Dirac fermion theory and spacetime geometry

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Outline

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Einstein's GR: metric = spacetime geometry. In modern GR, $g^{\mu\nu}$ is not fundamental: gravity enters Dirac equation via tetrads: 4×4 matrices e_a^μ – is the primary object. Metric = bilinear combination of tetrads. Index μ labels spacetime coordinates; index a describes internal spin $SO(1,3)$ space. Combining GR and Standard Model: symmetry groups $SO(1,7)$, $SO(3,11)$ [PRD **81** (2010) 025010], Clifford algebra $Cl(0,6)$ [IJGMMP **21** (2024) 2450089], higher spins [PLB **243** (1990) 378]. Internal dimension $n \neq D$ spacetime dimension $\Longrightarrow$ description of gravity via rectangular $D \times n$ frame. Frame components are not necessarily related to coordinate axes: e.g., for emerging fields in Akama-Dyakonov-Wetterich theory [PRD **60** (1978) 1900] and superfluid ^{3}He in B phase [Volovik: Physica ***B*162** (1990) 222], where dynamic frame is bilinear combination of fermion fields. Spin space dimension n can be greater than D spacetime dimension.

Geometric structure of spacetime

- Differential geometry: a crash-course
- Spacetime = 4-dimensional smooth manifold
- Local frame $e^\alpha_\mu(x)$ defines reference system of a physical observer (laboratory at point x)
- Metric $g_{\mu\nu}(x)$ introduces scalar product = (lengths and angles) $\Longrightarrow$ linear element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$
- Connection $\Gamma^\alpha{}_{\beta\mu}(x)$ defines parallel transport of geometric objects; ensures general covariance
- Metric-affine spacetime geometry $(g_{\mu\nu},\Gamma^\alpha{}_{\beta\mu})$ is characterized by curvature, torsion, non-metricity:

$$\begin{aligned} R^\alpha{}_{\beta\mu\nu} &= \partial_\mu\Gamma^\alpha{}_{\beta\nu} - \partial_\nu\Gamma^\alpha{}_{\beta\mu} + \Gamma^\alpha{}_{\lambda\mu}\Gamma^\lambda{}_{\beta\nu} - \Gamma^\alpha{}_{\lambda\nu}\Gamma^\lambda{}_{\beta\mu}, \\ T^\alpha{}_{\mu\nu} &= \Gamma^\alpha{}_{\nu\mu} - \Gamma^\alpha{}_{\mu\nu}, \\ Q_{\lambda\mu\nu} &= -\nabla_\lambda g_{\mu\nu} = -\partial_\lambda g_{\mu\nu} + \Gamma^\sigma{}_{\mu\lambda} g_{\sigma\nu} + \Gamma^\sigma{}_{\nu\lambda} g_{\mu\sigma} \end{aligned}$$

- In Einstein's general relativity: $T^\alpha{}_{\mu\nu} = 0$ and $Q_{\lambda\mu\nu} = 0$

Quintet of Dirac 4×4-matrices Γ^a with $a = 0, 1, 2, 3, 4$,

$$\{\Gamma^a, \Gamma^b\} = 2\eta^{ab}, \quad \eta^{ab} = \mathrm{diag}(1, -1, -1, -1, -1),$$

can be constructed in terms of usual 4×4 γ-matrices

$$\Gamma^0 := \gamma^0 \quad (a = 0), \quad \Gamma^a := \gamma^a \ (a = 1, 2, 3),$$
$$\Gamma^4 := -i\gamma_5 \ (a = 4), \quad \frac{1}{5!}\varepsilon_{abcde}\Gamma^a\Gamma^b\Gamma^c\Gamma^d\Gamma^e = 1$$

Here ε_{abcde} is five-dimensional Levi-Civita tensor; $\varepsilon_{01234} = +1$. Introduce 4×5 frame e_a^μ, and from 5-vector Γ^a in spin space we then construct 4×4 Dirac matrices $\gamma^\mu(x) = e_a^\mu(x)\Gamma^a$ (4-vector)

Generalised Dirac equation

$$(ie_a^\mu \Gamma^a \nabla_\mu - M)\,\psi = 0$$

is covariant with respect to diffeomorphisms, $x \longrightarrow x(x')$, and local $SO(1,4)$ spin transformations:

$$e_a^\mu \longrightarrow e_b^\mu \Lambda^b{}_a, \quad \psi \longrightarrow U^{-1}\psi, \quad U^{-1}\Gamma^a U = \Lambda^a{}_b\Gamma^b.$$

Fünfbein equation for fermion particle

Orthogonal 5×5 matrices $\Lambda^a{}_b(x)$ (i.e. $\Lambda^a{}_c\Lambda^b{}_d\eta^{cd}=\eta^{ab}$) generate $SO(1,4)$ spin transformations $U(x)$ by generators

$$S^{ab}=\frac{i}{4}[\Gamma^a,\Gamma^b]\,.$$

They determine spin-covariant derivatives

$$\nabla_\mu\psi=\partial_\mu\psi+\omega_\mu\psi,\qquad \omega_\mu=\frac{i}{2}\omega_{ab\mu}S^{ab}$$

in terms of spin-connection $\omega_{ab\mu}=-\,\omega_{ba\mu}$.
Main algebraic relations:

$$\begin{aligned}\left[\Gamma^a,S^{bc}\right] &= i\left(\eta^{ab}\Gamma^c-\eta^{ac}\Gamma^b\right),\quad \left\{\Gamma^a,S^{bc}\right\}=\varepsilon^{abcde}\,S_{de},\\ \left[S^{ab},S^{cd}\right] &= i(-\,S^{ac}\eta^{bd}+S^{ad}\eta^{bc}+\,S^{bc}\eta^{ad}-\,S^{bd}\eta^{ac}),\\ \left\{S^{ab},S^{cd}\right\} &= \frac{1}{2}\left(\eta^{ac}\eta^{bd}-\eta^{ad}\eta^{bc}+\varepsilon^{abcde}\,\Gamma_e\right).\end{aligned}$$

Induced spacetime geometry

Spin $SO(1,4)$ structure = 5-frame + spin-connection

$$(e^\mu_a, \quad {\omega^a}_{b\mu})$$

Evaluating commutator of covariant derivatives

$$\left(\nabla_\mu\nabla_\nu - \nabla_\nu\nabla_\mu\right)\psi = \frac{i}{2}\Omega_{ab\mu\nu}S^{ab}\psi,$$

we introduce field strength, or spin-curvature:

$${\Omega^a}_{b\mu\nu} = \partial_\mu{\omega^a}_{b\nu} - \partial_\nu{\omega^a}_{b\mu} + {\omega^a}_{c\mu}{\omega^c}_{b\nu} - {\omega^a}_{c\nu}{\omega^c}_{b\mu},$$

Usual $D = 4$ spacetime metric is expressed through 4×5 frame e^μ_a as in GR:

$$g^{\mu\nu} = e^\mu_a e^\nu_b \eta^{ab}.$$

Assuming non-degeneracy (i.e., $\det g^{\mu\nu} \neq 0$), one can get inverse tensor field $g_{\mu\nu}$.

Since rectangular frame connects 4- and 5-dimensional spaces, matrix e_a^μ cannot have an inverse. However, one can introduce a 5×4 matrix object

$$e_\mu^a := \eta^{ab} g_{\mu\nu} e_b^\nu,$$

which is semi-inverse to the original one, i.e.,

$$e_\mu^a e_a^\nu = \delta_\mu^\nu, \qquad e_\mu^a e_b^\mu = \Pi^a{}_b \neq \delta_b^a$$

From the definition we find that this is an idempotent object (hence a projector),

$$\Pi^a{}_c \Pi^c{}_b = \Pi^a{}_b.$$

In general, it depends on spacetime coordinates, $\Pi^a{}_b = \Pi^a{}_b(x)$.

Semi-inverse matrix determines spin-torsion:

$$\Theta^a{}_{\mu\nu} = \partial_\mu e^a_\nu - \partial_\nu e^a_\mu + \omega^a{}_{b\mu} e^b_\nu - \omega^a{}_{b\nu} e^b_\mu \,.$$

Together, the spin curvature and the spin torsion play the role of generalized structural relations. We postulate

$$\nabla_\mu e^\nu_a = \partial_\mu e^\nu_a + \Gamma^\nu{}_{\lambda\mu} e^\lambda_a - \omega^b{}_{a\mu} e^\nu_b = 0.$$

It is impossible to solve this equation with respect to spin connection $\omega^b{}_{a\mu}$. However, using semi-inverse matrix e^a_μ we find:

$$\Gamma^\alpha{}_{\beta\mu} = e^\alpha_a \omega^a{}_{b\mu} e^b_\beta + e^\alpha_a \partial_\mu e^a_\beta.$$

Spin structure induces spacetime geometry

$$(e^\mu_a, \quad \omega^a{}_{b\mu}) \quad \Longrightarrow \quad (g^{\mu\nu}, \quad \Gamma^\alpha{}_{\beta\mu})$$

Properties of induced spacetime geometry

A manifold with $\{g^{\mu\nu}, \Gamma^\alpha{}_{\beta\mu}\}$ is characterized by curvature, torsion, and nonmetricity. Differentiating $\nabla_\mu e_a^\nu = 0$, we find curvature

$$R^\alpha{}_{\beta\mu\nu} = \partial_\mu \Gamma^\alpha{}_{\beta\nu} - \partial_\nu \Gamma^\alpha{}_{\beta\mu} + \Gamma^\alpha{}_{\lambda\mu}\Gamma^\lambda{}_{\beta\nu} - \Gamma^\alpha{}_{\lambda\nu}\Gamma^\lambda{}_{\beta\mu} = \Omega^a{}_{b\mu\nu} e_a^\alpha e_\beta^b$$

By contraction, we derive Ricci tensor and curvature scalar:

$$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu} = e_a^\lambda e_\beta^b \, \Omega^a{}_{b\lambda\nu}, \qquad R = g^{\mu\nu} R_{\mu\nu} = e_a^\mu e_b^\nu \, \Omega^{ab}{}_{\mu\nu}.$$

Connection $\Gamma^\alpha{}_{\beta\mu}$ is compatible with metric

$$\nabla_\mu g^{\alpha\beta} = \partial_\mu g^{\alpha\beta} + \Gamma^\alpha{}_{\lambda\mu} g^{\lambda\beta} + \Gamma^\beta{}_{\lambda\mu} g^{\alpha\lambda} = 0.$$

Induced geometry thus has the vanishing nonmetricity.
However, induced spacetime torsion

$$T^\alpha{}_{\mu\nu} = \Gamma^\alpha{}_{\nu\mu} - \Gamma^\alpha{}_{\mu\nu} = e_a^\alpha \Theta^a{}_{\mu\nu}.$$

can be non-trivial, in general.

Understanding rectangular vielbein

Formally, rectangular frame is a 4×5 matrix

$$e_a^\mu = \left(\begin{array}{cccc|c} e^0_{\hat 0} & e^0_{\hat 1} & e^0_{\hat 2} & e^0_{\hat 3} & e^0_{\hat 4} \\ e^1_{\hat 0} & e^1_{\hat 1} & e^1_{\hat 2} & e^1_{\hat 3} & e^1_{\hat 4} \\ e^2_{\hat 0} & e^2_{\hat 1} & e^2_{\hat 2} & e^2_{\hat 3} & e^2_{\hat 4} \\ e^3_{\hat 0} & e^3_{\hat 1} & e^3_{\hat 2} & e^3_{\hat 3} & e^3_{\hat 4} \end{array}\right),$$

that naturally splits into 4×4 block $h^\mu_\alpha := e^\mu_\alpha$ $(\alpha = 0, 1, 2, 3)$ and column $k^\mu := e^\mu_{\hat 4}$ treated as 4-vector. Then induced metric

$$\boxed{g^{\mu\nu} = \widehat{g}^{\mu\nu} - k^\mu k^\nu}, \qquad \text{where} \qquad \widehat{g}^{\mu\nu} = h^\mu_\alpha h^\nu_\beta \eta^{\alpha\beta}.$$

This is a generalized Kerr-Schild ansatz! Inverting, we find

$$g_{\mu\nu} = \widehat{g}_{\mu\nu} + \frac{1}{1-k^2}\, k_\mu k_\nu, \qquad \text{where} \qquad \widehat{g}_{\mu\nu} = h^\alpha_\mu h^\beta_\nu \widehat{\eta}_{\alpha\beta},$$

with $k_\mu = \widehat{g}_{\mu\nu} k^\nu$, $k^2 = \widehat{g}_{\mu\nu} k^\mu k^\nu$. As a result, semi-inverse frame:

$$e^\alpha_\mu = h^\alpha_\mu + \frac{1}{1-k^2}\, k_\mu k^\alpha, \qquad e^{\hat 4}_\mu = -\frac{1}{1-k^2}\, k_\mu$$

Then projector $\Pi^a{}_b = e^a_\mu e^\mu_b$ reads explicity

$$\Pi^a{}_b = \delta^a_b - X^a{}_b, \qquad X^a{}_b = \frac{1}{1-k^2}\begin{pmatrix} 1 & k_\beta \\ -k^\alpha & -k^\alpha k_\beta \end{pmatrix}$$

One can check the idempotency $X^a{}_c X^c{}_b = X^a{}_b$.
Direct computation yields spin-curvature and spin-torsion:

$$\Omega^\alpha{}_{\beta\mu\nu} = \widehat{\Omega}^\alpha{}_{\beta\mu\nu} + b^\alpha{}_\mu b_{\beta\nu} - b^\alpha{}_\nu b_{\beta\mu}, \qquad \Omega^\alpha{}_{\hat{4}\mu\nu} = \widehat{D}_\mu b^\alpha{}_\mu - \widehat{D}_\nu b^\alpha{}_\mu$$

$$\Theta^\alpha{}_{\mu\nu} = \widehat{D}_\mu e^\alpha{}_\mu - \widehat{D}_\nu e^\alpha{}_\mu + b^\alpha{}_\mu \kappa_\nu - b^\alpha{}_\nu \kappa_\mu, \; \Theta^{\hat{4}}{}_{\mu\nu} = \partial_\mu \kappa_\nu - \partial_\nu \kappa_\mu + b_{\beta\mu} e^\beta{}_\nu - b_{\beta\nu} e^\beta{}_\mu$$

Here $\widehat{\Omega}^\alpha{}_{\beta\mu\nu}$ is the spin-curvature in the Lorentz sector, and we denoted $\kappa_\mu := e^{\hat{4}}_\mu$, $b^\alpha{}_\mu := \omega^\alpha{}_{\hat{4}\mu}$, and $b_{\alpha\nu} = \widehat{\eta}_{\alpha\beta} b^\beta{}_\mu$.

In representation $e^\mu_a = \{h^\mu_\alpha, k^\mu\}$, Dirac equation is recast into

$$\left(i e^\mu_\alpha \gamma^\alpha \nabla_\mu + \gamma_5 k^\mu \nabla_\mu - M\right)\psi = 0$$

Special case of Standard Model Extension, where k^μ manifests Lorentz symmetry breaking [Kostelecky, PRD **55** (1997) 6760].

De Sitter spacetime and de Sitter group

Rectangular 4×5 frame arises for de Sitter space as a $(1+3)$-hypersurface in the $(1+4)$-dimensional Minkowski spacetime $ds^2 = \eta_{ab} d\mathcal{X}^a d\mathcal{X}^b$. De Sitter spacetime $\Sigma_{1,3}$ = hyperboloid

$$\eta_{ab}\mathcal{X}^a\mathcal{X}^b = -\,\ell^2.$$

Embedding described in parametric form

$$\mathcal{X}^0 = \ell\, f(r)\,\mathbb{S}(t),\ \mathcal{X}^4 = \ell\, f(r)\mathbb{C}(t),\quad \mathcal{X}^1 = x,\quad \mathcal{X}^2 = y,\quad \mathcal{X}^3 = z,$$

with $r^2 = x^2 + y^2 + z^2$, $f = \sqrt{1 - \frac{r^2}{\ell^2}}$, $\mathbb{C} = \cosh(t/\ell)$, $\mathbb{S} = \sinh(t/\ell)$

On $\Sigma_{1,3}$ a metric $g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$ is induced via

$$e^a_\mu = \frac{\partial\mathcal{X}^a}{\partial x^\mu} = \begin{pmatrix} f\mathbb{C} & -\frac{x\,\mathbb{S}}{\ell\, f} & -\frac{y\,\mathbb{S}}{\ell\, f} & -\frac{z\,\mathbb{S}}{\ell\, f} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ f\mathbb{S} & -\frac{x\,\mathbb{C}}{\ell\, f} & -\frac{y\,\mathbb{C}}{\ell\, f} & -\frac{z\,\mathbb{C}}{\ell\, f} \end{pmatrix}.$$

Kerr-Schild representation:

$$h_\alpha^\mu = \left(\begin{array}{c|c} \frac{\mathbb{C}}{f} & 0 \\ \hline \frac{x^i}{\ell} f\mathbb{S} & \delta_j^i - \frac{x^i x_j}{\ell^2} \end{array}\right), \quad k^\mu = -\begin{pmatrix} \frac{\mathbb{S}}{f} \\ \frac{x^i}{\ell} f\mathbb{C} \end{pmatrix}$$

The resulting induced spacetime metric

$$g^{\mu\nu} = \widehat{g}^{\mu\nu} - k^\mu k^\nu = \begin{pmatrix} f^{-2} & 0 \\ 0 & -\delta^{ij} + \frac{x^i x^j}{\ell^2} \end{pmatrix}, \quad i,j = 1,2,3,$$

describes $\Sigma_{1,3}$ as a homogeneous 4-dimensional spacetime of constant curvature $R^\alpha{}_{\beta\mu\nu} = \frac{1}{\ell^2}\left(\delta^\alpha_\nu g_{\beta\mu} - \delta^\alpha_\mu g_{\beta\nu}\right)$. Replacing Cartesian coordinates (x, y, z) with spherical ones (r, θ, ϕ):

$$ds^2 = \left(1 - \frac{r^2}{\ell^2}\right) dt^2 - \frac{dr^2}{1 - \frac{r^2}{\ell^2}} - r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

Historic hint: Early models for spin $\frac{1}{2}$ in de Sitter space by Dirac [Ann. Math. **36** (1935) 657]; Gürsey-Lee [PNAS **49** (1963) 179]

Conclusions

The generalized Dirac equation suggests a nontrivial extension of the Lorentz group $SO(1,3)$ to de Sitter spin group $SO(1,4)$, introducing a rectangular 4×5 frame e_a^μ as a new geometric variable along with spin connection ${\omega^a}_{b\mu}$ (covariance!)

New fundamental spin variables $\{e_a^\mu, {\omega^a}_{b\mu}\}$ induce a geometric structure on spacetime, defining a metric $g^{\mu\nu}$ and a linear connection ${\Gamma^\alpha}_{\beta\mu}$. The induced Riemann-Cartan spacetime geometry is characterized by zero nonmetricity, but nontrivial curvature and torsion of spacetime are constructed from the spin-curvature and spin-torsion.

Resulting formalism opens new prospects for investigation of physically interesting problems related to hypothetic Lorentz symmetry violation effects.

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