(\nu'),$$

where

$$\frac{2\alpha}{\pi}=\frac{m^2}{f_0},\quad \lambda=\frac{M^2}{4\pi f_0}.$$

Bethe-Salpeter equation

'**t** Hooft equation for large N_c multiflavor 2D QCD has the form

$$\left(\frac{2\alpha}{\pi}+\nu\coth\frac{\pi\nu}{2}\right)\Psi(\nu|\beta)-\frac{2i\beta}{\pi}\fint_{-\infty}^{\infty}d\nu'\,\frac{1}{2\sinh\frac{\pi(\nu-\nu')}{2}}\Psi(\nu'|\beta)=$$

$$=\lambda\int_{-\infty}^{\infty}d\nu'\,\frac{\pi(\nu-\nu')}{2\sinh\frac{\pi(\nu-\nu')}{2}}\Psi(\nu'|\beta).$$

$$\alpha=\frac{\alpha_1+\alpha_2}{2},\quad \beta=\frac{\alpha_2-\alpha_1}{2},\quad \alpha_i=\frac{\pi m_i^2}{g^2}-1,\quad M_n^2=2\pi g^2\lambda_n.$$

Analytic tools have been developed

1. $\alpha=\beta=0$: Fateev, Lukyanov and Zamolodchikov 2009
2. $\alpha\neq 0$, $\beta=0$: Litvinov and Meshcheriakov 2024
3. $\alpha\neq 0$, $\beta\neq 0$: Artemev, Litvinov and Meshcheriakov 2025

TQ equation and integrability

We define the Q-function:

$$Q(\nu) \overset{\text{def}}{=} \cosh\frac{\pi\nu}{2}\left(\frac{2\alpha}{\pi} + \nu\tanh\frac{\pi\nu}{2}\right)\Psi(\nu)$$

It can be shown that $Q(\nu)$ satisfies Baxter's TQ equation

$$Q(\nu+2i) + Q(\nu-2i) - 2Q(\nu) = -\frac{2z}{\nu+\alpha x}Q(\nu),$$

$$z = 2\pi\lambda\coth\frac{\pi\nu}{2}, \quad x = \frac{2}{\pi}\coth\frac{\pi\nu}{2}.$$

1. $Q(\nu)$ is analytic in the strip $\mathrm{Im}\,\nu \in [-2,2]$
2. $Q(\nu)$ grows slower than any exponential as $|\mathrm{Re}\,\nu| \to \infty$

$$\forall\epsilon > 0 \quad Q(\nu) = \mathcal{O}(e^{\epsilon|\nu|}), \quad |\mathrm{Re}\,\nu| \to \infty;$$

3. the quantization condition:

$$Q(i) = -Q(-i) = \frac{i}{16}\int_{-\infty}^{\infty} d\nu' \frac{\nu'}{\cosh\frac{\pi\nu'}{2}} \frac{Q(\nu')}{\frac{2\alpha}{\pi}\cosh\frac{\pi\nu'}{2} + \nu'\sinh\frac{\pi\nu'}{2}}.$$

TQ equation and integrability

If the quantization condition is dropped, one can construct solutions $Q(\nu|\lambda)$ for arbitrary λ. In this case, the associated function $\Psi(\nu|\lambda)$ solves more general inhomogeneous integral equation

$$\left(\frac{2\alpha}{\pi}+\nu\tanh\frac{\pi\nu}{2}\right)\Psi(\nu)-\lambda\int_{-\infty}^{\infty}d\nu'\,\frac{\pi(\nu-\nu')}{2\sinh\frac{\pi(\nu-\nu')}{2}}\Psi(\nu')=F(\nu),$$

with the inhomogeneous part

$$F(\nu)\stackrel{\text{def}}{=}\frac{Q(i)+Q(-i)}{2}\,\frac{1}{\cosh\frac{\pi\nu}{2}}+\frac{Q(i)-Q(-i)}{2i}\,\frac{\nu}{\cosh\frac{\pi\nu}{2}}.$$

TQ equation and integrability

We are interested in the even/odd spectral sums, defined respectively as the traces over even/odd states:

$$\mathcal{G}_{+}^{(s)}(\alpha) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \left(\frac{1}{\lambda_{2n}^{s}} - \frac{\delta_{s,1}}{n+1} \right), \quad \mathcal{G}_{-}^{(s)}(\alpha) \stackrel{\text{def}}{==} \sum_{n=0}^{\infty} \left(\frac{1}{\lambda_{2n+1}^{s}} - \frac{\delta_{s,1}}{n+1} \right).$$

which can be conveniently packed into the spectral determinants

$$\mathcal{D}_{+}(\lambda) \stackrel{\text{def}}{=} \left(\frac{2\pi}{e} \right)^{\lambda} \prod_{n=0}^{\infty} \left(1 - \frac{\lambda}{\lambda_{2n}} \right) e^{\frac{\lambda}{n+1}},$$

$$\mathcal{D}_{-}(\lambda) \stackrel{\text{def}}{=} \left(\frac{2\pi}{e} \right)^{\lambda} \prod_{n=0}^{\infty} \left(1 - \frac{\lambda}{\lambda_{2n+1}} \right) e^{\frac{\lambda}{n+1}}.$$

as

$$\mathcal{D}_{\pm}(\lambda) = \left(\frac{2\pi}{e} \right)^{\lambda} \exp\left[- \sum_{s=1}^{\infty} s^{-1} \mathcal{G}_{\pm}^{(s)} \lambda^{s} \right].$$

TQ equation and integrability

The spectral determinants $\mathcal{D}_{\pm}(\lambda)$ are for our BS equation

$$\left(\frac{2\alpha}{\pi} + \nu \tanh \frac{\pi\nu}{2}\right)\Psi(\nu) - \frac{1}{16}\frac{\nu}{\cosh \frac{\pi\nu}{2}} \int\limits_{-\infty}^{\infty} d\nu' \frac{\nu'}{\cosh \frac{\pi\nu'}{2}} \Psi(\nu') =$$
$$= \lambda \int\limits_{-\infty}^{\infty} d\nu' \frac{\pi(\nu - \nu')}{2 \sinh \frac{\pi(\nu-\nu')}{2}} \Psi(\nu'),$$

Let us define similar objects $D_{\pm}(\lambda)$ for modified BS equation (even more similar to 't Hooft equation)

$$\left(\frac{2\alpha}{\pi} + \nu \tanh \frac{\pi\nu}{2}\right)\Psi(\nu) = \lambda \int\limits_{-\infty}^{\infty} d\nu' \frac{\pi(\nu - \nu')}{2 \sinh \frac{\pi(\nu-\nu')}{2}} \Psi(\nu'),$$

TQ equation and integrability

These two types of spectral determinants are related by

$$\mathcal{D}_+(\lambda) = D_+(\lambda), \quad \mathcal{D}_-(\lambda) = \frac{1}{1-\mathtt{v}(\alpha)}\left(1 + \frac{Q'_-(i)-1}{4\pi^2\lambda}\right) D_-(\lambda),$$

where

$$\mathtt{v}(\alpha) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty} \frac{t^2}{\cosh^2 t}\frac{dt}{\alpha + t\tanh t}$$

One has remarkable log-derivative relations (still unproven!)

$$\partial_\lambda \log D_-(\lambda) - \frac{\alpha \mathtt{i}_2(\alpha)}{4} = 2i\left(1 - \frac{\alpha}{\pi^2}\lambda^{-1}\right)\partial_\nu \log Q_+(\nu)\Big|_{2i},$$

$$\partial_\lambda \log D_+(\lambda) - \frac{\alpha \mathtt{i}_2(\alpha)}{4} = 2i\left(1 + \frac{1}{1-\frac{\pi^2\lambda}{\alpha}}\right)\partial_\nu \log Q_-(\nu)\Big|_{2i},$$

TQ equation and integrability

We note that the TQ equation

$$Q(\nu+2i)+Q(\nu-2i)-2Q(\nu)=-\frac{2z}{\nu+\alpha x}Q(\nu)$$

admits the following two solutions

$$\Xi(\nu|\lambda)=(\nu+\alpha x)\sum_{k=0}^{\infty}\frac{\left(1+\frac{i(\nu+\alpha x)}{2}\right)_k}{k!(k+1)!}(-iz)^k$$

$$\Sigma(\nu|\lambda)=1+\sum_{k=1}^{\infty}\frac{\left(\frac{i(\nu+\alpha x)}{2}\right)_k}{k!(k-1)!}\left(\psi_\alpha\left(\nu-2i(k-1)\right)-\psi(k)\right.$$
$$\left.-\psi(k+1)++\psi\left(\frac{1}{2}\right)+\log 16\right)(-iz)^k,$$

TQ equation and integrability

Where $\psi_\alpha(\nu)$ is given by the integral representation

$$\psi_\alpha(\nu+i) = -\gamma_E - \log 4 + \\ + \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{t+\frac{2\alpha}{\pi}\coth\frac{\pi t}{2}}\left(\tanh\frac{\pi t}{2} - \tanh\frac{\pi(t-\nu)}{2}\right) dt.$$

One can check that $\psi_\alpha(\nu)$ is a solution of the difference equation

$$\psi_\alpha(\nu+2i) = \psi_\alpha(\nu) + \frac{2i}{\nu+\frac{2\alpha}{\pi}\coth\frac{\pi\nu}{2}}.$$

TQ equation and integrability

Let us define two functions ($M_{\pm}(-\nu|\lambda) = \pm M_{\pm}(\nu|\lambda)$)

$$M_{-}(\nu|\lambda) = e^{\frac{iz}{2}}\Xi(\nu|\lambda), \quad M_{+}(\nu|\lambda) = \frac{1}{2}\left(e^{\frac{iz}{2}}\Sigma(\nu) + e^{-\frac{iz}{2}}\Sigma(-\nu)\right),$$

The functions $M_{\pm}(\nu|\lambda)$ solve the TQ-equation, but they do not satisfy the analytic properties that we require: the factors $(-iz)^k$ have poles at the points $0, \pm 2i$ of growing order.
We "improve" the solution of TQ equation as ($\tau \stackrel{\text{def}}{=} \frac{\pi^2}{4}\coth^2\left(\frac{\pi\nu}{2}\right)$)

$$Q_{\pm}(\nu|\lambda) = A_{\pm}(\tau|\lambda)M_{\pm}(\nu|\lambda) + B_{\pm}(\tau|\lambda)zM_{\mp}(\nu|\lambda),$$

where $A_{\pm}(\tau|\lambda)$ and $B_{\pm}(\tau|\lambda)$ admit expansions in the parameter λ with coefficients that are polynomials in τ

$$A_{\pm}(\tau|\lambda) = 1+\sum_{s=1}^{\infty} a_{\pm}^{(s)}(\tau)\lambda^s,\ B_{\pm}(\tau|\lambda) = -(1\mp 1)\frac{\alpha}{2\pi^2}\lambda^{-1}+\sum_{s=0}^{\infty} b_{\pm}^{(s)}(\tau)\lambda^s$$

TQ equation and integrability

The functions $a_{\pm}^{(s)}(\tau)$ and $b_{\pm}^{(s)}(\tau)$ are polynomials in τ of degree s and $s+1$ respectively. We determine these expansion coefficients by requiring the Q-function to be pole-free at $\nu = 0$ and $\nu = \pm 2i$:

$$a_{+}^{(1)}(\tau) = \frac{8\alpha}{\pi^2}\tau, \quad b_{+}^{(0)}(\tau) = \frac{\mathrm{u}_0(\alpha)}{2}$$

$$a_{+}^{(2)}(\tau) = -\left[2 + \frac{8\alpha\left(\pi^2 - \alpha\zeta(3)\right)}{\pi^4} - \frac{4\alpha^3\mathrm{u}_3(\alpha)}{\pi^2}\right]\tau + \frac{16\alpha^2}{\pi^4}\tau^2,$$

$$b_{+}^{(1)}(\tau) = 2 + \frac{\pi^2}{4\alpha} - \alpha^2\mathrm{u}_3(\alpha) - \frac{2\alpha\zeta(3)}{\pi^2} + \frac{4\alpha\left(\mathrm{u}_0(\alpha)+2\right)}{\pi^2}\tau, \ldots$$

where (similar integral representation for $\mathrm{u}_0(\alpha)$)

$$\mathrm{u}_{2k-1}(\alpha) \stackrel{\text{def}}{=} \fint_{-\infty}^{\infty} dt \frac{\cosh^2 t}{t\sinh^{2k-1} t \cdot (\alpha\cosh t + t\sinh t)}.$$

TQ equation and integrability

Using log-derivative relations one can derive analytic expressions for the spectral sums $\mathcal{G}_{\pm}^{(s)}$

$$\mathcal{G}_{+}^{(1)} = \log(2\pi) - 3 - \frac{\alpha}{4}\mathtt{i}_2(\alpha),$$

$$\mathcal{G}_{-}^{(1)} = \log(2\pi) - 1 - \frac{2\alpha\zeta(3)}{\pi^2} - \frac{\alpha}{4}\mathtt{i}_2(\alpha) - \alpha^2\mathtt{u}_3(\alpha) + \dots,$$

$$\mathcal{G}_{+}^{(2)} = 2 + \frac{\pi^2}{\alpha} - \frac{4\alpha}{3} - \frac{4\alpha^2\zeta(5)}{\pi^4} + \frac{8\alpha^2\zeta(3)}{3\pi^2} + 2\alpha^3\left(\mathtt{u}_3(\alpha) + \mathtt{u}_5(\alpha)\right),$$

$$\mathcal{G}_{-}^{(2)} = -\frac{\pi^2}{\alpha} + \frac{8\alpha}{3} - \frac{8\alpha}{3\pi^2}(3+2\alpha)\zeta(3) + \frac{4\alpha^2}{\pi^4}\left(\zeta^2(3) + 2\zeta(5)\right) - \\ - 4\alpha^2\mathtt{u}_3(\alpha)\left(1 + \alpha - \frac{\alpha\zeta(3)}{\pi^2}\right) + \alpha^4\mathtt{u}_3^2(\alpha) - 4\alpha^3\mathtt{u}_5(\alpha) + \dots,$$

etc

Singular points

All the spectral sums $\mathcal{G}_-^{(s)}(\alpha)$ contain transcendental functions

$$\mathrm{u}_{2k-1}(\alpha) \stackrel{\text{def}}{=} \fint_{-\infty}^{\infty} dt \frac{\cosh^2 t}{t \sinh^{2k-1} t \cdot (\alpha \cosh t + t \sinh t)}.$$

and

$$\mathrm{v}(\alpha) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \frac{t^2}{\cosh^2 t} \frac{dt}{\alpha + t \tanh t}$$

There are two types of singularities of $\mathcal{G}_-^{(s)}(\alpha)$

1. α_k^*: pinching effect in $\mathrm{u}_{2k-1}(\alpha)$
2. $\tilde{\alpha}_k$: solutions of $\mathrm{v}(\alpha) = 1$

Singular points

Integrand in

$$\mathtt{u}_{2k-1}(\alpha) \stackrel{\text{def}}{=} \fint_{-\infty}^{\infty} dt \frac{\cosh^2 t}{t \sinh^{2k-1} t \cdot (\alpha \cosh t + t \sinh t)}.$$

has poles at $t = \pm i t_k^*$ which are solutions of the transcendental equation

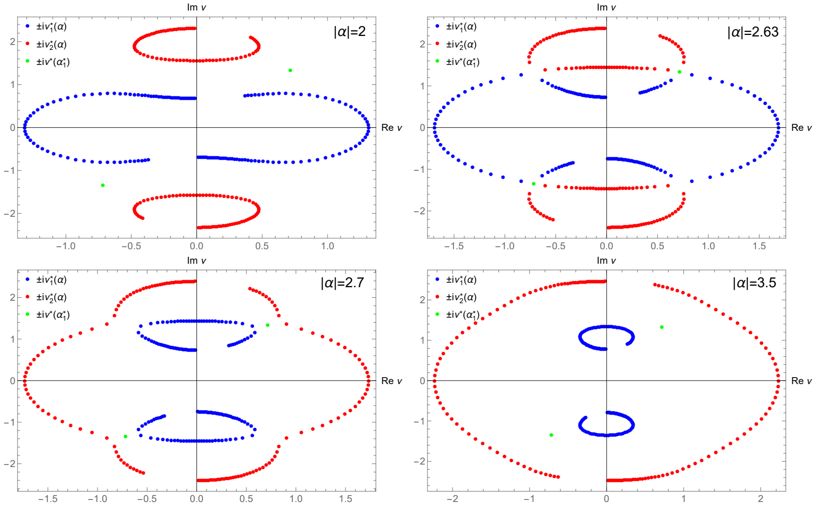
$$\alpha + t \tanh t = 0.$$

These $t_k^*(\alpha)$ are complicated functions of α. In particular, if we move α from the real line, $t_k^*(\alpha)$ start to travel in the complex plane. Consider the following paths parametrized by ϕ

$$\mathbb{R} \ni \alpha \quad \longmapsto \quad \alpha = e^{i\phi}|\alpha| \in \mathbb{C}.$$

Singular points

Evolution of $t_k^*(\alpha)$ in the complex plane



Singular points

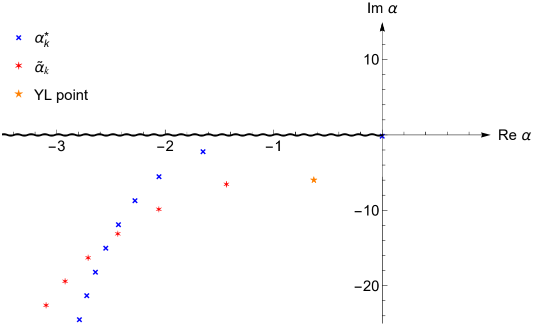


Figure: Critical points of λ on the second sheet of the α-plane correspond to the values α_k^* (marked in blue) and $\widetilde{\alpha}_k$ (marked in red). The point $\alpha = \alpha_0^* = 0$ is a square root branching point.

Did we really find new fixed points in IFT? (unclear)