Non-topological Solitons in CFT

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We recall free Schrödinger equation in arbitrary space dimensions

$$i\frac{\partial}{\partial t}\psi = -\frac{\nabla^2}{2m}\psi. \tag{1}$$

It is well known that this equation is invariant under space-time transformations of Schrödinger group.

Schrödinger group		
Subgroup	Transformations	Infinitesimal generators G
Time Translation	$t' = t + \beta$	$\frac{\partial}{\partial t}$
Space Translation	$x' = x + a$	$\frac{\partial}{\partial x}$
Rotation	$x' = x$	1
Galilean boost	$x' = x + v \cdot t$	$t\frac{\partial}{\partial x} - imx$
Dilatation	$t' = e^{2\sigma}t,\ x' = e^{\sigma}x$	$2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + \frac{1}{2}$
Special conformal symmetry	$t' = \frac{t}{1+\eta t},\ x' = \frac{x}{1+\eta t}$	$\frac{ix^2}{2} - \frac{t}{2} - xt\frac{\partial}{\partial x} - t^2\frac{\partial}{\partial t}$

Unbroken dilatation and conformal symmetry are present in a model with potential term $|\psi|^{2n}$ that satisfies relation[1]

$$nd = d + 2, \tag{2}$$

where d is the number of space dimensions.
The simplest case possible is to consider a $(1+1)$-dimensional theory with $|\psi|^6$ self-interaction term which Lagrangian is written as

$$\mathcal{L}_{GP6} = i\psi^*\dot{\psi} - \frac{1}{2m}\nabla\psi^*\nabla\psi + \frac{\lambda}{24m^3}\left(\psi^*\psi\right)^3. \tag{3}$$

The corresponding equation of motion is a quintic Gross-Pitaevskii equation

$$i\frac{\partial}{\partial t}\psi = \left[-\frac{\nabla^2}{2m} - \frac{\lambda}{8m^3}\left(\psi^*\psi\right)^2\right]\psi, \tag{4}$$

where coupling $\lambda > 0$ (attractive potential).

[1] M. O. deKok and J.W. van Holten, Nucl. Phys. B 803 (2008), arXiv: 0712.3686 [hep-th].

Bright Soliton

Quintic Gross-Pitaevskii equation (4) supports bright soliton solutions

$$\psi(\mathrm{t},\mathrm{x}) = \mathrm{e}^{\mathrm{i}\mu \mathrm{t}} \left(\frac{24\mathrm{m}^3\mu}{\lambda}\right)^{\frac{1}{4}} \sqrt{\operatorname{sech}\left(\sqrt{8\mathrm{m}\mu}\cdot \mathrm{x}\right)}. \tag{5}$$

It is worth studying the integral characteristics of these solutions, such as the U(1) charge and the energy functional. Thus, straightforward calculations show that

$$\begin{aligned} \mathrm{N} &= \int_{-\infty}^{\infty} \mathrm{dx} |\psi(\mathrm{t},\mathrm{x})|^2 = \frac{\sqrt{3}\pi \mathrm{m}}{\sqrt{\lambda}}, \\ \mathrm{H} &= \int_{-\infty}^{\infty} \mathrm{dx} \left[\frac{1}{2\mathrm{m}} |\nabla\psi(\mathrm{t},\mathrm{x})|^2 - \frac{\lambda}{24\mathrm{m}^3} |\psi(\mathrm{t},\mathrm{x})|^6\right] = 0. \end{aligned} \tag{6}$$

We support this result by considering scale invariance of theory (27) and the influence of unbroken conformal symmetry.

Dilatations: $e^{\sigma} = \sqrt{2m\mu}$, so that $t' = 2m\mu t$ and $x' = \sqrt{2m\mu}\cdot x$. The complex field $\psi' = (2m\mu)^{-\frac{1}{4}}\psi$.

$$\nabla^2\psi' = \psi' - \frac{\lambda}{4m^2}\left|\psi'\right|^4\psi'. \tag{7}$$

$$N = \frac{\sqrt{2m\mu}}{\sqrt{2m\mu}}\int_{-\infty}^{\infty} dx' \left|\psi'\left(t', x'\right)\right|^2, \quad H = 0. \tag{8}$$

The latter is a result of an unbroken scale invariance and conformal symmetry, the corresponding symmetry generators D and K

$$D = 2tH + \frac{i}{2}\int \vec{x}\left(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right)d^2x,$$
$$K = t^2H - tD - \frac{m}{2}\int \vec{x}^2(\psi^*\psi)d^2x.$$

are conserved in accordance with equations [2]

$$\frac{dK}{dt} = -t\frac{dD}{dt}, \qquad \frac{dD}{dt} = 2H. \tag{9}$$

[2] M. O. deKok and J.W. van Holten, Nucl. Phys. B 803 (2008), arXiv: 0712.3686 [hep-th].

Linear Perturbations

Relations (6) impose a constraint on both the energy and the U(1) charge of bright solitons (5).

$$\psi_{\mathrm{p}}(\mathrm{t},\mathrm{x}) = \psi(\mathrm{t},\mathrm{x}) + \delta\psi(\mathrm{t},\mathrm{x}) = \mathrm{e}^{\mathrm{i}\mu \mathrm{t}}\mathrm{f}(\mathrm{x}) + \delta\psi(\mathrm{t},\mathrm{x}) \tag{10}$$

one can derive linearized equation of motion

$$\begin{aligned} &\mathrm{i}\frac{\partial}{\partial \mathrm{t}}\delta\psi(\mathrm{t},\mathrm{x}) = -\frac{\nabla^2}{2\mathrm{m}}\delta\psi(\mathrm{t},\mathrm{x}) - \frac{\lambda}{8\mathrm{m}^3}\left(3\cdot\delta\psi(\mathrm{t},\mathrm{x})|\psi(\mathrm{t},\mathrm{x})|^4 + \right. \\ &\left. +2\cdot\delta\psi^*(\mathrm{t},\mathrm{x})\psi^3(\mathrm{t},\mathrm{x})\psi^*(\mathrm{t},\mathrm{x})\right). \end{aligned} \tag{11}$$

Symmetry-related zero modes have a simple form

$$\delta\psi_0(\mathrm{t},\mathrm{x}) = \mathrm{G}\psi(\mathrm{t},\mathrm{x}), \tag{12}$$

where G is an infinitesimal generator of Schrödinger group symmetry or a generator of U(1) symmetry $\mathrm{G} = \mathrm{i}$.

The general ansatz for linear perturbations of the complex field ψ can be written as

$$\delta\psi(\mathrm{t},\mathrm{x}) = \mathrm{e}^{\mathrm{i}\mu\mathrm{t}}\left(\mathrm{e}^{\mathrm{i}\gamma\mathrm{t}}\eta(\mathrm{t},\mathrm{x}) + \mathrm{e}^{-\mathrm{i}\gamma^*\mathrm{t}}\xi^*(\mathrm{t},\mathrm{x})\right). \tag{13}$$

By setting the parameter γ and the functions η, ξ to be real we study the vibrational modes of bright soliton.

$$\begin{aligned}\nabla^2\eta &= \left(1 + \frac{\gamma_{\mathrm{osc.}}}{\mu}\right)\eta - \frac{1}{4\mathrm{m}^2}\mathrm{f}^4(3\eta + 2\xi),\\ \nabla^2\xi &= \left(1 - \frac{\gamma_{\mathrm{osc.}}}{\mu}\right)\xi - \frac{1}{4\mathrm{m}^2}\mathrm{f}^4(3\xi + 2\eta).\end{aligned} \tag{14}$$

Considering decay modes requires redefinition $\gamma \to -\mathrm{i}\gamma, \gamma \in \mathbb{R}$ and $\xi \equiv (\eta + \xi^*)$.

$$\begin{aligned}\nabla^2\,\mathsf{Re}\,\xi &= \mathsf{Re}\,\xi + \frac{\gamma_{\mathrm{dec.}}}{\mu}\,\mathsf{Im}\,\xi - \frac{5}{4\mathrm{m}^2}\mathrm{f}^4\,\mathsf{Re}\,\xi,\\ \nabla^2\,\mathsf{Im}\,\xi &= \mathsf{Im}\,\xi - \frac{\gamma_{\mathrm{dec.}}}{\mu}\,\mathsf{Re}\,\xi - \frac{1}{4\mathrm{m}^2}\mathrm{f}^4\,\mathsf{Im}\,\xi.\end{aligned} \tag{15}$$

An extensive numerical scanning[3] of the modes with known boundary conditions have failed to find any modes at any value of the parameter μ other than zero modes.
Vakhitov-Kolokolov criterion of stability

$$\frac{\mu}{\mathrm{N}}\frac{\mathrm{d}}{\mathrm{d}\mu}\mathrm{N} < 0 \tag{16}$$

and instability

$$\frac{\mu}{\mathrm{N}}\frac{\mathrm{d}}{\mathrm{d}\mu}\mathrm{N} > 0. \tag{17}$$

[3] Yulia Galushkina et al., Phys. Lett. B 865 (2025), arXiv: 2411.13514 [hep-ph].

Relativistic Generalization

In order to provide relativistic generalization of the model (27) we use a simple relation between the relativistic field ϕ and the non-relativistic field ψ that has the form

$$\phi(\mathrm{t}, \mathrm{x}) = \frac{1}{\sqrt{2\mathrm{m}}} \mathrm{e}^{-\mathrm{imt}} \psi(\mathrm{t}, \mathrm{x}). \tag{18}$$

Thus, we are able to write down the following Lorentz-invariant Lagrangian

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - \mathrm{m}^2 \phi^* \phi + \frac{\lambda}{3} \left(\phi^* \phi\right)^3 . \tag{19}$$

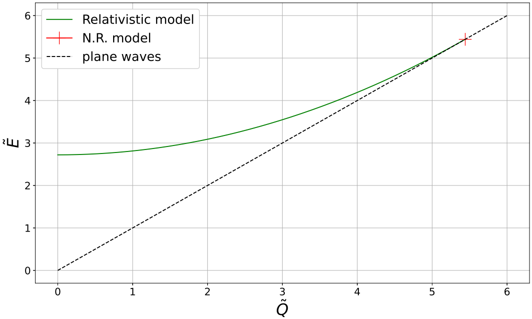
This theory also supports a soliton solution that can be written as

$$\phi(\mathrm{t}, \mathrm{x}) = \mathrm{e}^{-\mathrm{i}\omega \mathrm{t}} \mathrm{g}_\omega(\mathrm{x}) = \mathrm{e}^{-\mathrm{i}\omega \mathrm{t}} \left(\frac{3\left(\mathrm{m}^2 - \omega^2\right)}{\lambda} \right)^{\frac{1}{4}} \sqrt{\mathrm{sech}\left(2\sqrt{\mathrm{m}^2 - \omega^2} \cdot \mathrm{x}\right)}. \tag{20}$$

$$Q = 2\omega \int_{-\infty}^{\infty} dx \, |\phi(t,x)|^2 = \frac{\sqrt{3}\pi\omega}{\sqrt{\lambda}} \qquad (21)$$

$$E = \int_{-\infty}^{\infty} dx \left[\left|\dot{\phi}\right|^2 + |\nabla\phi|^2 + m^2 |\phi|^2 - \frac{\lambda}{3} |\phi|^6 \right] = \frac{\sqrt{3}\pi(m^2+\omega^2)}{2\sqrt{\lambda}}. \qquad (22)$$

It can be directly checked that the differential relation $\frac{dE}{dQ} = \omega$ is satisfied.



Decay Modes

Following scaling

$$\begin{aligned} \tilde{x} &= x\sqrt{m^2-\omega^2}; \\ \tilde{g} &= \frac{g_\omega \lambda^{\frac{1}{4}}}{(m^2-\omega^2)^{\frac{1}{4}}}, \end{aligned} \tag{23}$$

allows us to write linearized equations of motion for decay modes $\delta\phi(t,x) = e^{-i\omega t} e^{\gamma_{\text{dec.}} t} \left(\mathsf{Re}\,\xi(x) + i\,\mathsf{Im}\,\xi(x)\right)$

$$\begin{aligned} \tilde{\nabla}^2\,\mathsf{Re}\,\xi &= \frac{\left(m^2-\omega^2+\gamma_{\text{dec.}}^2\right)\mathsf{Re}\,\xi + 2\omega\gamma_{\text{dec.}}\,\mathsf{Im}\,\xi}{m^2-\omega^2} - 5\tilde{g}^4\,\mathsf{Re}\,\xi, \\ \tilde{\nabla}^2\,\mathsf{Im}\,\xi &= \frac{\left(m^2-\omega^2+\gamma_{\text{dec.}}^2\right)\mathsf{Im}\,\xi - 2\omega\gamma_{\text{dec.}}\,\mathsf{Re}\,\xi}{m^2-\omega^2} - \tilde{g}^4\,\mathsf{Im}\,\xi. \end{aligned} \tag{24}$$

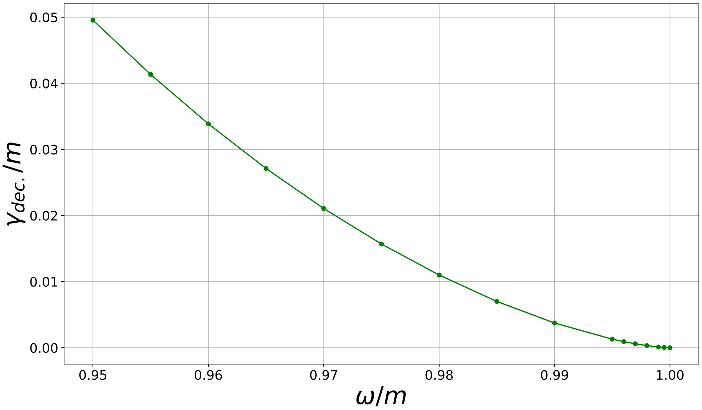


Figure 1: Spectrum of decay modes that are described by Eqs.(24). In the limit $\omega \to \mathrm{m}$ parameter $\gamma_{\mathrm{dec.}}$ tends to zero as $\mathrm{C} \cdot (\mathrm{m} - \omega)^{1.506}$.

It can be seen that in the limit $\omega \to \mathrm{m}$ parameter $\gamma_{\mathrm{dec.}}$ tends to zero. While $\frac{\gamma_{\mathrm{dec.}}}{\omega} \ll 1$ decay modes might be generated by expanding a soliton solution in perturbation series as

$$\mathrm{i}\phi_{\mathrm{p}}(\mathrm{t},\mathrm{x}) = \mathrm{i}\mathrm{e}^{-\mathrm{i}\left(1+\mathrm{i}\frac{\gamma}{\omega}\right)\omega \mathrm{t}} \mathrm{g}_{1+\mathrm{i}\frac{\gamma}{\omega}}(\mathrm{x}) \approx \mathrm{e}^{-\mathrm{i}\omega\mathrm{t}}\,(1+\gamma\mathrm{t}) \cdot \left(\mathrm{i}\mathrm{g}_{\omega}(\mathrm{x}) - \gamma \frac{\partial \mathrm{g}_{\omega}(\mathrm{x})}{\partial\omega}\right). \tag{25}$$

Comparison with the expansion of decay mode ansatz

$$\delta\phi(\mathrm{t},\mathrm{x}) = \mathrm{e}^{\mathrm{i}\omega\mathrm{t}}\mathrm{e}^{\gamma\mathrm{t}}\,(\mathsf{Re}\,\xi + \mathrm{i}\,\mathsf{Im}\,\xi) \approx \mathrm{e}^{\mathrm{i}\omega\mathrm{t}}(1+\gamma\mathrm{t})\,(\mathsf{Re}\,\xi + \mathrm{i}\,\mathsf{Im}\,\xi) \tag{26}$$

helps to evaluate that $\mathsf{Re}\,\xi = -\gamma\frac{\partial \mathrm{g}_{\omega}(\mathrm{x})}{\partial\omega}$ and $\mathsf{Im}\,\xi = \mathrm{g}_{\omega}(\mathrm{x})$.

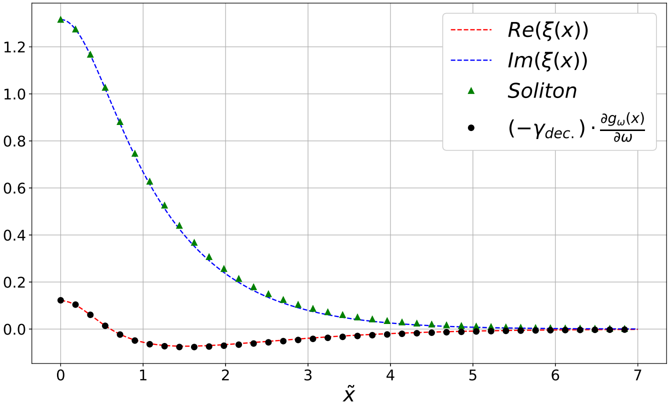


Figure 2: Decay mode profile at $\frac{\omega}{m} = 0.99$ and $\gamma_{\mathrm{dec.}} = 0.003739$. Scaled soliton profile and $(-\gamma_{\mathrm{dec.}})\frac{\partial g_\omega}{\partial \omega}$ are added for comparison.

$(2+1)$-dimensional CFT

In case of $\mathrm{d}=2$ space dimensions, non-relativistic CFT is written as

$$\mathcal{L}_{\mathrm{NR}} = \mathrm{i}\psi^*\dot{\psi} - \frac{1}{2\mathrm{m}}\nabla\psi^*\nabla\psi + \frac{\lambda}{8\mathrm{m}^2}\left(\psi^*\psi\right)^2 . \tag{27}$$

The corresponding equations of motion support conformal Q-tube solutions of the form $\psi(\mathrm{t},\mathrm{r}) = \mathrm{e}^{\mathrm{i}\mu\mathrm{t}}\mathrm{e}^{\mathrm{i}\mathrm{n}\theta}\mathrm{h}(\mathrm{r})$[4]. The equation of motions are written as

$$\mathrm{h}''(\mathrm{r}) + \frac{\mathrm{h}'(\mathrm{r})}{\mathrm{r}} - \frac{\mathrm{n}^2}{\mathrm{r}^2}\mathrm{h}(\mathrm{r}) = 2\mathrm{m}\mu\mathrm{h}(\mathrm{r}) - \frac{\lambda}{2\mathrm{m}}\mathrm{h}^3(\mathrm{r}). \tag{28}$$

This equation allows for scaling

$$\bar{\mathrm{r}} = \mathrm{r}\sqrt{2\mathrm{m}\mu}\ , \quad \bar{\mathrm{h}} = \mathrm{h}\sqrt{\frac{\lambda}{2\mathrm{m}\mu}} \tag{29}$$

and can be rewritten into

$$\bar{\mathrm{h}}''(\bar{\mathrm{r}}) + \frac{\bar{\mathrm{h}}'(\bar{\mathrm{r}})}{\bar{\mathrm{r}}} - \frac{\mathrm{n}^2}{\bar{\mathrm{r}}^2}\bar{\mathrm{h}}(\mathrm{r}) = \bar{\mathrm{h}}(\bar{\mathrm{r}}) - \frac{1}{2\mathrm{m}}\bar{\mathrm{h}}^3(\bar{\mathrm{r}}). \tag{30}$$

[4] M.Volkov and E.Wohnert 2002; P.Brax and P. Valageas, 2025

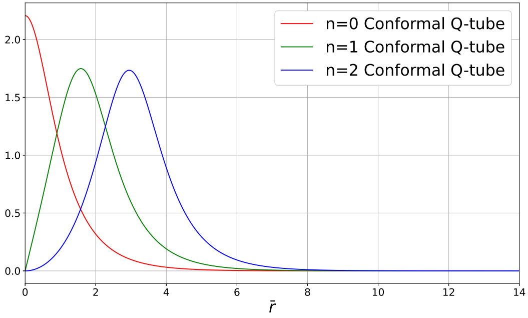


Figure 3: Conformal Q-tubes with winding number $n = 0, 1, 2$.

The integral characteristics of Q-tubes

$$\begin{aligned} H &= 2\pi \int_0^\infty \left[\frac{|\nabla\psi|^2}{2m} - \frac{\lambda}{8m^2}|\psi|^4 \right] r dr = 0, \\ N &= 2\pi \int_0^\infty |\psi|^2 r dr = \text{const}. \end{aligned} \tag{31}$$

Analytical Approximation

It can be noted that equation of motion (30) allows for following integral relation

$$\int_0^\infty d\bar{r} \frac{\left(\bar{h}'(\bar{r})\right)^2}{\bar{r}} = \int_0^\infty d\bar{r} \frac{n^2 \bar{h}^2(\bar{r})}{\bar{r}^3}. \tag{32}$$

A known asymptotic behavior of the Q-tube $\bar{h}(\bar{r}) \sim \bar{r}^n$ at $\bar{r} \to 0$ leads to approximation

$$\bar{h}''(\bar{r}) = \bar{h}(\bar{r}) \left(1 + \frac{n^2}{R^2}\right) - \frac{1}{2m} \bar{h}^3(\bar{r}), \text{ for } n \to \infty, \tag{33}$$

where R is the radius of the solution's peak. Soliton solution of Eq.(33) has an exact analytical form

$$\bar{h}(\bar{r}) = \sqrt{4m \left(1 + \frac{n^2}{R^2}\right)} \operatorname{sech} \left(\sqrt{1 + \frac{n^2}{R^2}} (\bar{r} - R)\right), \tag{34}$$

where $R = \sqrt{2}n$ in the limit of large n.

$$N \approx \frac{8\pi m}{\lambda} \ln\left(1 + e^{2\sqrt{3}n}\right) \xrightarrow{n\to\infty} \frac{16\sqrt{3}\pi m}{\lambda} n. \qquad (35)$$

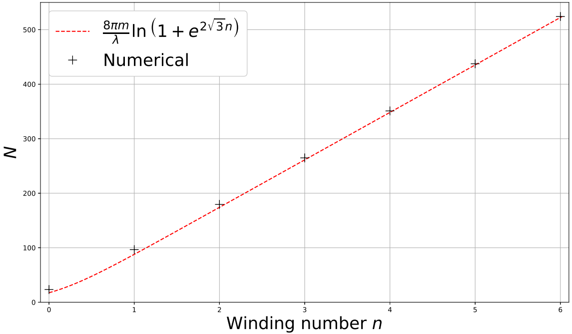


Figure 4: The value of global U(1) charge N plotted versus winding number n at $\lambda/m = 1$. Cross markers indicate results of numerical integration, while analytical estimations are represented by a dashed line.

Decay modes

$$\delta\psi(\mathrm{t},\mathrm{r}) = \mathrm{e}^{\mathrm{i}\mu\mathrm{t}+\gamma_{\mathrm{dec.}}\mathrm{t}}\mathrm{e}^{\mathrm{i}\mathrm{n}\theta}\left(\mathsf{Re}\,\xi + \mathrm{i}\,\mathsf{Im}\,\xi\right) \tag{36}$$

and linearized equations of motion

$$\begin{aligned}
\mathsf{Re}\,\xi'' + \frac{\mathsf{Re}\,\xi'}{\tilde{\mathrm{r}}} - \frac{\mathrm{n}^2}{\tilde{\mathrm{r}}}\,\mathsf{Re}\,\xi &= \mathsf{Re}\,\xi + \frac{\gamma_{\mathrm{dec.}}}{\mu}\,\mathsf{Im}\,\xi - \frac{3}{2\mathrm{m}}\bar{\mathrm{h}}^2\,\mathsf{Re}\,\xi, \\
\mathsf{Im}\,\xi'' + \frac{\mathsf{Im}\,\xi'}{\tilde{\mathrm{r}}} - \frac{\mathrm{n}^2}{\tilde{\mathrm{r}}}\,\mathsf{Im}\,\xi &= \mathsf{Im}\,\xi - \frac{\gamma_{\mathrm{dec.}}}{\mu}\,\mathsf{Re}\,\xi - \frac{1}{2\mathrm{m}}\bar{\mathrm{h}}^2\,\mathsf{Im}\,\xi.
\end{aligned} \tag{37}$$

The existence of a trivial spectrum is in accordance with Vakhitov-Kolokolov criterion

$$\frac{\mu}{\mathrm{N}}\frac{\mathrm{dN}}{\mathrm{d}\mu} = 0. \tag{38}$$

Breaking Conformal Symmetry

Consideration of relativistic corrections leads to violation of conformal symmetry. Stability analysis of non-topological solitons in the model

$$\mathcal{L} = \partial^\mu \phi^* \partial_\mu \phi - \mathrm{m}^2 \phi^* \phi + \frac{\lambda}{2} (\phi^* \phi)^2 \tag{39}$$

results in the following spectrum of decay modes

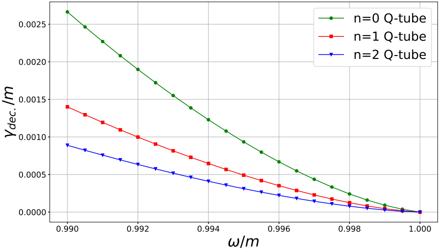


Figure 5: Decay parameter $\gamma_{\mathrm{dec.}}/\mathrm{m}$ for $\omega/\mathrm{m} \in [0.99, 1]$, $\mathrm{n} = 0, 1, 2$.

Near the isolated point

$$i\delta\phi(t,r) \approx e^{-i\omega t}e^{in\theta}\,(1+\gamma t)\left(if_\omega(r) - \gamma\frac{\partial f_\omega(r)}{\partial\omega}\right) = \\ = e^{-i\omega t}e^{in\theta}(1+\gamma_{dec.}t)\,(\mathrm{Re}\,\xi(\tilde{r}) + i\,\mathrm{Im}\,\xi(\tilde{r}))\,. \tag{40}$$

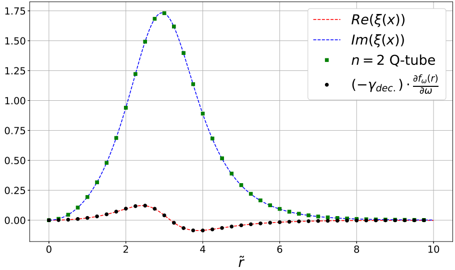


Figure 6: Decay mode profile at $\omega/m = 0.995$ and $\gamma_{dec.}/m = 3.11\cdot 10^{-4}$. Scaled soliton profile and $(-\gamma_{dec.})\dfrac{\partial f_\omega(r)}{\partial\omega}$ are added for comparison.

Summary

- We have found and examined bright solitons and Q-tube solutions in conformal field theory.
- In a relativistic generalization of our theory, the restoration of conformal symmetry leads to enhanced stability of non-topological solitons.
- The presence of conformal symmetry allowed for the Vakhitov-Kolokolov series expansion. The dynamical instability in relativistic model can be tracked using U(1) zero mode.

Thank you for attention![5]

[5] This work was supported by RSF 22-12-00215