New TMDs (AS-functions) and the twisted quark states

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based on the papers in collaborations with L. Szymanowski '22 and Xurong Chen '25

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From the factorization procedure applied to a given process, the hadron matrix elements of the quark-gluon nonlocal operators involve the interaction ($\mathbb{S}$-matrix) resulting in the explicit and implicit loop integrations.

- The explicit loop integrations lead to the evolution of parton distributions.
- The implicit loop integrations lead to the extension of the internal Lorentz covariants to be used for the Lorentz parametrization (new TMDs)

Full set of internal cov. due to $\mathbb{S}$-matrix in m.e. $\Rightarrow$ Extended L. param.-tion

- The conception of the twisted quark state provides the useful tool to realize the new TMD application.

- Factorization theorem/procedure in QCD:

$$\Big(\text{Direct Calculus of } \mathcal{A}\Big) \Rightarrow \Big(\text{Asym. Estimation of } \mathcal{A}\Big) \quad \text{if} \quad Q^2 \to \infty$$

The result of factorization is

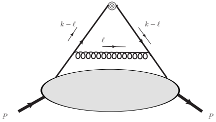
$$\boxed{\Big[E(x_1, ...) \otimes \Phi(x_1, ...)\Big] \oplus \Big[\Phi(y_1, ...) \otimes E(x_1, y_1; ....) \otimes \Phi(x_1, ...)\Big]}$$

- $E(x_1, ...)$ implies the parton subprocess (pQCD), while $\Phi(x_1, ...)$ is given by the quark-gluon correlators:

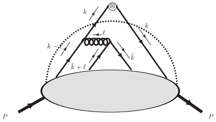
$$\langle P_1, ...|\mathrm{T}\,\mathcal{O}^{[\Gamma]}(\bar{\psi}, \psi, A)\,\mathbb{S}[\bar{\psi}, \psi, A]\,|..., P_1\rangle$$

which are non-pQCD (=, diagrammatically, the blob with incoming and outgoing parton legs).

- The diagram with the explicit loop integration:



- The diagram with the implicit loop integration:



- The Dirac (spinor) line, giving the Lorentz covariant $\mathbb{L}_j^{[\Gamma]}$, is defined as

$$\mathcal{L}_D^{[\Gamma]}(k,k') \stackrel{\text{def.}}{=} \left[\bar{u}_{\underline{\alpha}}(k)\,\Gamma_{\underline{\alpha\beta}}\,u_{\underline{\beta}}(k')\right]$$

- The Lorentz parametrization/expansion is given by

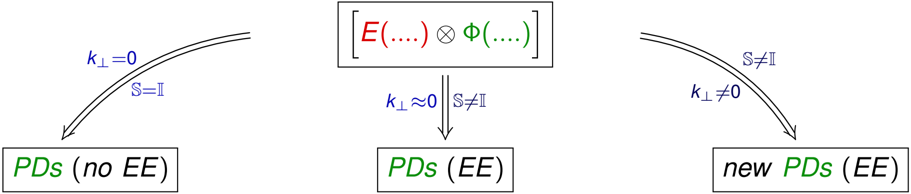
$$\boxed{\langle P,..|\mathrm{T}\,\mathcal{O}^{[\Gamma]}(\bar{\psi},\psi,A)\,\mathbb{S}[\bar{\psi},\psi,A]|..,P\rangle \stackrel{\mathcal{F}}{=} \sum_i \mathbb{L}_i^{[\Gamma]}(k,s|P,..)\,f^{(i)}(x,\{\vec{\mathbf{k}}_\perp\})}$$

where $\mathbb{L}_i^{[\Gamma]}(k,s|P)$ is the Lorentz covariant combination and the set of internal momenta is determined by the corresponding Dirac (spinor) lines $\mathcal{L}_D^{[\widetilde{\Gamma}]}$.

- The covariant (invariant) integrations are defined as

$$B_\perp^\alpha \int (d^4k) k^\alpha \mathcal{M}\left(k^2,(kP)\right) = (B_\perp P_\perp) \int (d^4k) \frac{(k_\perp P_\perp)}{P_\perp^2} \mathcal{M}\left(k^2,(kP)\right).$$

Contents, in a nutshell



A number of exterior and interior parameters (focus on the leading γ-projection):

- $k_\perp = 0$ and $\mathbb{S} = \mathbb{I}$: $\{P, S\}$- ext.,$\{k^+, s^+\}$-int.;
- $k_\perp \approx 0$ and $\mathbb{S} \neq \mathbb{I}$: $\{P, S\}$- ext., $\{k^+, s^+\}$-int.;
- $k_\perp \neq 0$ and $\mathbb{S} \neq \mathbb{I}$: $\{P, S\}$- ext., $\{k^+, s^+, k_\perp, s_\perp\}$-int.;

Consider the correlator with the essential $k_\perp$-dependence (TMDs) given by

$$\Phi^{[\gamma^+]}(x, k_\perp) = \int (dk^+ dk^-)\delta(x - k^+/P^+)\Phi^{[\gamma^+]}(k) \stackrel{\mathcal{F}}{=}$$

$$\boxed{\langle H|\mathrm{T}\,\bar{\psi}(0)\gamma^+\psi(0^+, z^-, \vec{\mathbf{z}}_\perp)\,\mathbb{S}[\bar{\psi}, \psi, A]|H\rangle \stackrel{\mathcal{F}}{=} \sum_i \mathbb{L}_i^+(k, s|P, ..)\, f^{(i)}\big(x, \{\vec{\mathbf{k}}_\perp\}\big)}$$

The summation is going over the different Lorentz combinations which are defined by the internal set, *i.e.* $[\bar{u}(k)\Gamma u(k)]$- spinor line, and by the external set, *i.e.* the hadron momentum P and so on.

As a rule, the internal Lorentz set have been generated by the corresponding spinor lines. Indeed, at the order of g^2, we have the following Anikin:2022

$$\langle P, S|\mathrm{T}\,\bar{\psi}(0)\gamma^+\psi(z)\,\mathbb{S}^{(2)}_{QCD}[\psi,\bar{\psi},A]\,|P,S\rangle\Big|_{\text{implicit loop integr.}} \sim$$
$$\left[\bar{u}(k)\gamma^+\hat{k}\gamma^\perp_\alpha u(k-\ell)\right]\left[\bar{u}(\tilde{k})\gamma^\perp_\alpha u(\tilde{k}+\ell)\right],$$

where $\sim$ implies “involves”.

$\mathbb{S}^{(2)}_{QCD}-$ order: The conditions for AS-function

- The kinematic regime for loop integrations:

$$|\ell| \ll \{|k|, |\tilde{k}|\} \text{ and } |\tilde{k}| \sim |k|,$$

- The same spin orientations for the partons (“aligned spin”),
- The Fierz transformations (Fi. Tr).

We can derive that (see Anikin:2021, Anikin:2022 for all details)

$$\langle P, S|\mathrm{T}\,\bar{\psi}(0)\gamma^+\psi(z)\,\mathbb{S}^{(2)}_{QCD}[\psi,\bar{\psi},A]\,|P,S\rangle\Big|_{\text{implicit loop integr.}}$$

$$\stackrel{\text{Fi. Tr.}}{\Longrightarrow} \left[\bar{u}^{(\uparrow_x)}(k)\gamma^+\gamma^\perp\gamma_5 u^{(\uparrow_x)}(k)\right] \sim s_\perp \quad \text{with} \;\; u^{(\uparrow_x)} = 1/2(1+\gamma_1\gamma_5)u$$

which is contributing additively to the other possible parametrizing functions.

Notice that there are no any “double counting" effects.

- Fierz transformations written in general form:

$$\left[\bar{u}^{(a)} O_1 u^{(b)}\right]\left[\bar{u}^{(c)} O_2 u^{(d)}\right] = \frac{1}{4}\sum_{A,R_1,R_2}\left\{\frac{1}{4}\mathrm{tr}\left[\Gamma_A O_1 \Gamma_{R_1}\right]\right\}\left\{\frac{1}{4}\mathrm{tr}\left[\Gamma^A O_2 \Gamma_{R_2}\right]\right\} \times \left[\bar{u}^{(c)}\Gamma^{R_1} u^{(b)}\right]\left[\bar{u}^{(a)}\Gamma^{R_2} u^{(d)}\right]$$

- With $O_1 = \gamma^+\gamma_j^\perp\gamma_5$, $O_2 = \mathbf{1}$, $\Gamma^A = \gamma_i^\perp$, $\Gamma^{R_1} = \gamma^+$, $\Gamma^{R_2} = \gamma_i^\perp$, we get

$$\left[\bar{u}^{(\uparrow_x)}(k)\gamma^+\gamma_j^\perp\gamma_5 u^{(\uparrow_x)}(k)\right]\left[\bar{u}^{(\uparrow_x)}(k) u^{(\uparrow_x)}(k)\right] = C_{ij}\left[\bar{u}^{(\uparrow_x)}(k)\gamma^+ u^{(\uparrow_x)}(k)\right]\left[\bar{u}^{(\uparrow_x)}(k)\gamma_i^\perp u^{(\uparrow_x)}(k)\right].$$

Here, for the fixed indices, $i \neq j$, the coefficient C_{ij} is given by

$$C_{ij} = \frac{1}{16}\mathrm{tr}\left[\gamma_i^\perp\gamma^+\gamma_j^\perp\gamma_5\gamma^-\right]\mathrm{tr}\left[\gamma_i^\perp\gamma_i^\perp\right].$$

Let us write the correlator function in the form of expansion as

$$\Phi^{[\gamma^+]}(k) \Rightarrow \int (d^4 z) e^{+ikz} \langle P, S| \mathrm{T}\, \bar{\psi}(0)\, \gamma^+\, \psi(z)\, \mathbb{S}^{(4)}_{QCD}[\psi, \bar{\psi}, A]\, |P, S\rangle =$$

$$\int (d^4 k_1)(d^4 \ell) \mathrm{tr}\big[\gamma^+ S(k) \gamma^\mu S(k-\ell) \gamma^\nu S(k)\big] \mathcal{D}_{\mu\mu'}(\ell) \mathcal{D}_{\nu\nu'}(\ell) \mathcal{F}^{\nu'\mu'}(k_1, \ell),$$

where

$$\mathcal{F}^{\nu'\mu'}(k_1, \ell) = \int (d^4 \xi) e^{-ik_1\xi} \langle H| \bar{\psi}(\xi) \gamma^{\nu'} S(k_1 - \ell) \gamma^{\mu'} \psi(0) |H\rangle.$$

Focusing on the axial-vector projection of Fierz decomposition of two fermions, the function $\mathcal{F}^{\nu'\mu'}(k_1, \ell)$ takes the form of

$$-4\,\mathcal{F}^{\nu'\mu'}_{(A)}(k_1, \ell) = \mathrm{tr}\big[\gamma^{\nu'} S(k_1-\ell)\gamma^{\mu'}\gamma^{\alpha}\gamma_5\big]\,\Phi^{[\gamma_\alpha\gamma_5]}(k_1)$$

with

$$\Phi^{[\gamma_\alpha\gamma_5]}(k_1) = \int (d^4\xi) e^{-ik_1\xi}\langle H|\bar{\psi}(\xi)\gamma_\alpha\gamma_5\psi(0)|H\rangle.$$

The (sub)structure function $\Phi^{[\gamma_\alpha\gamma_5]}(k_1)$ can be presented in the form of $\mathcal{M}$-amplitude written in the momentum representation:

$$\begin{aligned}&\delta^{(4)}\big(P_f - P_i\big)\mathcal{M}(k_1) = \delta^{(4)}(0)\Phi^{[\gamma_\alpha\gamma_5]}(k_1)\\ &= \langle H|b^+(k_1)b^-(k_1)|H\rangle\big[\bar{u}(k_1)\gamma_\alpha\gamma_5 u(k_1)\big],\end{aligned}$$

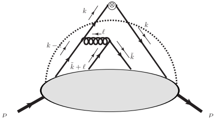
where the quark axial-vector combination gives the quark spin, *i.e.*

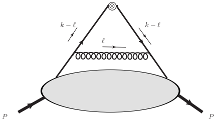
$$\big[\bar{u}(k_1)\gamma_\alpha\gamma_5 u(k_1)\big] \sim s_\alpha.$$

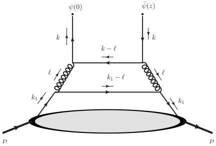
The principle proof:

It is shown (at $\mathbb{S}^{(2),\,(4)}_{QCD}-$ orders) that the presence of interactions in the correlator leads to the quark axial-vector combination giving the quark spin, *i.e.*

$$\boxed{\text{Interactions in } \Phi^{[\gamma^+]} \text{ with } \mathbb{S}^{(2),\,(4)}_{QCD}} \Rightarrow \boxed{\left[\bar{u}(k_1)\gamma_\alpha\gamma_5 u(k_1)\right] \sim s_\alpha .}$$







So, it explicitly shows that the function $\tilde{f}_1^{(1)}(x;\, k_\perp^2)$ and its analogues must appear in the parametrization of the hadron matrix element, *i.e.*

$$
\begin{aligned}
\Phi^{[\gamma^+]}(k) &\equiv \int (d^4 z) e^{+i(kz)} \langle P, S | \mathrm{T} \bar{\psi}(0)\, \gamma^+ \, \psi(z)\, \mathbb{S}[\psi, \bar{\psi}, A]\, | P, S \rangle \Big|_{k^+ = xP^+}^{k^- = 0, k_\perp \neq 0} \\
&= i\epsilon^{+-P_\perp s_\perp} \tilde{f}_1^{(1)}(x;\, k_\perp^2) + i\epsilon^{+-k_\perp s_\perp} f_{(2)}(x;\, k_\perp^2) + \ldots.,
\end{aligned}
$$

where the ellipse denotes the other possible terms of parametrization.

To see AS-functions, we deal with the IMPLICIT loop integrations !!

- For the existence of Lorentz vector defined as $\epsilon^{+-P_\perp s_\perp}$, it is necessary to assume that the quark spin $s_\perp$ is not a collinear vector to the hadron transverse momentum $P_\perp$.

Within the Collins-Soper frame, the hadron transverse momentum can be naturally presented as

$$\vec{\mathbf{P}}_\perp = (P_1^\perp, 0).$$

Since the hadron spin vector S can be decomposed on the longitudinal and transverse components as

$$S^L + S^\perp = \lambda P^+/m_N + S^\perp,$$

we get $P \cdot S = \vec{\mathbf{P}}^\perp \vec{\mathbf{S}}^\perp = 0$. Hence, it is natural to suppose that quark $s^\perp$ and hadron $S^\perp$ are collinear ones.

- This is a kinematical constraint (or evidence) for the nonzero Lorentz combination $\epsilon^{+-P_\perp s_\perp}$ and, therefore, for the existence of a new function $\tilde{f}_1^{(1)}(x;\, k_\perp^2)$.

- We also observe that Lorentz structure tensor, $\epsilon^{+-P_\perp s_\perp}$, associated with our function resembles the Sivers structure, $\epsilon^{+-P_\perp S_\perp}$ in which the nucleon spin vector $S_\perp$ is replaced by the quark spin vector $s_\perp$.

  However, despite this similarity the Sivers function and the introduced function $\tilde{f}_1^{(1)}(x;\, k_\perp^2)$ have totally different physical meaning.

- The angle ϕ_s cannot explicitly be measured in the experiment.

  However, the implementation of the covariant (invariant) integration of $f_{(2)}(x;\, k_1^{\perp\,2})$ gives the kinematical constraints on this angle relating the quark spin angle to the corresponding hadron angle. In particular, the orthogonality condition required by the covariant integration leads to the (anti)collinearity of $P_1^\perp$ and $s^\perp$, *i.e.* $\varphi_P = \varphi_s \pm n\pi$.

  It relates the hadron momentum with the quark spin vector and it can be treated as an additional condition for the existence of this new function.

There is the simplest way to observe the new TMDs thanks to the use of twisted particles. Before going further, we make two observations.

- First, one can see that if one fixes the spinor polarization along x-axis, the operator $\bar{\psi}(0)\,\gamma^+\,\psi(z)$ defining $\Phi^{[\gamma^+]}(x, k_\perp)$ can be written as

$$\boxed{\bar{\psi}^{(\uparrow_x)}\,\gamma^+\,\psi^{(\uparrow_x)} = \bar{\psi}^{(\uparrow_x)}\,\gamma^+\gamma_1\gamma_5\,\psi^{(\uparrow_x)}}$$

  where $\psi^{(\uparrow\downarrow_i)} = 1/2(1 \pm \gamma_i\gamma_5)\psi$ with $i = (1, 2) \equiv (x, y)$.

- Second, based on the simple algebra, one can conclude that the spinor line defined by $\big[\bar{u}(k)\gamma^{+}\hat{k}\gamma^{\perp}_{\alpha}u(k)\big]$ can be reduced to the spinor line such as $k^{\perp}_{\alpha}\big[\bar{u}(k)\gamma^{+}u(k)\big]$. On the other hand, as well-known the presence of $k_{\perp}$ in the spinor line or in the correlator signals on the non-trivial contributions of OAM (see for example Anikin:2015).

The excellent way to include OAM in the correlator is to use the conception of the twisted particles Jentschura:2011,Ivanov:2022,Bliokh:2017,Serbo:2015. In this case, we have a possibility to study the new kind of TMDs even at the leading order level.

Having said that, we go over to consideration of the correlator where one of quarks has been replaced on the twisted quark. We begin with the leading order of $\mathbb{S}$-expansion, we have

$$\Phi^{[\gamma^+]}_{\rm TW}(x, k_\perp) \stackrel{\mathcal{F}}{=} \langle P, S|\bar{\psi}(0)\,\gamma^+\,\Psi_{\rm TW}(z)\,|P, S\rangle^{l_z\neq 0},$$

where $\Psi_{\rm TW}(z)$ denotes the twisted (vortex) quark state and $\mathbb{S} = 1$, for the moment.

For the pedagogical reason, it is worth to recall the standard spherical $SL(2,C)$-spinor represented as

$$\boxed{\psi_{J\,M} \overset{\text{C-G}}{\rightsquigarrow} \big[R_{k\,l}(r)\; Y_{l\,l_z}(\theta,\varphi)\big] \circledast \varphi_{p\theta\phi,\,\lambda}}$$

where $\hat{J} = \hat{L} + \hat{S}$, $\hat{J}_3 \to M = l_z + s_z$, λ is an eigenvalue of spin operator, and $\vec{p} = (p_x, p_y, p_z) \equiv (p, \theta, \phi)$ with

$$\big\{p_x = p\,\cos\phi\,\sin\theta, \quad p_y = p\,\sin\phi\,\sin\theta, \quad p_z = p\,\cos\theta\big\}.$$

The OAM-part defined by the radial and spherical functions are given by

$$R_{k\,l}(r) = \sqrt{\frac{2\pi\,k}{r}}\,J_{l+1/2}(kr), \quad Y_{l\,l_z}(\theta,\phi) = \Theta_{l\,l_z}(\theta)\,e^{il_z\phi}.$$

Following Jentschura:2011, Ivanov:2022, Bliokh:2017, Serbo:2015, in the cylindric system the combination of the radial and spherical functions should be replaced by

$$\big[R_{k\,l}(r)\,Y_{l\,l_z}(\theta,\phi)\big] \stackrel{\text{repl.}}{\Longrightarrow} \sqrt{\frac{\varkappa}{2\pi}}J_{l_z}(\varkappa r)\,e^{il_z\phi}\Big|_{\text{twisted}}$$

and, in its turn, going over to the momentum representation, we have

$$\sqrt{\frac{\varkappa}{2\pi}}J_{l_z}(\varkappa r)\,e^{il_z\phi} \stackrel{p\text{-sp.}}{\Longrightarrow} e^{il_z\phi}\sqrt{\frac{2\pi}{\varkappa}}\delta\big(|\vec{\mathbf{k}}_\perp| - \varkappa\big).$$

Notice that

$$\varphi_{p\theta\phi,\pm\frac{1}{2}}\Big(\bar\theta=\frac{\pi}{2},\bar\phi=0\Big)\equiv U\Big(\mathcal{R}(\phi,\theta,0)\Big)\,\varphi_{p00,\pm\frac{1}{2}}\Big(\bar\theta=\frac{\pi}{2},\bar\phi=0\Big)=$$
$$\varphi_{p\theta\phi,-\frac{1}{2}}\Big(\bar\theta=0,\bar\phi=0\Big)\pm\varphi_{p\theta\phi,+\frac{1}{2}}\Big(\bar\theta=0,\bar\phi=0\Big),$$

where $U\big(\mathcal{R}(\phi,\theta,0)\big)$ denotes the rotation operator which is determined on the spinor representation.
Hence, it shows how the fermion state with the transverse polarization can be expressed through the corresponding helicity states,

$$\varphi_{p\theta\phi,\pm\frac{1}{2}}\Big(\bar\theta=\frac{\pi}{2},\bar\phi=0\Big)\rightsquigarrow\psi^{(\uparrow_x)},$$

which helps to prepare the corresponding twisted quark state.

Given that, we derive the twisted quark state with the transverse polarization in the form of (we adhere the Weyl representations for spinors)

$$\Psi^{(\uparrow_x)}_{\rm TW}(z) = \int (d^2 k_L)(d^2\vec{\mathbf{k}}_\perp) a_{\varkappa\, l_z}(\vec{\mathbf{k}}_\perp)\, e^{-ikz} u^{(\uparrow_x)}(k)\, b_{(\uparrow_x)}(k) + (\text{anti-quark term}),$$

where the weight function

$$a_{\varkappa\, l_z}(\vec{\mathbf{k}}_\perp) = (-i)^{j_z} e^{il_z\phi} \sqrt{\frac{2\pi}{\varkappa}}\delta\big(|\vec{\mathbf{k}}_\perp| - \varkappa\big)$$

has been inspired by the cylindric frame for the twisted particle.

One can see that

$$u^{(\uparrow_x)}(k) \sim \sum_{\lambda'=\pm 1/2} e^{-i\phi\lambda'}\, d_{+\frac{1}{2}\lambda'}(\theta) w^{(\lambda')}$$

with

$$w^{(\pm\frac{1}{2})} \equiv \varphi_{000,\pm\frac{1}{2}}(0,0),$$

where $d_{\lambda\lambda'}(\theta)$ implies the Wigner D-function.

- It is important to emphasize that one has the eigenfunction of $\hat{L}_z$-operator (z-projection of OAM) which is $\Phi_{l_z}(\phi) = e^{il_z\phi}$. In the momentum space, the weight function $a_{\varkappa\, l_z}(\vec{\mathbf{k}}_\perp)$ adopts the index $l_z \pm \lambda$ only after the expansion of $u(k)$-spinor over the helicity functions (pure spin states) $w^{(\lambda)}$.

That is, we write the following (the anti-quark term is not shown)

$$\Psi^{(\uparrow_x)}_{\rm TW}(z) = \int (d^2 k_L)(d^2\vec{\mathbf{k}}_\perp)\, e^{-ikz} \sum_{\lambda'=\pm 1/2} a_{\varkappa\, l_z \mp \lambda'}(\vec{\mathbf{k}}_\perp)\, d_{+\frac{1}{2}\lambda'}(\theta) w^{(\lambda')}\, b_{(\uparrow_x)}(k)$$

This expression shows directly the source of the imaginary part of $\Psi^{(\uparrow_x)}_{\rm TW}(z)$ which is induced by the phase factor of the weight function $a_{\varkappa\, l_z \mp \lambda'}(\vec{\mathbf{k}}_\perp)$ related to the twisted particle.

We are now inserting the twisted quark representation into the correlator to obtain that

$$\langle P,S|\bar{\psi}(0)\,\gamma^+\,\Psi^{(\uparrow_x)}_{\rm TW}(z)\,|P,S\rangle^{l_z\neq 0} = \int (d^4p)\int (d^2k_L)(d^2\vec{\mathbf{k}}_\perp) a_{\varkappa\, l_z}(\vec{\mathbf{k}}_\perp)\, e^{-ikz}$$
$$\times\big[\bar{u}^{(\lambda')}(p)\,\gamma^+\,u^{(\uparrow_x)}(k)\big]\;\langle P,S|b^+_{(\lambda')}(p)\,b_{(\uparrow_x)}(k)|P,S\rangle + (\text{anti-quark term}).$$

Notice that the fixed transverse polarization for the twisted quark singles out also the transverse polarization for the standard quark.

The simplest example is related to the well-known unpolarized Drell-Yan (DY) process, *i.e.* the lepton-production in nucleon-nucleon collision:

$$\begin{aligned} N(P_1) + N(P_2) &\to \gamma^*(q) + X(P_X) \\ &\to \ell(l_1) + \bar{\ell}(l_2) + X(P_X), \end{aligned}$$

with the initial unpolarized nucleons N.

The importance of the unpolarized DY differential cross section is due to the fact that it has been involved in the denominators of any spin asymmetries.

At LO, the hadron tensor which describes the unpolarized DY-process takes the following form:

$$\mathcal{W}^{(0)}_{\mu\nu} = \delta^{(2)}(\vec{\mathbf{q}}_\perp) \int (dx)(dy)\delta(xP_1^+ - q^+)\delta(yP_2^- - q^-) \times \mathrm{tr}\big[\gamma_\nu\, \gamma^+\, \gamma_\mu\, \gamma^-\big]\, \Phi^{[\gamma^-]}(y) \Big\{ \int (d^2\vec{\mathbf{k}}_1^\perp)\bar{\Phi}^{[\gamma^+]}_{\mathrm{TW}}(x, k_1^{\perp\,2}) \Big\},$$

where

$$\Phi^{[\gamma^-]}(y) = P_2^-\, f(y), \quad \bar{\Phi}^{[\gamma^+]}_{\mathrm{TW}}(x, k_1^{\perp\,2}) = i\epsilon^{+-k_1^\perp s^\perp}\, f^{\mathrm{TW}}_{(2)}(x;\, k_1^{\perp\,2}).$$

Calculating the contraction of hadron tensor with the unpolarized lepton tensor $\mathcal{L}^U_{\mu\nu}$, we derive that

$$d\sigma^{unpol.} \sim \int (d^2\vec{\mathbf{q}}_\perp)\mathcal{L}^U_{\mu\nu}\mathcal{W}^{(0)}_{\mu\nu} = \int (dx)(dy)\delta(xP_1^+ - q^+)\delta(yP_2^- - q^-)$$
$$\times \quad (1+\cos^2\theta)f(y)\int (d^2\vec{\mathbf{k}}_1^\perp)\epsilon^{P_2 - k_1^\perp s^\perp}\Im m f^{(\mathrm{TW})}_{(2)}(x;\, k_1^{\perp\,2}),$$

where $\epsilon^{+-k_1^\perp s^\perp} = \vec{\mathbf{k}}_1^\perp \wedge \vec{\mathbf{s}}^\perp \sim \sin(\phi_k - \phi_s)$ with ϕ_A, for $A = (k, s)$, denoting the angles between $\vec{\mathbf{A}}_\perp$ and $O\hat{x}$-axis in the Collins-Soper frame.

The imaginary part of $f^{(\mathrm{TW})}_{(2)}(x;\, k_1^{\perp\,2})$ is determined by the phase factor of the twisted quark which is involved in the correlator.

- we have demonstrated that the framework of twisted quarks Jentschura:2011, Ivanov:2022, Bliokh:2017, Serbo:2015 serves as a highly effective approach for investigating a new class of TMDs, specifically AS-functions, as introduced in previous works Anikin:2021, Anikin:2022, Anikin:2023. The proposed approach can be also adopted to study the standard TMDs.
- It is enough to be limited by the leading order of interaction to observe AS-functions. This is a simplest and more realible way compared to the traditional methods which are based on the $\mathbb{S}$-matrix expansion.
- The new $k_\perp$-dependent function $f_{(2)}(x;\, k_1^{\perp\,2})$ gives the additional and additive contribution to the depolarization factor $D_{[1+\cos^2\theta]}$ appeared in the differential cross section of unpolarized DY process.

Thank you for your attention !

Additional Materials to the Talk

The forward Compton scattering (CS), $\gamma^*(q) + \mathrm{h}(P) \to \gamma'(q) + \mathrm{h}'(P)$, amplitude is given by

$$\mathcal{A}_{\mu\nu} = \langle P | a^-_\nu(q)\, \mathbb{S}[\bar\psi, \psi, A]\, a^+_\mu(q) | P \rangle$$
$$= \int (d^4 z) e^{-iqz} \langle P |\, \mathrm{T}\Big\{ [\bar\psi(0)\gamma_\nu\psi(0)]\, [\bar\psi(z)\gamma_\mu\psi(z)]\, \mathbb{S}[\bar\psi, \psi, A] \Big\} | P \rangle.$$

Focusing on the connected diagrams only and using Wick's theorem, we derive the following contribution to the "hand-bag"-type of CS-diagrams:

$$\mathcal{A}_{\mu\nu}^{\text{hand-bag}} = \int (d^4 k)\, \text{tr}\big[E_{\mu\nu}(k,q)\,\Gamma\big]\, \Phi^{[\Gamma]}(k),$$

where Γ denotes the γ-matrix from the basis,

$$E_{\mu\nu}(k,q) = \gamma_\mu S(k+q)\gamma_\nu + (\textit{cross.}),$$
$$\Phi^{[\Gamma]}(k) = \int (d^4 z)\, e^{ikz} \langle P|\, \mathrm{T}\, \bar\psi(0)\, \Gamma\, \psi(z) \mathbb{S}[\bar\psi, \psi, A]|P\rangle.$$

In the similar manner, we can consider the hadron tensor of any one-photon processes like Drell-Yan (DY) process. We have

$$W = \int (d^4 k_1)(d^4 k_2) E(k_1, k_2, q; \Gamma_1, \Gamma_2)\, \Phi_1^{[\Gamma_1]}(k_1)\, \bar{\Phi}_2^{[\Gamma_2]}(k_2),$$

where

$$E(k_1, k_2, q; \Gamma_1, \Gamma_2) = \delta^{(4)}(k_1 + k_2 - q)\, \mathcal{E}(k_1, k_2, q; \Gamma_1, \Gamma_2)$$
$$\Phi_1^{[\Gamma_1]}(k_1) \stackrel{\mathcal{F}_1}{=} \langle \bar{\psi}(z_1)\Gamma_1 \psi(0)\rangle, \quad \bar{\Phi}_2^{[\Gamma_2]}(k_2) \stackrel{\mathcal{F}_2}{=} \langle \bar{\psi}(0)\Gamma_2 \psi(z_2)\rangle$$

and $\stackrel{\mathcal{F}_i}{=}$ denotes again the corresponding Fourier transforms.

For the CS-like amplitude, we write the following

$$A^{(0)} = \int (dx)\, E(xP^+; q; \Gamma) \Big\{ \int (d^4 k) \delta(x - k^+/P^+) \Phi^{[\Gamma]}(k) \Big\}$$

if $k_i^\perp$-terms are neglected in the expansion of $E(k, q)$;

and

$$A^{(k_\perp)} = \int (dx) \sum_i E^{(i)}(xP^+; q; \Gamma) \Big\{ \int (d^4k) \delta(x - k^+/P^+) \prod_{i'=1}^{i} k_{i'}^{\perp} \, \Phi^{[\Gamma]}(k) \Big\}$$

if $k_\perp$-terms are essential in the expansion Anikin:2020.

For the DY-like hadron tensor, we obtain that

$$W^{(0)} = \int (dx_1)(dx_2) E(x_1 P_1^+, x_2 P_2^-; q; \Gamma_1, \Gamma_2)$$
$$\times \Big\{ \int (d^4 k_1) \delta(x_1 - k_1^+/P_1^+) \Phi_1^{[\Gamma_1]}(k_1) \Big\} \Big\{ \int (d^4 k_2) \delta(x_2 - k_2^-/P_2^-) \bar{\Phi}_2^{[\Gamma_2]}(k_2) \Big\}$$

for the unessential (integrated out in the soft functions) $k_\perp$-dependence;

and

$$
\begin{aligned}
W^{(k_\perp)} = & \int (dx_1)(dx_2) \sum_{i,j} E^{(i,j)}(x_1 P_1^+, x_2 P_2^-; q; \Gamma_1, \Gamma_2) \\
\times & \Big\{ \int (d^4 k_1) \delta(x_1 - k_1^+/P_1^+) \prod_{i'=1}^{i} k^{\perp}_{1\,i'}\, \Phi_1^{[\Gamma_1]}(k_1) \Big\} \\
\times & \Big\{ \int (d^4 k_2) \delta(x_2 - k_2^-/P_2^-) \prod_{j'=1}^{j} k^{\perp}_{2\,j'}\, \bar{\Phi}_2^{[\Gamma_2]}(k_2) \Big\}
\end{aligned}
$$

for the essential $k_\perp$-dependence in the soft functions.

Let us begin with the well-know $k_\perp$-dependent function f_1:

$$\Phi^{[\gamma^+]}(k) = P^+ f_1\left(x;\, k_\perp^2, (k_\perp P_\perp)\right) =$$
$$P^+(k_\perp P_\perp) f_1^{(1)}(x;\, k_\perp^2) + \left\{\text{terms of } (k_\perp P_\perp)^n \,|\, n = 0, n \geq 2\right\},$$

where $k = (xP^+, k^-, \vec{\mathbf{k}}_\perp)$.

$f_1\left(x;\, k_\perp^2, (k_\perp P_\perp)\right)$ has been decomposed into the powers of $(k_\perp P_\perp)$. Keeping the term of decomposition with $n = 1$ represents the minimal necessary requirement for the manifestation of new functions.

Consider the second order of strong interactions, $\mathbb{S}^{(2)}_{QCD}$, in the correlator:

$$\langle P,S|\,T\bar{\psi}(0)\gamma^+\psi(z)\,\mathbb{S}^{(2)}_{QCD}[\psi,\bar{\psi},A]\,|P,S\rangle =$$

$$\int(d^4k)e^{-i(kz)}\Delta(k^2)\int(d^4\ell)\Delta(\ell^2)\int(d^4\tilde{k})\mathcal{M}\left(k^2,\ell^2,\tilde{k}^2,\ldots\right)$$

$$\times\left[\bar{u}(k)\gamma^+\hat{k}\gamma^\perp_\alpha u(k-\ell)\right]\left[\bar{u}(\tilde{k})\gamma^\perp_\alpha u(\tilde{k}+\ell)\right]\Longrightarrow$$

$$\left[\bar{u}(k)\gamma^+u(k)\right]\int(d^4\tilde{k})\left[\bar{u}(\tilde{k})\hat{k}_\perp u(\tilde{k})\right]$$

$$\times\int(d^4\ell)\Delta(\ell^2)\Delta(k^2)\mathcal{M}\left(k^2,\ell^2,\tilde{k}^2,\ldots\right)\sim P^+(k_\perp P_\perp)f_1^{(1)}(x;\,k_\perp^2)$$

$k_\perp$-dependence of PDs relates to the interactions encoded into correlators !

As explained in Anikin:2022, the essential $k_\perp$-dependence of any parton distributions ($k_\perp$-unintegrated functions), as a rule, stems from the interaction encoded in the correlator. So, we have (here, the notation $\tilde{z} = (z^-, \vec{\mathbf{z}}_\perp)$ has been used)

$$\Phi^{[\gamma^+]}(x, k_\perp) = \int (dk^+ dk^-) \delta\big(x - k^+/P^+\big) \Phi^{[\gamma^+]}(k)$$
$$\stackrel{\mathcal{F}}{=} \langle P| \mathrm{T}\, \bar{\psi}(0)\gamma^+ \psi(0^+, \tilde{z}) \mathbb{S}[\bar{\psi}, \psi, A] |P\rangle.$$

In QFT, the factorized (separated) form has been dictated by the fact that the OAM-operator commutes with SAM-operator, $[\,\hat{L},\ \hat{S}\,] = 0$, even if the spin-orbital interaction (which becomes rather sizeable in nuclei) is presented.

Indeed, the realization of the fictive internal space rotation giving the representation of $\hat{S}$-operator does not touch the rotation in $\mathbb{R}^3$ related to $\hat{L}$-operator. In other words, the generator that determines $\hat{S}$-operator excludes the space coordinate dependence.

Otherwise, it would be impossible to construct the Pauli-Lubansky vector W_α which is one of the Poincaré group characters.

It is necessary to extract the fermion (quark) states with the transverse polarization, $\psi^{(\uparrow_x)}$. For this goal, we assume the *x*-axis to be played a role of the spin quantization axis. The general representation of the spin quantization axis is given by $\vec{n}(\bar{\theta}, \bar{\phi})$ as a function of angles. That is, we fix the angular dependence of the original (at the rest frame) $SL(2, C)$-spinor as $\varphi_{000,\lambda}(\bar{\theta} = \pi/2, \bar{\phi} = 0)$, *i.e.*

$$\varphi_{000,+\frac{1}{2}}\Big(\bar{\theta} = \frac{\pi}{2}, \bar{\phi} = 0\Big) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \varphi_{000,-\frac{1}{2}}\Big(\bar{\theta} = \frac{\pi}{2}, \bar{\phi} = 0\Big) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Then, we implement the Lorentz boost along z-axis, $\vec{p} = (0, 0, p_z) \equiv (p, 0, 0)$. In other words, the spin quantazation axis differs from the moving direction (boost) of particle. That is, we have

$$\varphi_{p00,\pm\frac{1}{2}}\Big(\bar{\theta} = \frac{\pi}{2}, \bar{\phi} = 0\Big) \stackrel{N}{=} \begin{pmatrix} \pm(E + m + p_z) \\ (E + m - p_z) \end{pmatrix},$$

where, for the sake of shortness, the normalization factor given by $[2m(E+m)]^{-1/2}$ has been absorbed in the symbol $\stackrel{N}{=}$.

The state $\varphi_{p00,\lambda}(\pi/2, 0)$ can be re-expressed through the helicity states $\varphi_{p00,\lambda}(0, 0)$ as

$$\varphi_{p00,\pm\frac{1}{2}}\left(\bar{\theta} = \frac{\pi}{2}, \bar{\phi} = 0\right) = \\ \varphi_{p00,-\frac{1}{2}}\left(\bar{\theta} = 0, \bar{\phi} = 0\right) \pm \varphi_{p00,+\frac{1}{2}}\left(\bar{\theta} = 0, \bar{\phi} = 0\right).$$

We deal with TMDs (as are any kinds of parton distributions) which have been arisen from the factorization procedure, the quarks should be considered as massless objects.

The only difference is that there is no the rest system for the massless particle. Hence, the original $SL(2, C)$-spinor that has been above defined as $\varphi_{000,\lambda}(\bar{\theta} = \pi/2, \bar{\phi} = 0)$ should be replaced on the other $SL(2, C)$-spinor with the fixed momentum $\tilde{p}$, $\varphi_{\tilde{p}00,\lambda}(\bar{\theta} = \pi/2, \bar{\phi} = 0)$.

$SL(2, C)$-spinor should be then modified under the Lorentz boost along z-axis direction,

$$U\big(\mathcal{L}(p, 0, 0)\big)\varphi_{\tilde{p}00,\lambda}\big(\bar{\theta} = \frac{\pi}{2}, \bar{\phi} = 0\big) = \varphi_{p00,\lambda}\big(\bar{\theta} = \frac{\pi}{2}, \bar{\phi} = 0\big),$$

where $U\big(\mathcal{L}(p, 0, 0)\big) = \exp\{\sigma_3 p\, \tilde{\varphi}/2\}$ denotes the Lorentz boost along $(p, 0, 0)$.

The massless quark case leads to the independent $(\frac{1}{2}, 0)$- and $(0, \frac{1}{2})$-spinors, the left and right Weyl spinors respectively.

Indeed, in order to compensate the nullification of denominators in the normalization constants we have to suppose that

$$\big[E - \vec{\sigma}\vec{p}\big]\varphi^{(R)} = 0, \quad \big[E + \vec{\sigma}\vec{p}\big]\varphi^{(L)} = 0$$

independently. In its turns, it immediately leads to the helicity states:

$$\varphi^{(R)} \equiv \varphi_{p\theta\phi,+1/2}, \quad \varphi^{(L)} \equiv \varphi_{p\theta\phi,-1/2}.$$