On three-loop divergences in $6D, \mathcal{N} = (1,1)$ SYM

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Aims:

- Construction of the background superfield method for $6D, \mathcal{N} = (1,0)$ interacting non-Abelian vector multiplet with hypermultiplet
- Analysis of divergences in the $\mathcal{N} = (1,1)$ SYM theory up to three loops

The talk is based on:

I.L. Buchbinder, E.A. Ivanov, B.M., arXiv:2025.xxxxx

I.L. Buchbinder, E.A. Ivanov, K.V. Stepanayantz, B.M., *JHEP 2305 (***2023***)*; *Phys.Lett.B 820 (***2021***)*; *Nucl.Phys.B 921 (***2017***)*; *JHEP 1701 (***2017***)*; *Phys.Lett.B 763 (***2016***)*.

Plan

- General motivations
- $\mathcal{N} = (1,1)$ SYM theory in terms of $\mathcal{N} = (1,0)$ harmonic supersfields
- Background field method
- Divergent part of one-loop effective action
- Two-loop results
- Three-loop analysis
- Summary

The modern interest to $6D$ supersymmetric gauge theories is stipulated by the following reasons:

- The problem of describing the quantum structure of six-dimensional supersymmetric gauge theories dimensionally reduced from superstrings and the connection of effective action for the D5-branes at low energies with maximally supersymmetric Yang-Mills theory in six dimensions. [**N.Seiberg (1996), E. Witten (1996); N. Seiberg, (1997)**].
- Lagrangian description of the interacting multiple $M5$-branes is related to $6D$, $\mathcal{N} = (2,0)$ supersymmetric gauge theory. The theory includes self-dual non-Abelian antisymmetric tensor and it is not constructed still (see e.g. review [**J. Bagger, N. Lambert, S. Mikhu, C. Papageorgakis (2013)**]).

Motivations

► The problem of miraculous cancellation of on-shell divergences in higher dimensional maximally supersymmetric gauge theories (theories with 16 supercharges). All these theories are non-renormalizable by power counting.

- Field limit of superstring amplitude shows that $6D, \mathcal{N} = (1,1)$ SYM theory is on-shell finite at one-loop [**M.B. Green, J.H. Schwarz, L. Brink, (1982)**].
- Analysis based on on-shell supersymmetries, gauge invariance and field redefinitions [**P.S. Howe, K.S. Stelle, (1984), (2003); G. Bossard, P.S. Howe, K.S. Stelle, (2009)**].
- Direct one-loop and two-loop component calculations (mainly in bosonic sector and mainly on-shell) [**E.S. Fradkin, A.A. Tseytlin, (1983); N. Marcus, A. Sagnotti, (1984), (1985)**].
- Direct calculations of scattering amplitudes in $6D$ theory up to five loops and in $D8, 10$ theories up to four loops [**L.V. Bork, D.I. Kazakov, M.V. Kompaniets, D.M. Tolkachev, D.E. Vlasenko, (2015)**.]

Results: On-shell divergences in $6D$ theory start at three loops.

Purpose

To study one-, two- and three-loop divergences of the superfield effective action in $\mathcal{N} = (1,1)$ SYM theory.

Properties

$6D, \mathcal{N} = (1,1)$ SYM theory possesses some properties close or analogous to $4D, \mathcal{N} = 4$ SYM theory.

- The $6D, \mathcal{N} = (1,1)$ SYM theory can be formulated in harmonic superspace as well as the $4D, \mathcal{N} = 4$ SYM theory.
- The $6D, \mathcal{N} = (1,1)$ SYM theory possesses the manifest $\mathcal{N} = (1,0)$ supersymmetry and additional hidden $\mathcal{N} = (0,1)$ supersymmetry analogous to $4D, \mathcal{N} = 4$ SYM theory where there is the manifest $\mathcal{N} = 2$ supersymmetry and additional hidden $\mathcal{N} = 2$ supersymmetry.
- The $6D, \mathcal{N} = (1,1)$ SYM theory is anomaly free as well as the $4D, \mathcal{N} = 4$ SYM theory and satisfies some non-renormaization theorems.

$4D$

A.Galperin, E. Ivanov, S. Kalitsyn, V. Ogievetsky, E. Sokatchev, (1984).

A.Galperin, E. Ivanov, V. Ogievetsky, E. Sokatchev, *Harmonic Superspace*, **(2001).**

General purpose:
to formulate $\mathcal{N} = 2$ models in terms of unconstrained $\mathcal{N} = 2$ superfields.
General idea:
to use the parameters $u^{\pm i} (i = 1, 2)$ (harmonics) related to $SU(2)$ automorphism group of the $\mathcal{N} = 2$ superalgebra and parameterizing the 2-sphere, $u^{+i} u_i^- = 1$.

It allows to introduce the $\mathcal{N} = 2$ superfields and formulate the theory with manifest $\mathcal{N} = 2$ supersymmetry in harmonic superspace. Price for this is a presence of extra bosonic variables, harmonics $u^{\pm i}$.

$6D$

P.S. Howe, K.S. Stelle, P.C. West, (1985).
B.M. Zupnik, (1986); (1999).
G. Bossard, E. Ivanov, A. Smilga, (2015).

Note! Pure spinor approach to describe $6D$ SYM theories, [**M. Cederwall, (2018)**].

- The classical action of $6D$, $\mathcal{N}=(1,1)$ SYM model $S_0[q^+, V^{++}]$ in the harmonic superspace is written as

$$S_0 = \frac{1}{f^2}\sum_{n=2}^{\infty}\frac{(-i)^n}{n}\mathrm{tr}\int d^{14}z\, du_1 \ldots du_n \frac{V^{++}(z,u_1)\ldots V^{++}(z,u_n)}{(u_1^+u_2^+)\ldots(u_n^+u_1^+)}$$
$$-\frac{1}{2f^2}\int d\zeta^{(-4)}\, q^{+A}\nabla^{++}q_A^+ .$$

- Gauge transformations

$$V^{++\prime} = -ie^{i\lambda}D^{++}e^{-i\lambda} + e^{i\lambda}V^{++}e^{-i\lambda}, \qquad q^{+\prime} = e^{i\lambda}q^+ \tag{1}$$

Addition supersymmetry

The $\mathcal{N}=(1,0)$ SYM theory is manifestly $\mathcal{N}=(1,0)$ supersymmetric and possesses the extra hidden $\mathcal{N}=(0,1)$ supersymmetry if the hypermultiplet align in adjoint representation.

- Action

$$S[V^{++},q^{+}]=S_{SYM}[V^{++}]+S_{HYP}[q^{+},V^{++}]$$

- The action is manifestly $\mathcal{N}=(1,0)$ supersymmetric.
- The action is invariant under the transformations of extra hidden $\mathcal{N}=(0,1)$ supersymmetry

$$\delta V^{++}=\epsilon^{+}q^{+},\qquad \delta q^{+}=-(D^{+})^{4}(\epsilon^{-}V^{--})$$

where the transformation parameter $\epsilon^{\pm}_{A}=\epsilon_{aA}\theta^{\pm A}$.

Aim:gauge invariant effective action, (see, e.g., [**B.DeWitt (1965)**]).

- Background-quantum splitting

$$V^{++} \to V^{++} + \mathrm{f}v^{++}, \qquad q^{+} \to Q^{+} + q^{+}$$

- The gauge-fixing function

$$\mathcal{F}^{(+4)} = \nabla^{++} v^{++}$$

- Faddev-Popov procedure
  Analogous to one in $4D, \mathcal{N} = 2$ SYM theory[**I.L.Buchbinder, E.I. Buchbinder, S.M. Kuzenko, B.A. Ovrut, (1998)**].

Background field method

- The effective action $\Gamma[V^{++}, Q^+]$ is written in terms of path integral

$$e^{i\Gamma[V^{++},Q^+]} = \mathrm{Det}^{1/2}\, \widehat{\Box} \int \mathcal{D}v^{++}\, \mathcal{D}q^+\, \mathcal{D}\mathbf{b}\, \mathcal{D}\mathbf{c}\, \mathcal{D}\varphi\; e^{iS_{quant}}$$

- The quantum action S_{quant} has the structure

$$S_{quant} = S + S_{GF}[v^{++}, V^{++}] + S_{FP}[\mathbf{b}, \mathbf{c}, v^{++}, V^{++}] + S_{NK}[\varphi, V^{++}].$$

- Gauge fixing action $S_{GF}[v^{++}, V^{++}]$, Faddeed-Popov ghost action $S_{FP}[\mathbf{b}, \mathbf{c}, v^{++}, V^{++}]$, Nelson-Kalosh ghost action $S_{NK}[\varphi, V^{++}]$
- Operator $\widehat{\Box} = \frac{1}{2}(D^+)^4(\nabla^{--})^2$

$$\widehat{\Box} = \eta^{MN}\nabla_M\nabla_N + W^{+a}\nabla_a^- + F^{++}\nabla^{--} - \frac{1}{2}(\nabla^{--}F^{++})$$

- All ghosts are the analytic superfields

- The gauge fixing action

$$S_{GF}[v^{++}, V^{++}] = -\frac{1}{2\xi_0}\mathrm{tr}\int d^{14}z du_1 du_2 \frac{v_\tau^{++}(1)v_\tau^{++}(2)}{(u_1^+ u_2^+)^2} + \frac{1}{4\xi_0}\mathrm{tr}\int d^{14}z du\, v_\tau^{++}(D^{--})^2 v_\tau^{++}. \qquad (2)$$

- Faddeed-Popov ghosts and Nelson-Kalosh ghost actions

$$S_{FP} = \mathrm{tr}\int d\zeta^{(-4)} du\, b\nabla^{++}(\nabla^{++} + iv^{++})c\,, \qquad (3)$$

$$S_{NK} = \frac{1}{2}\mathrm{tr}\int d\zeta^{(-4)} du\, \varphi(\nabla^{++})^2\varphi\,. \qquad (4)$$

In what follows we assume gauge fixing parameter $\xi_0 = 1$. The case $\xi_0 \neq 1$ can be considered separately [**I.L. Buchbinder, E.A. Ivanov, K.V. Stepanayantz, B.M., Phys.Lett.B (2019)**]

Supergraphs

- Perturbation theory can be given in terms of Feynman diagrams formulated in superspace
- Vector multiplet propagator

$$G^{(2,2)}(1|2) = -2\frac{(D_1^+)^4}{\widehat{\Box}_1}\delta^{14}(z_1 - z_2)\delta^{(-2,2)}(u_1, u_2)$$

- Hypermultiplet propagator

$$G^{(1,1)}(1|2) = \frac{(D_1^+)^4(D_2^+)^4}{\widehat{\Box}_1}\frac{\delta^{14}(z_1 - z_2)}{(u_1^+ u_2^+)^3}$$

- Ghost propagators have the analogous structure
- Superspace delta-function

$$\delta^{14}(z_1 - z_2) = \delta^6(x_1 - x_2)\delta^8(\theta_1 - \theta_2)$$

- The vertices are taken from the superfield action as usual

The one-loop approximation is given by the quadratic action S_2:

$$\begin{aligned} S_2 \quad = \quad & \frac{1}{2}\int d\zeta^{(-4)}du\, v^{++\,A}\,\widehat{\Box}^{AB}\, v^{++\,B} + \int d\zeta^{(-4)}du\,\mathbf{b}^A(\nabla^{++})^{2\,AB}\mathbf{c}^B \\ & +\frac{1}{2}\int d\zeta^{(-4)}du\,\varphi^A(\nabla^{++})^{2\,AB}\varphi^B - \int d\zeta^{(-4)}du\,\tilde{q}^{+\,m}(\nabla^{++})_m{}^n q^+_n \\ & -i\mathrm{f}\int d\zeta^{(-4)}du\Big\{\widetilde{Q}^{+\,m}(v^{++})^C(T^C)_m{}^n q^+_n + \tilde{q}^{+\,m}(v^{++})^C(T^C)_m{}^n Q^+_n\Big\}, \end{aligned} \tag{5}$$

We consider the special change of hypermultiplet variables [**I.L. Buchbinder, N.G. Pletnev,** *JHEP* 0704; **S. M. Kuzenko, S. J. Tyler,** *JHEP* 0705] in the one-loop effective action

$$q^+_n(1) \to q^+_n(1) - \mathrm{f}\int d\zeta_2^{(-4)}du_2\,G^{(1,1)}(1|2)_n{}^p\,iv^{++\,C}(2)\,(T^C)_p{}^l\,Q^+_l(2)\,, \tag{6}$$

where $G^{(1,1)}$ is the hypermultiplet Green function.

Structure of one-loop counterterm

According to the general analysis[**G. Bossard, E. Ivanov, A. Smilga,(2015)**] the one-loop logarithmic divergences have a structure

$$\Gamma^{(1)}_{\rm div} = \int d\zeta^{(-4)}\, du \left[c_1 (F^{++A})^2 + ic_2 F^{++A} (\widetilde{q}^{+})^m (T^A)_m{}^n (q^+)_n + c_3 \Big((\widetilde{q}^{+})^m (q^+)_m \Big)^2 \right], \tag{7}$$

where c_1, c_2, and c_3 are numerical real coefficients.

Superficial degree of divergence ω

- One can prove that any supergraph for effective action can be written through the integrals over full $\mathcal{N}=(1,0)$ superspace and contains only a single integral over $d^8\theta$ (non-renormalization theorem).
- $\omega_{\rm L-loop}(G) = 2L - N_Q - \frac{1}{2} N_D$
- The possible one-loop divergences correspond to $\omega_{\rm 1-loop} = 2$ and $\omega_{\rm 1-loop} = 0$

Calculations of ω are analogous to ones in $4D, \mathcal{N}=2$ gauge theory [**I.L. Buchbinder, S.M. Kuzenko, B.A. Ovrut, (1998)**].

The one-loop quantum correction $\Gamma^{(1)}[V^{++}, Q^+]$ to the classical action, which has the following formal expression

$$\Gamma^{(1)}[V^{++}, Q] = \frac{i}{2}\mathsf{Tr}\ln\Big\{\, \widehat{\Box}^{AB} - 4\mathrm{f}^2\widetilde{Q}^{+\,m}\big(T^A G T^B\big)_m{}^n Q^+_n \Big\} - \frac{i}{2}\mathsf{Tr}\ln\widehat{\Box}_{\mathrm{Adj}}$$
$$-i\mathsf{Tr}\ln(\nabla^{++})^2_{\mathrm{Adj}} + \frac{i}{2}\mathsf{Tr}\ln(\nabla^{++})^2_{\mathrm{Adj}} + i\mathsf{Tr}\ln\nabla^{++}_{\mathrm{R}}\,. \qquad (8)$$

The divergent contributions read

$$\Gamma^{(1)}_{F^2} = \frac{C_2 - T(R)}{6(4\pi)^3\varepsilon}\int d\zeta^{(-4)} du\,(F^{++A})^2\,. \qquad (9)$$

$$\Gamma^{(1)}_{QFQ}[V^{++}, Q^+] = -\frac{2i\mathrm{f}^2}{(4\pi)^3\varepsilon}\int d\zeta^{(-4)} du$$
$$\widetilde{Q}^{+\,m}(C_2\delta^l_m - C(R)_m{}^l)(F^{++})^A\ (T^A)_l{}^n\,Q^+_n. \qquad (10)$$

Two-loop loop divergences

In general, the off-shell two-loop divergent contributions to effective action may include following terms

$$\Gamma^{(2)}_{\rm div} = {\rm tr} \int d\zeta^{(-4)} \Big(c_1 F^{++} \,\widehat{\Box}\, F^{++} + c_2 i\, F^{++} \,\widehat{\Box}\, [Q^{+A}, Q^{+}_{A}]$$

$$+ c_3\, [Q^{+A}, Q^{+}_{A}] \,\widehat{\Box}\, [Q^{+B}, Q^{+}_{B}] \Big) + \text{terms vanishing on eq.o.m. for } Q^{+}. \quad (11)$$

Inexplicit $\mathcal{N} = (0,1)$ supersymmetry restricts the structure of the divergent contribution (11) and leads to the absence of two-loop divergences on-shell. Hence we obtain non-trivial constraints on the coefficients:

$$2c_2 + 4c_3 = c_1 \,.$$

In case $Q = 0$, the calculation was provided in [**I.L. Buchbinder, E.A. Ivanov, K.V. Stepanayantz, B.M., (2021)**]

$$c_1 = \frac{\mathrm{f}^2 (C_2)^2}{8(2\pi)^6 \varepsilon^2} \,, \qquad \varepsilon \to 0 \,, \quad (12)$$

Vertices

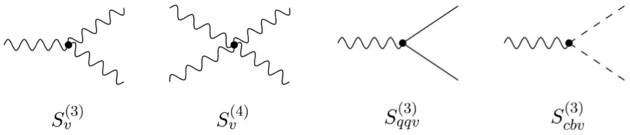

Figure: Standard vertices

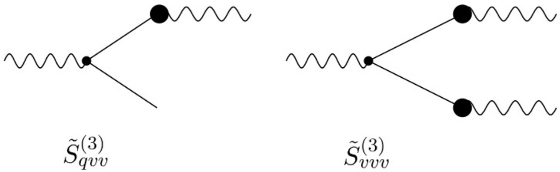

Figure: New type vertices

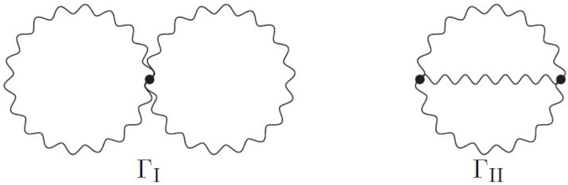

Figure: Two-loop Feynman supergraphs with the gauge self-interactions vertices.

Figure: Two-loop Feynman supergraphs with the hypermultiplet and ghosts vertices.

Two-loop diagrams

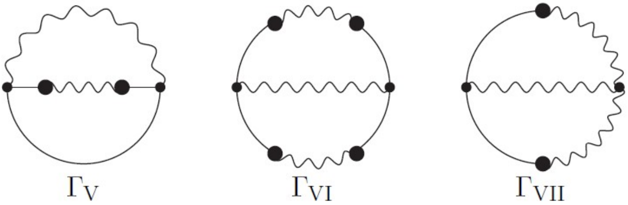

Figure: Two-loop Feynman supergraphs with the new 'non-local' vertices.

Two-loop calculation

Summing up the divergent contributions which contain $F^{++}Q^+Q^+$ from Γ_{I}, Γ_{V} and Γ_{VII} we obtain

$$\Gamma^{\mathrm{FQQ}}_{\mathrm{I,\,div}} + \Gamma^{\mathrm{FQQ}}_{\mathrm{V,\,div}} + \Gamma^{\mathrm{FQQ}}_{\mathrm{VII,\,div}} \tag{13}$$

$$= -\frac{i\mathrm{f}^2(C_2)^2}{8(2\pi)^6\,\varepsilon^2}\mathrm{tr}\int d^{14}z\,\frac{du_1du_2du_3}{(u^+_1u^+_2)(u^+_3u^+_1)}F^{++}_{1,\tau}[Q^{+A}_{2,\tau},Q^+_{3A,\tau}]$$
$$+\frac{2i\mathrm{f}^2(C_2)^2}{(4\pi)^6\varepsilon^2}\mathrm{tr}\int d^{14}z\frac{du_1du_2}{(u^+_1u^+_2)^2}F^{++}_{1,\tau}[Q^{+A}_{2,\tau},Q^+_{2A,\tau}]$$
$$-\frac{4i\mathrm{f}^2(C_2)^2}{(4\pi)^6\varepsilon^2}\mathrm{tr}\int d^{14}z\frac{du_1du_2du_3(u^-_1u^+_2)}{(u^+_1u^+_2)(u^+_2u^+_3)}F^{++}_{1,\tau}[Q^{+A}_{2,\tau},Q^+_{3A,\tau}] \tag{14}$$

Here we also use the property

$$G^{(2,2)}(z_1,u_1|z_2,u_2)\Big|_{z_2\to z_1} = \frac{2i}{(4\pi)^3}\,\frac{1}{\varepsilon}\Big(F^{++}_{1,\tau}\,(u^-_1u^+_2)(u^+_1u^+_2)^3(D^{--}_1)^2\delta^{(2,-2)}(u_1,u_2)$$
$$+2(u^+_1u^+_2)\,Q^{+A}_{1,\tau}Q^+_{2A,\tau}\Big) + \text{finite terms}\,,\quad \varepsilon\to 0\,. \tag{15}$$

Two-loop calculation

Final expression for the divergent part of the two-loop effective action in the $\mathcal{N}=(1,1)$ SYM theory reads

$$\Gamma^{(2)}_{\rm div} = \frac{\mathrm{f}^2(C_2)^2}{8(2\pi)^6\varepsilon^2}\mathrm{tr}\int d\zeta^{(-4)}\Big(F^{++}\,\widehat{\Box}\,F^{++} - \frac{i}{2}\,F^{++}\,\widehat{\Box}\,[Q^{+A},Q^{+}_{A}] + \frac{1}{2}\,[Q^{+A},Q^{+}_{A}]\,\widehat{\Box}\,[Q^{+B},Q^{+}_{B}]\Big) + \text{terms proportional to eq.o.m. for } Q^{+}.$$

The last two coefficient have the form

$$c_2 = -\frac{\mathrm{f}^2(C_2)^2}{16(2\pi)^6\varepsilon^2}\,, \qquad c_3 = \frac{\mathrm{f}^2(C_2)^2}{16(2\pi)^6\varepsilon^2}\,, \qquad \varepsilon\to 0\,. \tag{16}$$

For one-and two-loop divergences we have

$$\Gamma^{(1)}_{\rm div} \sim \frac{1}{\varepsilon}{\rm tr} \int d\zeta^{(-4)} F^{++} F^{++} + \text{terms with}\, q^{+}$$

$$\Gamma^{(2)}_{\rm div} \sim \frac{1}{\varepsilon^2}{\rm tr} \int d\zeta^{(-4)} F^{++} \,\widehat{\Box}\, F^{++} + \text{terms with}\, q^{+}$$

$$+\frac{1}{\varepsilon}{\rm tr} \int d\zeta^{(-4)} F^{++} g^{++}\,, \qquad (17)$$

where $g^{++} = G^{(2,2)}(z,u|z,u)$. All such contributions vanish **on-shell**!

As we mentioned above, the UV divergences in $\mathcal{N}=(1,1)$ starts from three loops [**L.V. Bork, D.I. Kazakov, M.V. Kompaniets, D.M. Tolkachev, D.E. Vlasenko, (2015)**.].
We will consider the simplest background superfields

$$F^{++}=0 \qquad q^{+}=0\,. \tag{18}$$

The only possible non-vanishing on-shell divergent contribution is

$$\Gamma^{(3)}_{\rm div}\sim\frac{1}{\varepsilon}{\rm tr}\int d^6x(F^4+\dots), \qquad \varepsilon\to+0\,. \tag{19}$$

Here we have used the notation

$$F^4\equiv\frac{1}{4}(F_{MN}F^{MN})^2+\frac{1}{8}F_{MN}F_{PQ}F^{MN}F^{PQ}-\frac{1}{2}F^{NM}F_{MR}F^{RS}F_{SN}-F^{NM}F_{MR}F^{RS}F_{SN}\,,$$

with F_{MN} being the gauge field strength.
The superfield form of the contribution (19) ia also known [**G. Bossard, E. Ivanov, A. Smilga,**(2015)] and has the form

$$\Gamma^{(3)}_{\rm div}\sim\frac{1}{\varepsilon}{\rm tr}\int d\zeta^{(-4)}(W^{+})^4 \qquad \varepsilon\to+0\,. \tag{21}$$

The superfield $W^{+a}=-\frac{i}{6}\varepsilon^{abcd}D^{+}_bD^{+}_cD^{+}_dV^{--}$ is analytic under the condition $F^{++}=0$.

Three-loop diagrams

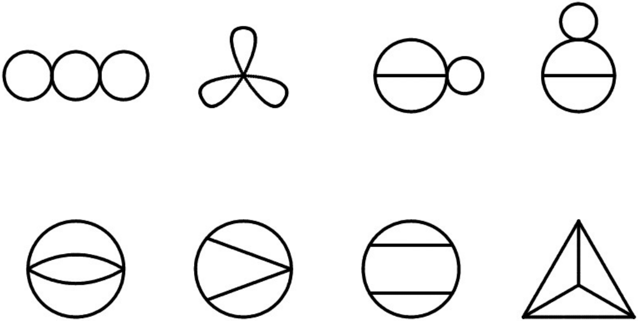

Figure: Topology of three-loop diagrams

Summary

- Background field method in $\mathcal{N}=(1,0)$ harmonic superspace was developed.
- Superficial degree of divergence was evaluated and structure of one and two-loop counterterms were studied.
- The one-loop divergences in the $6D, \mathcal{N}=(1,0)$ SYM theory were calculated *off-shell.*
- Two-loop divergences in the $6D, \mathcal{N}=(1,1)$ SYM theory are calculated.
- Three-loop analysis is provided.

Thank you for your attention!