Indecomposable multiplets in supersymmetric mechanics

Stepan Sidorov (BLTP JINR, Dubna)

Based on:
E. Ivanov, S. S., *Phys. Rev. D* **93** (2016) 065052, `arXiv:1509.05561 [hep-th]`,
S. S., *J. Phys. A* **58** (2025) 025210, `arXiv:2410.11618 [hep-th]`,
S. Fedoruk, E. Ivanov, S. S., *in progress.*

Contents

There are several different names for not fully reducible multiplets in supersymmetric mechanics, such as “long”, “non-minimal”, “tuned”, “indecomposable”. I will use the term “indecomposable”.

- $\mathcal{N} = 2$ supermultiplets
- $\mathcal{N} = 4$ supermultiplets
- $\mathcal{N} = 8$ supermultiplets

$\mathcal{N}=2$ supersymmetric mechanics

Supersymmetric quantum mechanics has been continuously developed since the supersymmetry breaking mechanism was introduced by Edward Witten (E. Witten, *Nucl. Phys. B* **188** (1981) 513; *Nucl. Phys. B* **202** (1982) 253).

However, Hermann Nicolai was the first (H. Nicolai, *J. Phys. A* **9** (1976) 1497) to introduce the simplest $\mathcal{N}=2$, $d=1$ algebra of supersymmetry as

$$\{Q,\bar{Q}\}=2H, \qquad [H,Q]=0, \qquad [H,\bar{Q}]=0. \tag{1}$$

where the central charge generator H was identified with the Hamiltonian. He treated supersymmetric mechanics as the simplest supersymmetric $d=1$ Lagrangian field theory in the framework of superfield approach. The time coordinate t is extended by additional Grassmann coordinates θ and $\bar{\theta}$. The superspace coordinates transform as

$$\delta\theta=\epsilon, \qquad \delta\bar{\theta}=\bar{\epsilon}, \qquad \delta t=i\left(\epsilon\,\bar{\theta}+\bar{\epsilon}\,\theta\right). \tag{2}$$

The $\mathcal{N}=2$ covariant derivatives D, $\bar{D}$ are defined by

$$D=\frac{\partial}{\partial\theta}-i\,\bar{\theta}\,\partial_t\,, \qquad \bar{D}=-\frac{\partial}{\partial\bar{\theta}}+i\,\theta\,\partial_t\,. \tag{3}$$

Irreducible $\mathcal{N}=2$, $d=1$ supermultiplets

Irreducible $d=1$ multiplets of the ranks $\mathcal{N}=2,4,8$ are conveniently denoted as $(\mathbf{k},\mathcal{N},\mathcal{N}-\mathbf{k})$, where $\mathbf{k}$ takes the values from $\mathbf{0}$ to $\mathcal{N}$ and stands for the number of physical bosonic fields, the second number $\mathcal{N}$ is the number of fermionic fields and $\mathcal{N}-\mathbf{k}$ corresponds to the number of auxiliary bosonic fields (A. Pashnev, F. Toppan, *J. Math. Phys.* **42** (2001) 5257-5271).

- The multiplet $(\mathbf{1},\mathbf{2},\mathbf{1})$ is described by an unconstrained real superfield X:

$$X\left(t,\theta,\bar\theta\right)=x\,(t)+\theta\,\eta\,(t)-\bar\theta\,\bar\eta\,(t)+\theta\bar\theta\,A\,(t)\,. \tag{4}$$

- The multiplet $(\mathbf{2},\mathbf{2},\mathbf{0})$ is described by a chiral superfield Z:

$$\bar D Z=0,\qquad \begin{aligned} &Z\,(t_{\rm L},\theta)=z\,(t_{\rm L})+\sqrt{2}\,\theta\,\xi\,(t_{\rm L})\,,\\ &Z\left(t,\theta,\bar\theta\right)=z\,(t)+\sqrt{2}\,\theta\,\xi\,(t)-i\,\theta\bar\theta\,\dot z\,(t)\,. \end{aligned} \tag{5}$$

- The multiplet $(\mathbf{0},\mathbf{2},\mathbf{2})$ is described by a fermionic chiral superfield Ψ:

$$\bar D\Psi=0,\qquad \begin{aligned} &\Psi\,(t_{\rm L},\theta)=\psi\,(t_{\rm L})+\sqrt{2}\,\theta\,B\,(t_{\rm L})\,,\\ &\Psi\left(t,\theta,\bar\theta\right)=\psi\,(t)+\sqrt{2}\,\theta\,B\,(t)-i\,\theta\bar\theta\,\dot\psi\,(t)\,. \end{aligned} \tag{6}$$

Unconstrained complex superfield

Let us consider an unconstrained complex superfield that has $4+4$ components:

$$Z = z + \sqrt{2}\,\theta\,\xi + \sqrt{2}\,\bar\theta\,\chi + \theta\bar\theta\,C. \tag{7}$$

We define a new fermionic superfield as

$$\Psi = \frac{1}{\sqrt{2}\,\kappa}\,\bar D Z \qquad \Rightarrow \qquad \bar D Z = \sqrt{2}\,\kappa\,\Psi, \qquad \bar D\Psi = 0, \tag{8}$$

where κ is a real parameter. The fermionic superfield is chiral, so it describes the multiplet $(\mathbf{0},\mathbf{2},\mathbf{2})$:

$$\Psi = \psi + \sqrt{2}\,\theta B - i\,\theta\bar\theta\,\dot\psi, \qquad \chi = -\,\kappa\,\psi, \quad C = 2\kappa B - i\dot z. \tag{9}$$

Then the complex superfield is represented as

$$Z = z + \sqrt{2}\,\theta\,\xi - i\,\theta\bar\theta\,\dot z - \sqrt{2}\,\kappa\,\bar\theta\,\Psi. \tag{10}$$

Indecomposable $\mathcal{N} = 2$ multiplet

The condition (8) identifies an irreducible subrepresentation of Z with the chiral superfield Ψ. The the components z and ξ transform through ψ and B:

$$\delta\psi = -\sqrt{2}\,\epsilon\, B, \qquad \delta B = \sqrt{2}\, i\,\bar{\epsilon}\,\dot{\psi}, \qquad \delta z = -\sqrt{2}\,\epsilon\,\xi + \underline{\sqrt{2}\,\kappa\,\bar{\epsilon}\,\psi}, \qquad \delta\xi = \sqrt{2}\, i\,\bar{\epsilon}\,\dot{z} - \underline{\sqrt{2}\,\kappa\,\bar{\epsilon}\, B}. \tag{11}$$

The explicit dependence on the underlined terms cannot be removed by any field redefinition. Both superfields Z and Ψ form one reducible but indecomposable representation of supersymmetry. They become independent chiral superfields in the limit $\kappa \to 0$, *i.e.* the multiplet is fully reducible: $(\mathbf{2}, \mathbf{2}, \mathbf{0}) \oplus (\mathbf{0}, \mathbf{2}, \mathbf{2})$.

Hidden $\mathcal{N}=2$ supersymmetry

In fact, we can define a hidden $\mathcal{N}=2$ supersymmetry $\tilde{\epsilon}$-transformations that commute with the manifest $\mathcal{N}=2$ transformations (11):

$$\delta\xi=\sqrt{2}\,\tilde{\epsilon}\,B,\qquad \delta B=-\sqrt{2}\,i\,\bar{\tilde{\epsilon}}\,\dot{\xi},\qquad \delta z=-\sqrt{2}\,\tilde{\epsilon}\,\psi-\sqrt{2}\,\kappa\,\bar{\tilde{\epsilon}}\,\xi,\qquad \delta\psi=\sqrt{2}\,i\,\bar{\tilde{\epsilon}}\,\dot{z}-\sqrt{2}\,\kappa\,\tilde{\epsilon}\,B. \tag{12}$$

Both supersymmetries form the $\mathcal{N}=4$ supersymmetry $\{Q^i,\bar{Q}_j\}=2\,\delta^i_j H$. The indecomposable multiplet under the $\mathcal{N}=4$ supersymmetry becomes the irreducible chiral multiplet $(\mathbf{2},\mathbf{4},\mathbf{2})$:

$$\left(\bar{D}-\kappa\,\tilde{D}\right)Z=0,\qquad \left(\bar{\tilde{D}}+\kappa\,D\right)Z=0\qquad\Rightarrow\qquad \frac{1}{\sqrt{1+\kappa^2}}\left(\bar{D}_j-\kappa\,D_j\right)Z=\bar{\mathbf{D}}_j Z=0. \tag{13}$$

Redefining infinitesimal parameters and components as

$$\begin{aligned}
\eta_1 &:= \frac{1}{\sqrt{1+\kappa^2}}\left(\epsilon+\kappa\,\bar{\tilde{\epsilon}}\right), \qquad \chi^1 := \sqrt{1+\kappa^2}\,\xi,\\
\eta_2 &:= \frac{1}{\sqrt{1+\kappa^2}}\left(\tilde{\epsilon}-\kappa\,\bar{\epsilon}\right), \qquad \chi^2 := \sqrt{1+\kappa^2}\,\psi, \qquad \mathcal{B}:=\left(1+\kappa^2\right)B-i\kappa\,\dot{z},
\end{aligned} \tag{14}$$

we find the $\mathcal{N}=4$ supersymmetry transformations in the familiar form:

$$\delta z=-\sqrt{2}\,\eta_i\,\chi^i,\qquad \delta\chi^i=\sqrt{2}\,i\,\bar{\eta}^i\,\dot{z}-\sqrt{2}\,\eta^i\mathcal{B},\qquad \delta\mathcal{B}=-\sqrt{2}\,i\,\bar{\eta}_i\,\dot{\chi}^i. \tag{15}$$

Non-linear multiplet

Another example comes from a non-linear modification of the chiral condition [1]:

$$\bar{D}Z = \kappa\, Z\, \Psi, \qquad \bar{D}\Psi = 0. \tag{16}$$

Both superfields are unified into the non-linear $\mathcal{N}=4$ chiral superfield (E. Ivanov, S. Krivonos, F. Toppan, *Phys. Lett. B* **405** (1997) 85–94, S. Bellucci, A. Beylin, S. Krivonos, A. Nersessian, E. Orazi, *Phys. Lett. B* **616** (2005) 228-232) that satisfies

$$\bar{D}_j Z = \kappa\, Z D_j Z. \tag{17}$$

[1] Recently it was studied by D. Bykov, V. Krivorol, A. Kuzovchikov, arXiv:2412.21024 [hep-th].

Indecomposable multiplet with the twisted Grassmann parity

By swapping the roles of bosonic and fermionic superfields, we can consider another indecomposable multiplet:

$$\bar{D}\Psi = \sqrt{2}\, m\, Z, \qquad \bar{D}Z = 0, \qquad [m] = t^{-1}. \tag{18}$$

The components transform as

$$\delta\psi = -\sqrt{2}\,\epsilon\, B + \underline{\sqrt{2}\, m\,\bar{\epsilon}\, z}, \qquad \delta B = \sqrt{2}\, i\,\bar{\epsilon}\,\dot{\psi} - \underline{\sqrt{2}\, m\,\bar{\epsilon}\,\xi}, \qquad \delta z = -\sqrt{2}\,\epsilon\,\xi, \qquad \delta\xi = \sqrt{2}\, i\,\bar{\epsilon}\,\dot{z}. \tag{19}$$

So-called "$\mathcal{N}=2$ long multiplet" was obtained from $\mathrm{SU}(2|1)$ irreducible chiral multiplets (B. Assel, D. Cassani, L. Di Pietro, Z. Komargodski, J. Lorenzen, D. Martelli, *JHEP* **07** (2015) 043). The centrally-extended $su(2|1)$ superalgebra ("Weak supersymmetry algebra" A. V. Smilga, *Phys. Lett. B* **585** (2004) 173–179) can be treated as a deformation of the $\mathcal{N}=4$ superalgebra:

$$\{Q^i, \bar{Q}_j\} = 2m\left(I^i_j - \delta^i_j F\right) + 2\,\delta^i_j H. \tag{20}$$

Chiral $\mathrm{SU}(2|1)$ superfields can carry non-zero external spins s and charge λ with respect to the $\mathrm{U}(2)$ subgroup of $\mathrm{SU}(2|1)$ (E. Ivanov, S. S., *Phys. Rev. D* **93** (2016) 065052):

$$\bar{D}_j \Phi^{\lambda}_{(i_1\,\ldots\,i_{2s})} = 0, \qquad s = 1/2, 1, 3/2\,\ldots. \tag{21}$$

The number of components in Φ carrying non-zero external spins increases by a factor of $2s+1$.

Indecomposable $\mathcal{N}=4$ supermultiplets

- A classification of $\mathcal{N}=4$, $d=1$ reducible ("non-minimal") multiplets consisting of 8 bosonic and 8 fermionic fields was given at the component level in M. Gonzales, S. Khodaee, F. Toppan, *J. Math. Phys.* **52** (2011) 013514 and M. Gonzales, K. Iga, S. Khodaee, F. Toppan, *J. Math. Phys.* **53** (2012) 103513.

- We gave a superfield description for some examples (E. Ivanov, S. S., *Phys. Rev. D* **93** (2016) 065052, E. Ivanov, A. Rivasplata Mendoza, S. S., *J. Phys. Conf. Ser.* **804** (2017) 012021).

- Below I will consider a reducible $\mathcal{N}=4$, $d=1$ multiplet described by an unconstrained real superfield as a coupling of the mirror $(\mathbf{1},\mathbf{4},\mathbf{3})$ and ordinary $(\mathbf{3},\mathbf{4},\mathbf{1})$ multiplets (S. S., *J. Phys. A* **58** (2025) 025210). Employing this multiplet, I will construct a coupled system of dynamical and semi-dynamical multiplets. I will show that the corresponding *on-shell* model reproduces the model constructed by S. Fedoruk, E. Ivanov, O. Lechtenfeld, *JHEP* **06** (2012) 147 in the framework of the $\mathcal{N}=4$, $d=1$ harmonic superspace.

Basics of $\mathcal{N}=4$ supersymmetric mechanics

The standard $\mathcal{N}=4$, $d=1$ superalgebra:

$$\left\{Q^i_\alpha, Q^\beta_j\right\} = 2\,\delta^i_j\,H. \tag{22}$$

The supercharges Q^i_α carry fundamental indices ($i=1,2$ and $\alpha=1,2$) of the automorphism group $\mathrm{SO}(4)\sim\mathrm{SU}(2)_{\mathrm{L}}\times\mathrm{SU}(2)_{\mathrm{R}}$. The superspace is

$$\zeta := \left\{t, \theta^{i\alpha}\right\}, \qquad \overline{(\theta^{i\alpha})} = -\,\theta_{i\alpha}\,, \tag{23}$$

and transforms as

$$\delta\theta^{i\alpha} = \epsilon^{i\alpha}, \qquad \delta t = -\,i\,\epsilon^{i\alpha}\theta_{i\alpha}\,, \qquad \overline{(\epsilon^{i\alpha})} = -\,\epsilon_{i\alpha}\,. \tag{24}$$

The covariant derivatives are

$$D^{i\alpha} = \frac{\partial}{\partial\theta_{i\alpha}} + i\,\theta^{i\alpha}\partial_t\,. \tag{25}$$

The superalgebra can be rewritten in more familiar notation as

$$\left\{Q^i, \bar{Q}_j\right\} = 2\,\delta^i_j\,H, \qquad Q^i := Q^{i1}, \quad \bar{Q}_j := -\,Q_{j1}\,. \tag{26}$$

Irreducible $\mathcal{N} = 4$ multiplets

An arbitrary unconstrained real superfield contains 8 bosonic and 8 fermionic component fields in its general θ-expansion. The mirror multiplet $(\mathbf{1}, \mathbf{4}, \mathbf{3})$ is described by a real superfield X satisfying

$$D^{(i}_{\alpha} D^{j)\alpha} X = 0. \tag{27}$$

The constraint kills the half of component fields, so the field content is reduced to 4 bosonic and 4 fermionic fields. The multiplet $(\mathbf{3}, \mathbf{4}, \mathbf{1})$ is described by a triplet superfield V^{ij} that satisfies

$$D^{(i}_{\alpha} V^{jk)} = 0, \qquad \overline{(V^{ij})} = V_{ij}\,, \qquad V^{ij} = V^{ji}. \tag{28}$$

Their θ-expansions are given by

$$\begin{aligned}
X &= x - \theta_{i\alpha}\psi^{i\alpha} + \frac{1}{2}\,\theta_{i(\alpha}\theta^{i}_{\beta)}A^{\alpha\beta} + \frac{i}{3}\,\theta^{\beta}_{i}\theta_{j\beta}\theta^{i}_{\alpha}\dot{\psi}^{j\alpha} - \frac{1}{12}\,\theta^{i}_{\alpha}\theta^{j\alpha}\theta_{i\beta}\theta^{\beta}_{j}\,\ddot{x},\\
&\quad \overline{(x)} = x, \qquad \overline{(\psi^{i\alpha})} = \psi_{i\alpha}\,, \qquad \overline{(A^{\alpha\beta})} = -\,A_{\alpha\beta}\,.\\
V^{ij} &= v^{ij} - i\,\theta^{(i}_{\alpha}\chi^{j)\alpha} - \frac{i}{2}\,\theta^{i}_{\alpha}\theta^{j\alpha}C + i\,\theta^{\alpha}_{k}\theta^{(i}_{\alpha}\dot{v}^{j)k} - \frac{1}{3}\,\theta^{k\alpha}\theta^{\beta}_{k}\theta^{(i}_{\alpha}\dot{\chi}^{j)}_{\beta} + \frac{1}{12}\,\theta^{k}_{\alpha}\theta^{l\alpha}\theta_{k\beta}\theta^{\beta}_{l}\,\ddot{v}^{ij},\\
&\quad \overline{(v^{ij})} = v_{ij}\,, \qquad v^2 = \frac{1}{2}\,v^{ij}v_{ij}\,, \qquad \overline{(\chi^{i\alpha})} = -\,\chi_{i\alpha}\,, \qquad \overline{(C)} = C.
\end{aligned} \tag{29}$$

Indecomposable $\mathcal{N}=4$ multiplet

We couple the mirror multiplet $(\mathbf{1},\mathbf{4},\mathbf{3})$ to the multiplet $(\mathbf{3},\mathbf{4},\mathbf{1})$ by modifying the quadratic constraint as

$$\boxed{D^{(i}_{\alpha}D^{j)\alpha}X_{\kappa}=4i\kappa\,V^{ij}\,,\qquad D^{(i}_{\alpha}V^{jk)}=0.} \tag{30}$$

The new superfield X_{κ} is a deformation of X as

$$X_{\kappa}=X+i\kappa\,\theta^{\beta}_{(i}\theta_{j)\beta}\left(v^{ij}-\frac{2i}{3}\,\theta^{i}_{\alpha}\chi^{j\alpha}-\frac{i}{6}\,\theta^{i}_{\alpha}\theta^{j\alpha}C\right). \tag{31}$$

Here the superfield V^{ij} corresponds to the irreducible multiplet $(\mathbf{3},\mathbf{4},\mathbf{1})$. In the limit $\kappa\to 0$ the multiplet become fully reducible. The component transformations are

$$\begin{aligned}
&\delta x=\epsilon_{i\alpha}\psi^{i\alpha},\qquad \delta\psi^{i\alpha}=\epsilon^{i}_{\beta}A^{\alpha\beta}+i\,\epsilon^{i\alpha}\dot{x}-\underline{2i\kappa\,\epsilon^{\alpha}_{j}v^{ij}},\qquad \delta A^{\alpha\beta}=2i\,\epsilon^{i(\alpha}\dot{\psi}^{\beta)}_{i}+\underline{2\kappa\,\epsilon^{i(\alpha}\chi^{\beta)}_{i}},\\
&\delta v^{ij}=i\,\epsilon^{(i}_{\alpha}\chi^{j)\alpha},\qquad \delta\chi^{i}_{\alpha}=-2\,\epsilon_{j\alpha}\dot{v}^{ij}-\epsilon^{i}_{\alpha}C,\qquad \delta C=-i\,\epsilon_{k\alpha}\dot{\chi}^{k\alpha}.
\end{aligned} \tag{32}$$

Lagrangian

Now the Lagrangian is modified as

$$\mathcal{L}_{\text{kin.}} = \frac{1}{2}\int d^4\theta\, f\left(X_\kappa\right), \tag{33}$$

and in components it is written as

$$\begin{aligned}\mathcal{L}_{\text{kin.}} &= \left(\frac{\dot{x}^2}{2} + \frac{i}{2}\,\psi^{i\alpha}\dot{\psi}_{i\alpha} - \frac{A^{\alpha\beta}A_{\alpha\beta}}{4}\right) g\left(x\right) - \frac{1}{4}\,A^{\alpha\beta}\psi^{i}_{\alpha}\psi_{i\beta}\,g'\left(x\right) - \frac{1}{24}\,\psi^{i}_{\alpha}\psi_{i\beta}\psi^{j\alpha}\psi^{\beta}_{j}\,g''\left(x\right) \\ &\quad - \kappa\,C\,f'\left(x\right) - \kappa^2\,v^{ij}v_{ij}\,g\left(x\right) + \kappa\,\psi^{i\alpha}\chi_{i\alpha}\,g\left(x\right) - \frac{i}{2}\,\kappa\,v^{ij}\psi^{\alpha}_{i}\psi_{j\alpha}\,g'\left(x\right),\end{aligned} \tag{34}$$

where $g\left(x\right) = f''\left(x\right)$. In this Lagrangian the field v^{ij} is an auxiliary field, so it can be eliminated by its equation of motion.

Total Lagrangian

To avoid this elimination, we make these fields semi-dynamical by adding the Wess-Zumino (WZ) Lagrangian [2] of the multiplet $(\mathbf{3}, \mathbf{4}, \mathbf{1})$. We suggest the following total Lagrangian:

$$\mathcal{L}_{\text{tot.}} = \mathcal{L}_{\text{kin.}} + \mathcal{L}_{\text{WZ}}\,. \tag{35}$$

The second Lagrangian for the analytic superfield V^{++} was constructed as an analytic superpotential (E. Ivanov, O. Lechtenfeld, *JHEP* **09** (2003) 073). Without going into details, we write the total Lagrangian as

$$\begin{aligned} \mathcal{L}_{\text{tot.}} \;=\; & \left(\frac{\dot{x}^2}{2} + \frac{i}{2}\,\psi^{i\alpha}\dot{\psi}_{i\alpha} - \frac{A^{\alpha\beta}A_{\alpha\beta}}{4}\right) g\,(x) - \frac{1}{4}\,A^{\alpha\beta}\psi^i_\alpha\psi_{i\beta}\,g'\,(x) - \frac{1}{24}\,\psi^i_\alpha\psi_{i\beta}\psi^{j\alpha}\psi^\beta_j\,g''\,(x) \\ & - \kappa\,C\,f'\,(x) - \kappa^2\,v^{ij}v_{ij}\,g\,(x) + \kappa\,\psi^{i\alpha}\chi_{i\alpha}\,g\,(x) - \frac{i}{2}\,\kappa\,v^{ij}\psi^\alpha_i\psi_{j\alpha}\,g'\,(x) \\ & + C\,\mathcal{U}\,(v) - \dot{v}^{ij}\mathcal{A}_{ij}\,(v) - \frac{i}{2}\,\chi^{i\alpha}\chi^j_\alpha\,\mathcal{R}_{ij}\,(v)\,. \end{aligned} \tag{36}$$

[2] Lagrangians describing semi-dynamical (or spin) multiplets have only first-order time derivatives of bosonic fields and are known in supersymmetric mechanics as Wess-Zumino (or Chern-Simons) type Lagrangians.

On-shell Lagrangian

Then we eliminate the auxiliary fields $A^{\alpha\beta}$ and $\chi^{i\alpha}$ by their equations of motion. We keep the auxiliary field C as a Lagrange multiplier. Then the on-shell Lagrangian is written as

$$\begin{aligned}
\mathcal{L}^{\text{on-shell}}_{\text{tot.}} = & \left(\frac{\dot{x}^2}{2} + \frac{i}{2}\,\psi^{i\alpha}\dot{\psi}_{i\alpha} - \kappa^2 v^{ij}v_{ij}\right) g\,(x) + C\left[\mathcal{U}\,(v) - \kappa\, f'\,(x)\right] - \dot{v}^{ij}\mathcal{A}_{ij}\,(v) \\
& - i\,\psi^{\alpha}_{i}\psi_{j\alpha}\left[\frac{\kappa}{2}\,v^{ij}g'\,(x) + \frac{\kappa^2 g^2\,(x)\,\mathcal{R}^{ij}\,(v)}{\mathcal{R}^{kl}\,(v)\,\mathcal{R}_{kl}\,(v)}\right] - \frac{1}{24}\,\psi^{i}_{\alpha}\psi_{i\beta}\psi^{j\alpha}\psi^{\beta}_{j}\left[g''\,(x) - \frac{3\,g'\,(x)\,g'\,(x)}{2\,g\,(x)}\right].
\end{aligned} \tag{37}$$

The on-shell transformations are

$$\begin{aligned}
&\delta x = \epsilon_{i\alpha}\psi^{i\alpha}, \qquad \delta\psi^{i\alpha} = -\frac{g'\,(x)}{2g\,(x)}\,\epsilon^{i}_{\beta}\psi^{j\alpha}\psi^{\beta}_{j} + i\,\epsilon^{i\alpha}\dot{x} - 2i\kappa\,\epsilon^{\alpha}_{j}v^{ij}, \\
&\delta v^{ij} = -\frac{2\kappa g\,(x)}{\kappa\,\mathcal{R}^{kl}\,(v)\,\mathcal{R}_{kl}\,(v)}\,\epsilon^{(i}_{\alpha}\mathcal{R}^{j)m}\,(v)\,\psi^{\alpha}_{m}, \qquad \delta C = \epsilon_{i\alpha}\,\partial_t\left[\frac{2\kappa g\,(x)\,\psi^{\alpha}_{j}\mathcal{R}^{ij}\,(v)}{\kappa\,\mathcal{R}^{kl}\,(v)\,\mathcal{R}_{kl}\,(v)}\right].
\end{aligned} \tag{38}$$

Comparison

We compare it with the on-shell Lagrangian and transformations found in S. Fedoruk, E. Ivanov, O. Lechtenfeld, *JHEP* **06** (2012) 147. The equivalence of these two models can be established by performing the duality transformations (E.A. Ivanov, S.O. Krivonos, A.I. Pashnev, *Class. Quant. Grav.* **8** (1991) 19):

$$y = f'(x)\,, \quad \dot{x}(t) = x'(y)\,\dot{y}(t)\,, \quad \tilde{g}(y) = x'(y) = \frac{1}{y'(x)} = \frac{1}{f''(x)}\,, \quad \psi^{i\alpha} = \eta^{i\alpha} x'(y)\,. \tag{39}$$

Thus, the model has a dual superfield approach:

I		II
$(y, \eta^{i\alpha}, \underline{A^{ij}}) + (v^{ij}, \chi^{i\alpha}, C)$		$(x, \psi^{i\alpha}, \underline{A^{\alpha\beta}}, (v^{ij}, \chi^{i\alpha}, C))$
$\downarrow$		$\downarrow$
$\mathcal{L}_{\text{kin.}}(Y) + \kappa\,\mathcal{L}_{\text{int.}}(Y, V^{ij}) + \mathcal{L}_{\text{WZ}}(V^{ij})$		$\mathcal{L}_{\text{kin.}}(X_\kappa) + \mathcal{L}_{\text{WZ}}(V^{ij})$
$\downarrow$		$\downarrow$
$y, \eta^{i\alpha}, v^{ij}, C$	$\overset{\text{on-shell}}{\Longleftrightarrow}$	$x, \psi^{i\alpha}, v^{ij}, C$

SU(2|2) multiplet $(\mathbf{4},\mathbf{8},\mathbf{4})$

Models of SU(2|2) supersymmetric mechanics as deformations of $\mathcal{N}=8$ models were studied at the superfield level in E. Ivanov, O. Lechtenfeld, S. S., *JHEP* **11** (2016) 031. The corresponding superalgebra $su(2|2)$ is written as

$$\left\{Q^i_\alpha, Q^\beta_j\right\} = 2\delta^i_j\delta^\beta_\alpha H, \qquad \left\{S^i_\alpha, S^\beta_j\right\} = 2\delta^i_j\delta^\beta_\alpha H,$$
$$\left\{Q^i_\alpha, S^\beta_j\right\} = -\,2im\left(\delta^\alpha_\beta I^i_j + \delta^i_j F^\alpha_\beta\right). \tag{40}$$

Under the hidden supersymmetry S^i_α , the component fields transform as

$$\delta x = i\,\hat{\epsilon}_{i\alpha}\chi^{i\alpha}, \qquad \delta\psi^i_\alpha = -\,2i\,\hat{\epsilon}_{j\alpha}\dot{v}^{ij} - i\,\hat{\epsilon}^i_\alpha C, \qquad \delta A^{\alpha\beta} = -\,2\,\hat{\epsilon}^{i(\alpha}\dot{\chi}^{\beta)}_i + 2i\kappa\,\hat{\epsilon}^{i(\alpha}\psi^{\beta)}_i,$$
$$\delta v^{ij} = \hat{\epsilon}^{(i}_\alpha\psi^{j)\alpha}, \qquad \delta\chi^{i\alpha} = -\,i\,\hat{\epsilon}^i_\beta A^{\alpha\beta} + \hat{\epsilon}^{i\alpha}\dot{x} + 2\kappa\,\hat{\epsilon}^\alpha_j v^{ij}, \qquad \delta C = -\,\hat{\epsilon}_{k\alpha}\dot{\psi}^{k\alpha}. \tag{41}$$

Under the SU(2|2) supersymmetry, they form the multiplet $(\mathbf{4},\mathbf{8},\mathbf{4})$ described by a pair of superfields $\mathcal{V}^{ij}$ and $\mathcal{X}$ satisfying the constraints

$$\left.\begin{array}{ll} D^{(i}_\alpha\mathcal{V}^{jk)} = 0, & D^{i\alpha}\mathcal{V}^{jk} = -\,\varepsilon^{i(j}\nabla^{k)\alpha}\mathcal{X} \\ \nabla^{(i}_\alpha\mathcal{V}^{jk)} = 0, & \nabla^{i\alpha}\mathcal{V}^{jk} = -\,\varepsilon^{i(j}D^{k)\alpha}\mathcal{X} \end{array}\right\} \;\Rightarrow\; D^{(i}_\alpha D^{j)\alpha}\mathcal{X} = -\,4im\,\mathcal{V}^{ij}, \qquad \nabla^{(i}_\alpha\nabla^{j)\alpha}\mathcal{X} = 4im\,\mathcal{V}^{ij}.$$

$\mathcal{N} = 8$ multiplets

The problem of generalising indecomposable multiplets to $\mathcal{N} = 8$ supersymmetric mechanics certainly deserves attention.

- The $\mathcal{N} = 8$ invariant model with semi-dynamical degrees of freedom was presented in terms of $\mathcal{N} = 4$ superfields by S. Fedoruk, E. Ivanov, *Phys. Rev. D* **109** (2024) 085007. As it turns out, the model is OSp(8|2) superconformal (E. Khastyan, S. Krivonos, A. Nersessian, *Int. J. Mod. Phys. A* **40** (2025) 2450165).
- New $\mathcal{N} = 7, 8$ superconformal models associated with the superalgebras $G(3)$, $osp(8|2)$, $F(4)$, $osp(4^*|4)$ and $su(1,1|4)$ were considered in the recent papers S. Krivonos, A. Nersessian, *Phys. Lett. B* **863** (2025) 139373; *Phys. Rev. D* **111** (2025) 125025.
- There were indications that the Fedoruk-Ivanov model is also based on an indecomposable supermultiplet. In particular, it has two subrepresentations corresponding to the multiplets $(\mathbf{8}, \mathbf{8}, \mathbf{0})$ non-linearly coupled to the multiplet $(\mathbf{1}, \mathbf{8}, \mathbf{7})$.

Non-linear multiplet

To describe the multiplet mentioned above, we can assume constraints written in the SU(4) covariant formulation as

$$
\begin{aligned}
& D^I \bar{D}_J X_\kappa - \frac{\delta^I_J}{4}\, D^K \bar{D}_K X_\kappa = i\kappa \left[\left(Z^{1I} \bar{Z}^2_J - Z^{2I} \bar{Z}^1_J \right) - \frac{\delta^I_J}{4} \left(Z^{1K} \bar{Z}^2_K - Z^{2K} \bar{Z}^1_K \right) \right], \\
& D^I D^J X_\kappa - \frac{1}{2}\, \varepsilon^{IJKL}\, \bar{D}_K \bar{D}_L X_\kappa = 2i\kappa \left(Z^{1[I} Z^{2J]} - \frac{1}{2}\, \varepsilon^{IJKL}\, \bar{Z}^1_K \bar{Z}^2_L \right), \\
& D^I Z^{aJ} = \frac{1}{2}\, \varepsilon^{IJKL}\, \bar{D}_K \bar{Z}^a_L\,, \qquad D^J \bar{Z}^a_I = \frac{\delta^J_I}{4}\, D^K \bar{Z}^a_K\,, \qquad \bar{D}_J Z^{aI} = \frac{\delta^I_J}{4}\, \bar{D}_K Z^{aK}, \\
& D^{(I} Z^{aJ)} = 0, \qquad \bar{D}_{(I} \bar{Z}^a_{J)} = 0, \qquad \overline{(Z^{aI})} = \bar{Z}^a_I\,, \qquad a = 1, 2.
\end{aligned} \tag{42}
$$

Here, the covariant derivatives are

$$
D^I = \frac{\partial}{\partial \theta_I} - i\,\bar{\theta}^I \partial_t\,, \qquad \bar{D}_J = -\frac{\partial}{\partial \bar{\theta}^J} + i\,\theta_J\,\partial_t\,, \qquad I = 1, 2, 3, 4. \tag{43}
$$

However, it is the unsolved problem to write a superfield action for X_κ even when $\kappa = 0$.

Linear multiplet

Let us consider an indecomposable multiplet described by a chiral superfield Φ ($\bar{D}_I \Phi = 0$) with $16 + 16$ number of component fields. We impose the following linear constraints:

$$
\begin{aligned}
&D^I \bar{\Phi} = 0, \qquad \bar{D}_I \Phi = 0, \qquad D^I D^J \Phi - \frac{1}{2}\,\varepsilon^{IJKL}\,\bar{D}_K \bar{D}_L\,\bar{\Phi} = \kappa\, V^{IJ}, \\
&D^{(I} V^{J)K} = 0, \qquad \bar{D}_{(I} V_{J)K} = 0, \qquad \overline{(V^{IJ})} = V_{IJ} = \frac{1}{2}\,\varepsilon_{IJKL}\, V^{KL}.
\end{aligned} \tag{44}
$$

The irreducible multiplet $(\mathbf{6}, \mathbf{8}, \mathbf{2})$ is described by the superfield $V^{IJ} = -\,V^{JI}$ with antisymmetric indices. In the limit $\kappa = 0$ the multiplet becomes fully reducible and splits into $(\mathbf{2}, \mathbf{8}, \mathbf{6})$ and $(\mathbf{6}, \mathbf{8}, \mathbf{2})$.

The chiral superfield is given by the expression

$$\begin{aligned}\Phi\,(t_{\rm L}\,,\theta_I) \;&=\; \phi+\sqrt{2}\,\theta_I\psi^I+\theta_I\theta_J\left(i\,A^{IJ}-\frac{\kappa}{4}\,v^{IJ}\right)-\frac{\sqrt{2}}{3}\,\varepsilon^{IJKL}\,\theta_I\theta_J\theta_K\left(i\,\dot{\bar\psi}_L-\frac{\sqrt{2}}{4}\,\kappa\,\bar\chi_L\right)\\ &+\frac{1}{6}\,\varepsilon^{IJKL}\,\theta_I\theta_J\theta_K\theta_L\left(\ddot{\bar\phi}+\frac{\sqrt{2}\,i}{4}\,\kappa\,\bar C\right),\\ &\overline{(\phi)}=\bar\phi,\qquad \overline{(\psi^I)}=\bar\psi_I\,,\qquad \overline{(A^{IJ})}=A_{IJ}=\frac{1}{2}\,\varepsilon_{IJKL}\,A^{KL},\\ &\overline{(v^{IJ})}=v_{IJ}=\frac{1}{2}\,\varepsilon_{IJKL}\,v^{KL},\qquad \overline{(C)}=\bar C,\qquad \overline{(\chi^I)}=\bar\chi_I\,. \end{aligned}\tag{45}$$

The components transform as

$$\begin{aligned} &\delta\phi=-\sqrt{2}\,\epsilon_I\psi^I,\qquad \delta\bar\phi=\sqrt{2}\,\bar\epsilon^I\bar\psi_I\,,\\ &\delta\psi^I=\sqrt{2}\,i\,\bar\epsilon^I\dot\phi-\sqrt{2}\,i\,\epsilon_J A^{IJ}+\underline{\frac{\sqrt{2}}{4}\,\kappa\,\epsilon_J v^{IJ}},\qquad \delta\bar\psi_I=-\sqrt{2}\,i\,\epsilon_I\dot{\bar\phi}+\sqrt{2}\,i\,\bar\epsilon^J A_{IJ}+\underline{\frac{\sqrt{2}}{4}\,\kappa\,\bar\epsilon^J v_{IJ}}\,,\\ &\delta A^{IJ}=\sqrt{2}\,\varepsilon^{IJKL}\,\epsilon_K\dot{\bar\psi}_L-2\sqrt{2}\,\bar\epsilon^{[I}\dot\psi^{J]}+\underline{\frac{i\kappa}{4}\left(\varepsilon^{IJKL}\,\epsilon_K\bar\chi_L+2\,\bar\epsilon^{[I}\chi^{J]}\right)},\\ &\delta v^{IJ}=\varepsilon^{IJKL}\,\epsilon_K\bar\chi_L-2\,\bar\epsilon^{[I}\chi^{J]},\qquad \delta C=-\sqrt{2}\,\epsilon_I\dot\chi^I,\qquad \delta\bar C=\sqrt{2}\,\bar\epsilon^I\dot{\bar\chi}_I\,,\\ &\delta\chi^I=\sqrt{2}\,i\,\bar\epsilon^I C-2i\,\epsilon_J\dot v^{IJ},\qquad \delta\bar\chi_I=-\sqrt{2}\,i\,\epsilon_I\bar C+2i\,\bar\epsilon^J\dot v_{IJ}\,. \end{aligned}\tag{46}$$

Lagrangian

The simplest superfield action is written as

$$S_1 = \int dt\, \mathcal{L}_1 = -\frac{1}{16}\left[\int dt_{\rm L}\, d^4\theta\, \Phi^2 + \int dt_{\rm R}\, d^4\bar\theta\, \bar\Phi^2\right]. \tag{47}$$

The corresponding Lagrangian reads

$$\mathcal{L}_1 = \dot\phi\dot{\bar\phi} + \frac{i}{2}\left(\psi^I\dot{\bar\psi}_I - \dot\psi^I\bar\psi_I\right) + \frac{1}{4}A^{IJ}A_{IJ} + \frac{\sqrt{2}\,i}{8}\,\kappa\,(C\bar\phi - \phi\bar C) - \frac{\sqrt{2}\,\kappa}{8}\left(\psi^I\bar\chi_I + \chi^I\bar\psi_I\right) - \frac{\kappa^2}{64}\,v^{IJ}v_{IJ}\,. \tag{48}$$

We need to add to it the Lagrangian for the multiplet $(\mathbf{6}, \mathbf{8}, \mathbf{2})$ [3]:

$$\mathcal{L}_2 = \frac{1}{2}\,\dot v^{IJ}\dot v_{IJ} + \frac{i}{2}\left(\chi^I\dot{\bar\chi}_I - \dot\chi^I\bar\chi_I\right) + C\bar C. \tag{49}$$

[3] The construction can be given within the framework of the SU(4)/[SU(2) × SU(2) × U(1)] harmonic superspace (E. Ivanov, O. Lechtenfeld, S. S., *JHEP* **11** (2016) 031).

Superpotential

Surprisingly, we can write a superpotential as

$$S_{\text{pot.}} = -\frac{i}{2\sqrt{2}\,\kappa}\left[\int dt_{\rm L}\, d^4\theta\, \Phi^2 - \int dt_{\rm R}\, d^4\,\bar{\theta}\,\bar{\Phi}^2\right]. \tag{50}$$

The component Lagrangian is written as

$$\mathcal{L}_{\text{pot.}} = C\bar{\phi} + \phi\,\bar{C} - \frac{1}{\sqrt{2}}\,A^{IJ}v_{IJ} + i\left(\chi^I\bar{\psi}_I - \psi^I\bar{\chi}_I\right). \tag{51}$$

Another surprise is that this Lagrangian stays $\mathcal{N}=8$ invariant in the limit $\kappa=0$.

Conclusion

- Indecomposable multiplets combine irreducible multiplets via special parameters. They provide construction of new Lagrangians with interaction and potential terms.

Thank you for your attention!

Conclusion

- Indecomposable multiplets combine irreducible multiplets via special parameters. They provide construction of new Lagrangians with interaction and potential terms.
- Indecomposable multiplets can form irreducible multiplets under the action of higher-rank supersymmetry.

Thank you for your attention!

Conclusion

- Indecomposable multiplets combine irreducible multiplets via special parameters. They provide construction of new Lagrangians with interaction and potential terms.
- Indecomposable multiplets can form irreducible multiplets under the action of higher-rank supersymmetry.
- There may be more interesting examples of such multiplets.

Thank you for your attention!

Conclusion

- Indecomposable multiplets combine irreducible multiplets via special parameters. They provide construction of new Lagrangians with interaction and potential terms.
- Indecomposable multiplets can form irreducible multiplets under the action of higher-rank supersymmetry.
- There may be more interesting examples of such multiplets.

Thank you for your attention!