Scale and scheme dependence in applications of QED parton distribution and fragmentation functions

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To-do list for QED

For modern and future experiments we need new higher-order calculations in QED

- Compute 2-loop QED radiative corrections to differential distributions of key processes: Bhabha scattering, muon decay, $e^+e^- \to \mu^+\mu^-$,$e^+e^- \to \pi^+\pi^-$, $e^+e^- \to ZH$ etc.
- Estimate higher-order contributions within some approximations
- Account for interplay with QCD and electroweak effects
- Match with parton showers
- Construct reliable Monte Carlo codes

Perturbative QED (I)

Fortunately, in our case the general perturbation theory can be applied:

$$\frac{\alpha}{2\pi} \approx 1.2 \cdot 10^{-3}, \quad \left(\frac{\alpha}{2\pi}\right)^2 \approx 1.4 \cdot 10^{-6}$$

Moreover, other effects: hadronic vacuum polarization, (electro)weak contributions, hadronic pair emission, etc. are small in, e.g., Bhabha scattering and can be treated one-by-one separately

Nevertheless, there are some enhancement factors:

1) First of all, the large logarithm $L \equiv \ln \frac{\Lambda^2}{m_e^2}$ where $\Lambda^2 \sim Q^2$ is the momentum transferred squared, e.g., $L(\Lambda = 1\,\mathrm{GeV}) \approx 16$ and $L(\Lambda = M_Z) \approx 24$.

2) The energy region at the Z boson peak ($s \sim M_Z^2$) requires a special treatment since factor M_Z/Γ_Z appears in the annihilation channel

Perturbative QED (II)

Methods of resummation of higher-order QED corrections

- Resummation of vacuum polarization corrections (geometric series): $\alpha(0) \to \alpha(\mu_F^2)$
- Yennie–Frautschi–Suura (YFS) soft photon exponentiation and its extensions, see, e.g., PHOTOS
- Leading logarithms via QED structure functions or QED PDFs (E.Kuraev and V.Fadin 1985;
A. De Rujula, R.Petronzio, A.Savoy-Navarro 1979)

N.B. Resummation of real photon radiation is good only for sufficiently inclusive observables . . .

Leading and next-to-leading logs in QED

The QED leading (LO) logarithmic corrections

$$\sim \left(\frac{\alpha}{2\pi}\right)^n \ln^n \frac{s}{m_e^2}$$

were relevant for LEP measurements of Bhabha, $e^+e^- \to \mu^+\mu^-$ etc.
for $n \leq 3$ since $\ln(M_Z^2/m_e^2) \approx 24$

NLO contributions

$$\sim \left(\frac{\alpha}{2\pi}\right)^n \ln^{n-1} \frac{s}{m_e^2}$$

with at least $n = 3, 4, 5$ are required for future e^+e^- colliders

In the collinear approximation we can get them within
the NLO QED structure function formalism

- F.A.Berends, W.L. van Neerven, G.J.Burgers, NPB'1988
- A.A., K.Melnikov, PRD'2002; A.A. JHEP'2003

QED NLO DGLAP evolution equations

$$\mathcal{D}_{ba}\left(x, \frac{\mu_R^2}{\mu_F^2}\right) = \delta_{ab}\delta(1-x) + \sum_{c=e,\gamma,\bar{e}} \int\limits_{\mu_R^2}^{\mu_F^2} \frac{dt}{t} \int\limits_{x}^{1} \frac{dy}{y} P_{bc}(y,t) \mathcal{D}_{ca}\left(\frac{x}{y}, \frac{\mu_R^2}{t}\right)$$

a, b, c are massless partons ($\sim e^{\pm}, \gamma$)

μ_F is a factorization (energy) scale

μ_R is a renormalization (energy) scale

D_{ba} is a parton density function (PDF)

P_{bc} is a splitting function or kernel of the DGLAP equation

N.B. In QED $\mu_R = m_e \approx 0$ is the natural choice well motivated by known analytic results

QED splitting functions

The perturbative splitting functions are

$$P_{ba}(x, \bar{\alpha}(t)) = \frac{\bar{\alpha}(t)}{2\pi} P^{(0)}_{ba}(x) + \left(\frac{\bar{\alpha}(t)}{2\pi}\right)^2 P^{(1)}_{ba}(x) + \mathcal{O}(\alpha^3)$$

$$\text{e.g.} \quad P^{(0)}_{ee}(x) = \left[\frac{1+x^2}{1-x}\right]_+$$

They come from loop calculations, e.g., $P^{(1)}_{ba}(x)$ comes from 2-loops

The splitting functions can be obtained by reduction of the ones known in QCD to the abelian case of QED

$\bar{\alpha}(t)$ is the QED running coupling constant in the $\overline{\text{MS}}$ scheme

N.B. Factorization in $\bar{\alpha}(t) \times P_{ba}(x)$ is not unique

Iterative solution

The NLO “electron in electron” PDF reads [A.A., U.Voznaya, JPG 2023]

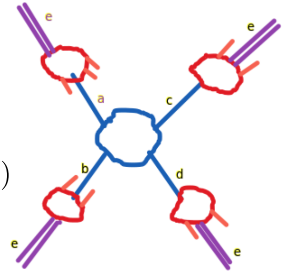
$$
\begin{aligned}
\mathcal{D}_{ee}(x,\mu_F,m_e) &= \delta(1-x) + \frac{\alpha}{2\pi} L P^{(0)}_{ee}(x) + \frac{\alpha}{2\pi} d^{(1)}_{ee}(x,m_e,m_e) \\
&+ \left(\frac{\alpha}{2\pi}\right)^2 L^2 \left(\frac{1}{2} P^{(0)}_{ee} \otimes P^{(0)}_{ee}(x) + \frac{1}{3} P^{(0)}_{ee}(x) + \frac{1}{2} P^{(0)}_{e\gamma} \otimes P^{(0)}_{\gamma e}(x) \right) \\
&+ \left(\frac{\alpha}{2\pi}\right)^2 L \left(P^{(0)}_{e\gamma} \otimes d^{(1)}_{\gamma e}(x,m_e,m_e) + P^{(0)}_{ee} \otimes d^{(1)}_{ee}(x,m_e,m_e) - \frac{10}{9} P^{(0)}_{ee}(x) + P^{(1)}_{ee}(x) \right) \\
&+ \left(\frac{\alpha}{2\pi}\right)^3 L^3 \left(\frac{1}{6} P^{(0)}_{ee} \otimes P^{(0)}_{ee} \otimes P^{(0)}_{ee}(x) + \frac{1}{6} P^{(0)}_{e\gamma} \otimes P^{(0)}_{\gamma\gamma} \otimes P^{(0)}_{\gamma e}(x) + \dots \right) \\
&+ \left(\frac{\alpha}{2\pi}\right)^3 L^2 \left(P^{(0)}_{ee} \otimes P^{(1)}_{ee}(x) + P^{(0)}_{ee} \otimes P^{(0)}_{ee} \otimes d^{(1)}_{ee}(x,m_e,m_e) + \frac{1}{3} P^{(1)}_{ee}(x) - \frac{10}{9} P^{(0)}_{ee} \otimes P^{(0)}_{ee}(x) + \dots \right) \\
&+ \mathcal{O}(\alpha^2 L^0, \alpha^3 L^1) + \dots
\end{aligned}
$$

The large logarithm $L \equiv \ln \frac{\mu_F^2}{\mu_R^2}$, $\alpha \equiv \alpha(\mu_R)$ and $\mu_R = m_e$.

QED NLO master formula

The NLO Bhabha cross section reads

$$d\sigma = \sum_{a,b,c,d=e,\bar{e},\gamma} \int_{\bar{z}_1}^{1} dz_1 \int_{\bar{z}_2}^{1} dz_2 \mathcal{D}^{\rm str}_{ae}(z_1)\mathcal{D}^{\rm str}_{b\bar{e}}(z_2)$$
$$\times \left[d\sigma^{(0)}_{ab\to cd}(z_1,z_2) + d\bar{\sigma}^{(1)}_{ab\to cd}(z_1,z_2) \right]$$
$$\times \int_{\bar{y}_1}^{1} \frac{dy_1}{Y_1} \int_{\bar{y}_2}^{1} \frac{dy_2}{Y_2} \mathcal{D}^{\rm frg}_{ec}\left(\frac{y_1}{Y_1}\right) \mathcal{D}^{\rm frg}_{\bar{e}d}\left(\frac{y_2}{Y_2}\right)$$
$$+\mathcal{O}\left(\alpha^n L^{n-2}, \frac{m_e^2}{s}\right)$$



$\mathcal{O}(\alpha)$ matching

The expansion of the master formula for ISR gives

$$d\sigma^{(1)}_{e\bar{e}\to\gamma^*} = \frac{\alpha}{2\pi}\left\{2LP^{(0)}_{ee}\otimes d\sigma^{(0)}_{e\bar{e}\to\gamma^*} + 2d^{(1)}_{ee}\otimes d\sigma^{(0)}_{e\bar{e}\to\gamma^*}\right\} + d\,\bar{\sigma}^{(1)}_{e\bar{e}\to\gamma^*}$$

We know the massive $d\sigma^{(1)}$ and massless $d\,\bar{\sigma}^{(1)}$ ($m_e \equiv 0$ with $\overline{\text{MS}}$ subtraction) results in $\mathcal{O}(\alpha)$ $\Rightarrow$

$$d^{(1)}_{ee} = \left[\frac{1+z^2}{1-z}\left(\ln\frac{\mu_R^2}{m_e^2} - 1 - \ln(1-z)\right)\right]_+, \quad P^{(0)}_{ee}(z) = \left[\frac{1+z^2}{1-z}\right]_+, \quad L = \ln\frac{\mu_F^2}{\mu_R^2}$$

Scheme dependence comes from here

Factorization and renormalization scale dependence is also from here

Factorization scale choice

The final result of calculation in all orders in α and L would not depend on μ_F

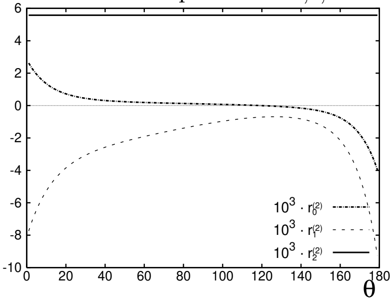
But for a fixed-order result for an observable does depend on μ_F

Many different methods for choosing μ_F were proposed:

- CSS — Conventional Scale Setting ($\mu_F =$ hard momentum transfer)
- FAC — Fastest Apparent Convergence [G. Grunberg]
- PMS — Principle of Minimal Sensitivity [R.J. Stevenson]
- BLM — Brodsky-Lepage-Mackenzie (absorb β_0-dependent terms)
- PMC — Principle of Maximal Conformality [S.Brodsky et al.]
- ...

Factorization scale choice — Bhabha scattering

Let's look at soft + virtual $\mathcal{O}\left(\alpha^2\right)$ RC [A. Penin, PRL'2005, NPB'2006]: $\Delta_{2-\text{loop}} = \sum_{n=0,1,2} C_n \ln^n \frac{\mu_F^2}{m_e^2} = \sum_{n=0,1,2} r_n$



Soft and virtual second order photonic relative radiative corrections in permil versus the scattering angle in degrees for $\Delta = 1$, $\sqrt{s}$=1 GeV; $\mu_F = \sqrt{s}$

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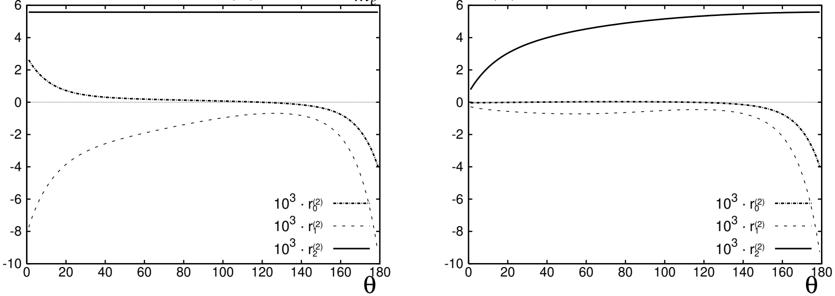
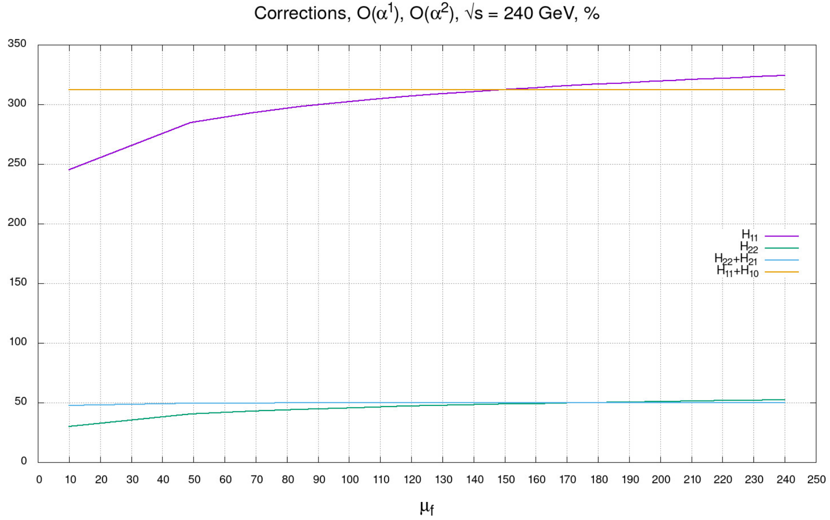


Soft and virtual second order photonic relative radiative corrections in permil versus the scattering angle in degrees for $\Delta = 1$, $\sqrt{s}$=1 GeV; $\mu_F = \sqrt{s}$ on the left side and $\mu_F = \sqrt{-t}$ on the right side.

Factorization scale choice – $e^+e^- \to \mu^+\mu^-$



Factorization scale — conclusions

The sensitivity to the factorization scale choice is relevant numerically

More higher-order calculations are required to reduce the dependency

The comparison of several concrete schemes shows:

- CSS — Conventional Scale Setting (μ_F = hard momentum transfer) fails
- FAC — Fastest Apparent Convergence looks good
- PMS — Principle of Minimal Sensitivity looks reasonable
- BLM — Brodsky-Lepage-Mackenzie not applicable
- PMC — Principle of Maximal Conformality not applicable

ISR corrections to $e^+e^- \to Z(\gamma^*)$ ($\sqrt{s} = M_Z$)

LO $\mathcal{O}(\alpha^n L^n)$ and NLO $\mathcal{O}(\alpha^n L^{n-1})$ ISR corr. [%] at Z-peak, $z_{\min} = 0.1$

Type / n	1	2	3	4	5
LO γ	−32.7365	4.8843	−0.3776	0.0034	0.0032
NLO γ	2.0017	−0.5952	0.0710	−0.0019	
LO pair	—	−0.3057	0.0875	0.0016	−0.0001
NLO pair	—	0.1585	−0.0460	0.0038	
Σ	−30.7348	4.1419	−0.2651	0.0069	0.0031

Even higher orders seem to be relevant numerically $\Longrightarrow$ exponentiation

Exponentiation of the leading logs is straightforward and known [Gribov-Lipatov, Kuraev-Fadin, . . .]

Factorization (subtraction) scheme choice

NLO exponentiation in the MSbar scheme is ambiguous:

explicit solution for $D_{ee}(x)$ in the $\overline{\mathrm{MS}}$ scheme in the limit $x \to 1$ doesn't match the (pure photonic) exact solution by Gribov and Lipatov '1972

$$\mathcal{D}_{ee}^{(\gamma)}(x)\Big|_{x\to 1} = \frac{\beta}{2}\,\frac{(1-x)^{\beta/2-1}}{\Gamma(1+\beta/2)}\exp\left\{\frac{\beta}{2}\left(\frac{3}{4}-C\right)\right\}$$

where $\beta = 2\alpha/\pi(L-1)$ and C is the Euler constant.

See also [A.V. Kotikov et al., "α_s from DIS data with large x resummation," arXiv:2403.13360]

We suggest a DIS-like scheme with the following modification of the NLO initial condition

$$d_{ee}^{(1)}\Big|_{\overline{\mathrm{MS}}} = \left[\frac{1+x^2}{1-x}\left(\ln\frac{\mu_R^2}{m_e^2} - 1 - \ln(1-x)\right)\right]_+ \to \tilde{d}_{ee}^{(1)} = \left[\frac{1+x^2}{1-x}\ln\frac{\mu_R^2}{m_e^2}\right]_+ = 0$$

for $m_R = m_e$ with subsequent modification of $\sigma^{(1)}$ to preserve the NLO matching. Fixed-order results for total cross-sections remain unchanged.

Scale variation test: $\mu_F \to \mu_F/2, \quad \mu_F \times 2$

True (Δ) shifts and the ones estimated (δ) by factorization scale variation by factor 2 in $\mathcal{O}(\alpha^2)$ for $\mu_F = \sqrt{s}$

	LO		NLO	
	Δ	δ	Δ	δ
$\sqrt{s} = M_z$, $z_{min} = 0.1$	0.436689	0.524911	0.003416	0.025032
$\sqrt{s} = M_z$, $z_{min} = 0.5$	0.4365967	0.5246878	0.0033886	0.0250268
$\sqrt{s} = M_z$, $z_{min} = 0.9$	0.440478	0.528603	0.0033499	0.025249
$\sqrt{s} = 240$ GeV, $z_{min} = 0.1$	2.468049	5.568990	0.697615	0.147786
$\sqrt{s} = 240$ GeV, $z_{min} = 0.5$	0.114240	0.105660	0.007085	0.006063
$\sqrt{s} = 240$ GeV, $z_{min} = 0.9$	0.072996	0.040264	0.002663	0.003874

$$\Delta^{\mathrm{LO}} = h_{21}, \qquad \Delta^{\mathrm{NLO}} = h_{20}$$

$$\delta^{\mathrm{LO}} = \frac{|h_{22} - h_{22}(1/2)| + |h_{22} - h_{22}(2)|}{2}$$

$$\delta^{\mathrm{NLO}} = \frac{|h_{22} + h_{21} - (h_{22} + h_{21})(1/2)| + |h_{22} + h_{21} - (h_{22} + h_{21})(2)|}{2}$$

Outlook

- Current and future high-precision HEP experiments challenge theory
- New calculations of two-loop and higher-order corrections within QED and full SM are required
- We have a progress in NLO QED PDFs and fragmentation functions
- QED provides explicit results and serves for cross checks of QCD
- Optimization of factorization scale and scheme choices is important as in QCD as well as in QED
- There is no perfect choice, compromises are inevitable