Derivation of functional relations for multi-loop Feynman integrals

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Introduction

Essential results were obtained at the one-loop level.
A method of functional reduction for the dimensionally regularized one-loop Feynman integrals with massive propagators was proposed.
The method is based on a repeated application of the functional relations. Explicit formulae were given for reducing one-loop scalar integrals to a simpler ones, the arguments of which are the ratios of polynomials in the masses and kinematic invariants.

It was shown that a general scalar n-point integral, depending on $n(n+1)/2$ generic masses and kinematic variables, can be expressed as a linear combination of integrals depending only on n variables. The latter integrals were given explicitly in terms of hypergeometric functions of $(n-1)$ dimensionless variables.

What about multi-loop integrals?

FR from recurrence relations

Functional relations (FR) for Feynman integrals were proposed in O.V.T. Phys.Lett. B670 (2008) 67.

Feynman integrals satisfy recurrence relations which can be written as

$$\sum_j Q_j I_{j,n} = \sum_{k,r<n} R_{k,r} I_{k,r}$$

where Q_j, R_k are polynomials in masses, scalar products of external momenta, space-time dimension d, and powers of propagators. $I_{k,r}$ - are integrals with r external lines. In recurrence relations some integrals are more complicated than the others: $I_{j,n}$ on the l.h.s have more arguments than $I_{k,r}$ on the r.h.s.

General method for deriving functional equations:

By choosing kinematic variables, masses, indices of propagators remove most complicated integrals, i.e. impose conditions :

$$Q_j = 0$$

keeping at least some other coefficients $R_k \neq 0$.

This method can be used for deriving FR for multi-loop integrals.

The problem: it is difficult to derive recurrence relations for integrals with many masses and momenta.

To obtain FR for n -point integral one need recurrence relations for $n+1$ point integral.

FR from algebraic relations

The simplest is the method based on algebraic relations for propagators.
The following algebraic relation between the products of n propagators was discovered:

$$\prod_{r=1}^{n} \frac{1}{D_r} = \frac{1}{D_0} \sum_{r=1}^{n} x_r \prod_{\substack{j=1 \\ j \neq r}}^{n} \frac{1}{D_j},$$

where

$$D_j = (k_1 - p_j)^2 - m_j^2 + i\eta.$$

This equation can be fulfilled for arbitrary k_1 by imposing conditions on $x_j, \ m_0, \ p_0$. The resulting algebraic relation for the product of n propagators depends on $n-1$ arbitrary parameters.
For $n=2$ and $n=3$ we get

$$\frac{1}{D_1 D_2} = \frac{x_1}{D_2 D_3} + \frac{x_2}{D_1 D_3},$$

$$\frac{1}{D_1 D_2 D_3} = \frac{x_1}{D_4 D_2 D_3} + \frac{x_2}{D_1 D_4 D_3} + \frac{x_3}{D_1 D_2 D_4},$$

Considering p_j as external momenta and integrating w.r.t. k_1 we get functional relations

$$\int \frac{d^d k_1}{D_1 D_2} \rightarrow I_2^{(d)}(m_1^2, m_2^2; s_{12}) = x_1 I_2^{(d)}(m_0^2, m_2^2; s_{02}) + x_2 I_2^{(d)}(m_1^2, m_0^2; s_{10}),$$

$$\int \frac{d^d k_1}{D_1 D_2 D_3} \rightarrow I_3^{(d)}(m_1^2, m_2^2, m_3^2; s_{23}, s_{13}, s_{12}) = x_1 I_3^{(d)}(m_0^2, m_2^2, m_3^2; s_{23}, s_{03}, s_{02})$$
$$+ x_2 I_3^{(d)}(m_1^2, m_0^2, m_3^2; s_{03}, s_{13}, s_{10}) + x_3 I_3^{(d)}(m_1^2, m_2^2, m_0^2; s_{20}, s_{10}, s_{12}),$$

where

$$s_{ij} = (p_i - p_j)^2.$$

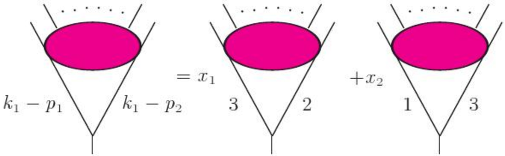
Algebraic relations for propagators

Algebraic relations for products of 2,3,... propagators multiplied by products of any number of propagators raised to arbitrary power ν_j

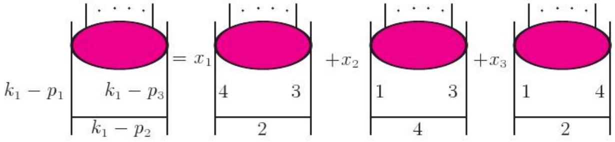
$$\prod_{j=n_0}^{N} \frac{1}{[(k_1 - p_j) - m_j^2]^{\nu_j}}$$

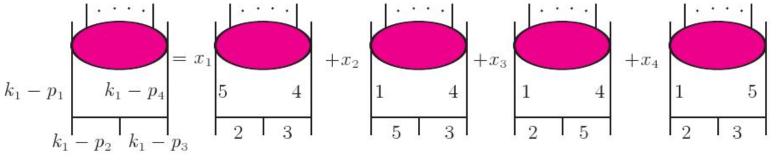
and integrated with respect to k_1 yield functional relations for one-loop integrals with any number of external legs.

Algebraic relations for products of any number of propagators can be used for derivation functional relations for integrals with any number of loops. Multiplying algebraic relation by function corresponding to Feynman integral depending on k_1 and any number of external momenta and integrating with respect to k_1 will produce functional relations. Diagrammatic representation of such relations based on 2-, 3- and 4- propagator relations:



Algebraic relation for propagators





The blob on these pictures correspond to either product of propagators raised to arbitrary powers or to an integral with any number of loops. One of the external momenta of this multiloop integral should be k_1.

Example.

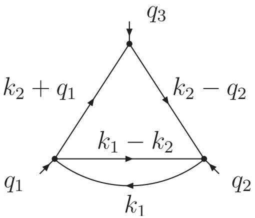
We can use algebraic relation for two propagators multiply it by the integral

$$\int \frac{d^d k_2}{[k_2^2 - m_4^2][(k_1-k_2)^2 - m_5^2]}$$

and integrate over momentum k_1.

Thus we will obtain functional relation for the following integral:

$$R(m_1^2, m_2^2, m_3^2, m_4^2;\ q_1^2, q_2^2, q_3^2)$$
$$= \int\int \frac{d^d k_1 d^d k_2}{(i\pi^{d/2})^2} \frac{1}{[(k_2+q_1)^2 - m_1^2][(k_2-q_2)^2 - m_2^2][k_1^2 - m_3^2][(k_1-k_2)^2 - m_4^2]}.$$



Using arbitrary parameter in the two-terms relation for propagators we get the following functional relation

$$
\begin{aligned}
&R(m_1^2, m_2^2, m_3^2, m_4^2;\ q_1^2, q_2^2, q_3^2) \\
&\qquad = \alpha R(0, m_2^2, m_3^2, m_4^2;\ Q^2, q_2^2, (m_2^2 - m_1^2 + q_3^2)\alpha - m_2^2) \\
&\qquad + (1-\alpha) R(m_1^2, 0, m_3^2, m_4^2;\ q_1^2, Q^2, (m_2^2 - m_1^2 - q_3^2)\alpha + q_3^2 - m_2^2),
\end{aligned}
$$

where

$$
\begin{aligned}
Q^2 &= (q_1^2 - q_2^2 - m_1^2 + m_2^2)\alpha + q_2^2 - m_2^2, \\
\alpha &= \frac{q_3^2 - m_1^2 + m_2^2 \pm \sqrt{\Delta}}{2q_3^2}, \\
\Delta &= q_3^4 + m_1^4 + m_2^4 - 2q_3^2 m_1^2 - 2q_3^2 m_2^2 - 2m_1^2 m_2^2.
\end{aligned}
$$

Instead of the integral $R(m_1^2, m_2^2, m_3^2, m_4^2;\ q_1^2, q_2^2, q_3^2)$ one can consider it's derivatives w.r.t. m_3^2 or m_4^2. The derivatives will correspond to diagrams with dot on 3rd or 4-th line and will be UV finite.

These derivatives will correspond to the following integrals:

$$R_3(m_1^2, m_2^2, m_3^2, m_4^2;\ q_1^2, q_2^2, q_3^2)$$
$$= \int\int \frac{d^dk_1 d^dk_2}{(i\pi^{d/2})^2} \frac{1}{[(k_2+q_1)^2-m_1^2][(k_2-q_2)^2-m_2^2][k_1^2-m_3^2]^2[(k_1-k_2)^2-m_4^2]}.$$

$$R_4(m_1^2, m_2^2, m_3^2, m_4^2;\ q_1^2, q_2^2, q_3^2)$$
$$= \int\int \frac{d^dk_1 d^dk_2}{(i\pi^{d/2})^2} \frac{1}{[(k_2+q_1)^2-m_1^2][(k_2-q_2)^2-m_2^2][k_1^2-m_3^2][(k_1-k_2)^2-m_4^2]^2}.$$

Functional relation for two-loop vertex

Integrals R_3 and R_4 are UV finite and satisfy the following functional relations:

$$\begin{aligned} & R_3(m_1^2, m_2^2, m_3^2, m_4^2;\ q_1^2, q_2^2, q_3^2) \\ & \quad = \alpha R_3(0, m_2^2, m_3^2, m_4^2;\ Q^2, q_2^2, (m_2^2 - m_1^2 + q_3^2)\alpha - m_2^2) \\ & \quad + (1-\alpha) R_3(m_1^2, 0, m_3^2, m_4^2;\ q_1^2, Q^2, (m_2^2 - m_1^2 - q_3^2)\alpha + q_3^2 - m_2^2), \end{aligned}$$

$$\begin{aligned} & R_4(m_1^2, m_2^2, m_3^2, m_4^2;\ q_1^2, q_2^2, q_3^2) \\ & \quad = \alpha R_4(0, m_2^2, m_3^2, m_4^2;\ Q^2, q_2^2, (m_2^2 - m_1^2 + q_3^2)\alpha - m_2^2) \\ & \quad + (1-\alpha) R_4(m_1^2, 0, m_3^2, m_4^2;\ q_1^2, Q^2, (m_2^2 - m_1^2 - q_3^2)\alpha + q_3^2 - m_2^2), \end{aligned}$$

These functional relations can be used for calculating basis integral encountered in the orthopositronium decay. In particular one of the basis integrals corresponds to kinematics $m_1^2 = m_2^2 = m_3^2 = m_4^2$, $q_1^2 = q_2^2 = m^2$, $q_3^2 = 0$. In this case functional relation for R_3 reads

$$R_3(m^2, m^2, m^2, m^2;\; m^2, m^2, 4m^2) = R_3(0, m^2, m^2, m^2; 0, m^2, m^2).$$

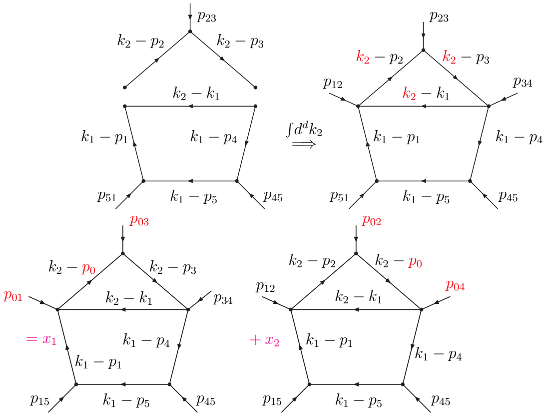
Integral on the right hand side is in fact propagator type integral with one massless line. Applying recurrence relations this integral can be reduced to simpler integrals:

$$\begin{aligned} & R_3(0, m^2, m^2, m^2; 0, m^2, m^2) \\ & = \frac{1}{(i\pi^{d/2})^2} \int\int \frac{d^d k_1 d^d k_2}{k_1^2(k_2^2 - m^2)[(k_1-k_2)^2 - m^2]^2[(k_1+q_1)^2 - m^2]} \\ & = \frac{2}{3(d-3)} J_{111}^{(d-2)}(m^2), \end{aligned}$$

where

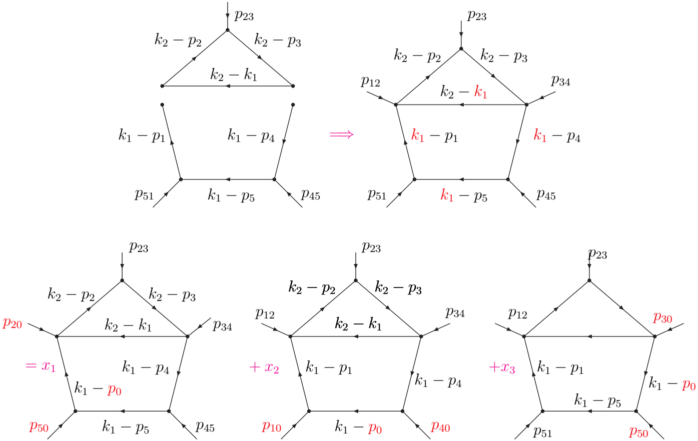
$$J_{111}^{(d)}(q^2) = \frac{1}{(i\pi^{d/2})^2} \int\int \frac{d^d k_1 d^d k_2}{(k_1^2 - m^2)[(k_1-k_2)^2 - m^2][(k_2-q)^2 - m^2]}.$$

Consider more complicated example. Integrating the two term relation with one-loop box type integral depending on k_2 one can get FR for the two-loop pentagon integral:

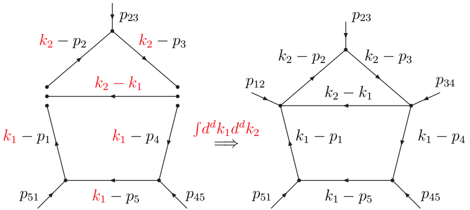


Two more FR can be generated using the two-term relation.

Integrating the three term relation with one-loop vertex type integral depending on k_1 one can get another FR for the two-loop pentagon integral:



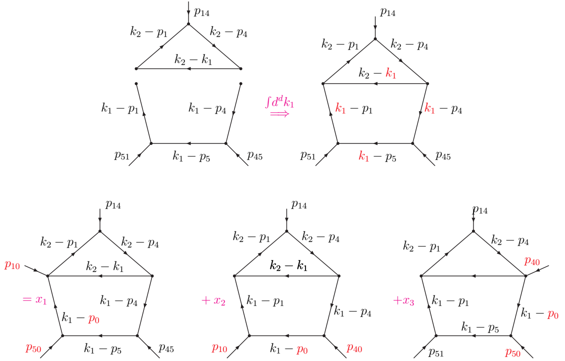
Integrating the three terms relation multiplied by two terms relation and by the factor $1/((k_1 - k_2)^2 - m_7^2)$ one can get 6 terms FR for the two-loop pentagon integral:



FR for the two-loop planar vertex

Problems:

Integrating the three terms relation multiplied by one-loop vertex one can get FR for the two-loop planar integral:



Thus we get mixture of different topologies. Therefore we must write FR for box type integrals and then to pentagon type integral.

FR by deforming propagators

The methods presented before does not work for arbitrary integrals. For example, we did not found functional equation for the two-loop vacuum type integral.
To find functional relation for the L-loop Feynman integral depending on E- external momenta we will start from the relation of the form

$$\prod_{r=1}^{n} \frac{1}{\widetilde{D}_r} = \frac{1}{\widetilde{D}_{n+1}} \sum_{r=1}^{n} x_r \prod_{\substack{j=1 \\ j \neq r}}^{n} \left(\frac{1}{\widetilde{D}_j} \right), \qquad (1)$$

where $\widetilde{D}_j$ is defined as:

$$\widetilde{D}_j = \widetilde{Q}_j^2 - m_j^2 + i\epsilon.$$

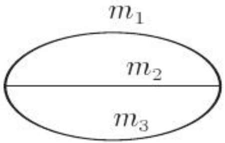
with

$$\widetilde{Q}_j = \sum_{l=1}^{L} a_{jl} k_l + \sum_{l=1}^{E+1} b_{jl} p_l,$$

and a_{jl}, b_{jl} for the time being are arbitrary scalar parameters. Some of these parameters as well as x_r and m_{n+1}^2 can be fixed from the above equation. Part of these parameters will be fixed from the requirement that the product of propagators in the equation should correspond to the integrand of the integral with the considered topology.

FR by deforming propagators

In general the integrals with modified propagators will not correspond to usual Feynman integrals. Further restrictions of parameters may be needed to obtain relations between integrals corresponding to Feynman integrals coming from a realistic quantum field theory models.



As an example let us consider derivation of FR for the two-loop vacuum type integral

$$J_0^{(d)}(m_1^2, m_2^2, m_3^2) = \int\int \frac{\mathrm{d}^d k_1 \mathrm{d}^d k_2}{(i\pi^{d/2})^2} \frac{1}{(k_1^2 - m^2)((k_1 - k_2)^2 - m^2)(k_2^2 - m^2)}.$$

FR by deforming propagators

For the product of three propagators one can try to find an algebraic relation of the form:

$$\frac{1}{\widetilde{D}_1\widetilde{D}_2\widetilde{D}_3}=\frac{x_1}{\widetilde{D}_4\widetilde{D}_2\widetilde{D}_3}+\frac{x_2}{\widetilde{D}_1\widetilde{D}_4\widetilde{D}_3}+\frac{x_3}{\widetilde{D}_1\widetilde{D}_2\widetilde{D}_4},$$

where

$$\widetilde{D}_1=(a_1k_1+a_2k_2)^2-m_1^2+i\epsilon,\quad \widetilde{D}_2=(b_1k_1+b_2k_2)^2-m_2^2+i\epsilon,$$
$$\widetilde{D}_3=(h_1k_1+h_2k_2)^2-m_3^2+i\epsilon,\quad \widetilde{D}_4=(r_1k_1+r_2k_2)^2-m_4^2+i\epsilon,$$

m_k are arbitrary masses, $x_k,\ a_j,\ b_i,\ h_s,\ r_l$ are undetermined unknowns and $k_1,\ k_2$ will be integration momenta. Bringing all the terms to a common denominator and equating coefficients in front of different powers of $k_1^2,\ k_2^2,\ k_1k_2$ and free term to zero leads to a nonlinear system of equations:

$$r_1^2-x_1a_1^2-x_2b_1^2-x_3h_1^2=0,\quad r_1r_2-x_1a_1a_2-x_2b_1b_2-x_3h_1h_2=0,$$
$$r_2^2-x_1a_2^2-x_2b_2^2-x_3h_2^2=0,\quad m_4^2-x_1m_1^2-x_2m_2^2-x_3m_3^2=0.$$

FR by deforming propagators

Solution of this system reads:

$$
\begin{aligned}
r_1 &= r_2\lambda, \\
Ax_1 &= r_2^2(h_1h_2m_2^2 - b_2b_1m_3^2) + b_2h_2(b_1h_2 - b_2h_1)m_4^2 - r_2^2(h_2^2m_2^2 - m_3^2b_2^2)\ \lambda, \\
Ax_2 &= -r_2^2(m_1^2h_2h_1 - a_2a_1m_3^2) - a_2h_2(a_1h_2 - a_2h_1)m_4^2 + r_2^2(h_2^2m_1^2 - m_3^2a_2^2)\ \lambda, \\
Ax_3 &= r_2^2(m_1^2b_2b_1 - a_2a_1m_2^2) + a_2b_2(a_1b_2 - a_2b_1)m_4^2 - r_2^2(b_2^2m_1^2 - m_2^2a_2^2)\ \lambda.
\end{aligned}
$$

where λ is a root of the quadratic equation

$$A\lambda^2 + B\lambda + C = 0,$$

and

$$
\begin{aligned}
A &= b_2h_2(b_1h_2 - h_1b_2)m_1^2 + a_2h_2(h_1a_2 - a_1h_2)m_2^2 + a_2b_2(a_1b_2 - b_1a_2)m_3^2, \\
B &= (h_1b_2 - b_1h_2)(b_1h_2 + h_1b_2)m_1^2 + (a_1h_2 - h_1a_2)(a_1h_2 + h_1a_2)m_2^2 \\
&\quad + (b_1a_2 - a_1b_2)(a_1b_2 + b_1a_2)m_3^2, \\
C &= b_1h_1(b_1h_2 - h_1b_2)m_1^2 + a_1h_1(a_2h_1 - a_1h_2)m_2^2 + a_1b_1(a_1b_2 - b_1a_2)m_3^2 \\
&\quad + (b_2h_1 - b_1h_2)(a_1h_2 - a_2h_1)(a_1b_2 - a_2b_1)m_4^2.
\end{aligned}
$$

FR by deforming propagators

To obtain FR for the two-loop vacuum integral, first we integrate algebraic relation with respect to k_1, k_2 and then transform these integrals into the α -parametric representation. Transforming all propagators into a parametric form

$$\frac{1}{(k^2-m^2+i\epsilon)^{\nu}} = \frac{i^{-\nu}}{\Gamma(\nu)} \int_0^{\infty} d\alpha\ \alpha^{\nu-1} \exp\left[i\alpha(k^2-m^2+i\epsilon)\right],$$

and using the d- dimensional Gaussian integration formula

$$\int d^d k \exp\left[i(ak^2+2(pk))\right] = i\left(\frac{\pi}{ia}\right)^{\frac{d}{2}} \exp\left[-\frac{ip^2}{a}\right],$$

we evaluate the integrals over loop momenta. The final result is:

$$\widetilde{J}_0^{(d)}(m_1^2,m_2^2,m_3^2) = \left(\frac{\pi}{i}\right)^d \frac{1}{i} \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} \frac{d\alpha_1 d\alpha_2 d\alpha_3}{[\widetilde{D}(\alpha)]^{\frac{d}{2}}} \exp\left[-i\sum_{l=1}^{3}\alpha_l(m_l^2-i\epsilon)\right],$$

where

$$\widetilde{D} = (a_1b_2-a_2b_1)^2\alpha_1\alpha_2 + (a_1h_2-a_2h_1)^2\alpha_1\alpha_3 + (b_1h_2-b_2h_1)^2\alpha_2\alpha_3,$$

differs from the D form of the original two-loop vacuum integral

$$D = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3.$$

FR by deforming propagators

Rescaling $\alpha_j \to \alpha_j \theta_j^2$ with

$$\theta_1 = b_1 h_2 - b_2 h_1, \quad \theta_2 = a_1 h_2 - a_2 h_1, \quad \theta_3 = a_1 b_2 - a_2 b_1,$$

leads to the relation:

$$\widetilde{D} = (b_1 h_2 - b_2 h_1)^2 (a_1 h_2 - a_2 h_1)^2 (a_1 b_2 - a_2 b_1)^2 \, (\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3)$$

and therefore

$$\int\int \frac{d^d k_1 \; d^d k_2}{\widetilde{D}_1 \widetilde{D}_2 \widetilde{D}_3} = [\theta_1^2\theta_2^2\theta_3^2]^{\frac{2-d}{2}} \int_0^\infty \int_0^\infty \int_0^\infty \frac{d\alpha_1 d\alpha_2 d\alpha_3}{D^{\frac{d}{2}}} \exp\left[-i\mathcal{M}\right]$$

$$= [\theta_1^2\theta_2^2\theta_3^2]^{\frac{2-d}{2}} J_0^{(d)}(\theta_1^2 m_1^2, \theta_2^2 m_2^2, \theta_3^2 m_3^2)$$

where

$$\mathcal{M} = \alpha_1 \theta_1^2 m_1^2 + \alpha_2 \theta_2^2 m_2^2 + \alpha_3 \theta_3^2 m_3^2.$$

FR by deforming propagators

Integrating algebraic relation w.r.t. k_1, k_2 and using the above relation we get:

$$[\theta_1^2\theta_2^2\theta_3^2]^{\frac{2-d}{2}} J_0^{(d)}(\theta_1^2 m_1^2, \theta_2^2 m_2^2, \theta_3^2 m_3^2) = x_1[\theta_1^2\theta_6^2\theta_4^2]^{\frac{2-d}{2}} J_0^{(d)}(\theta_1^2 m_4^2, \theta_6^2 m_2^2, \theta_4^2 m_3^2)$$
$$+x_2[\theta_2^2\theta_5^2\theta_6^2]^{\frac{2-d}{2}} J_0^{(d)}(\theta_6^2 m_1^2, \theta_2^2 m_4^2, \theta_5^2 m_3^2)$$
$$+x_3[\theta_3^2\theta_4^2\theta_5^2]^{\frac{2-d}{2}} J_0^{(d)}(\theta_4^2 m_1^2, \theta_5^2 m_2^2, \theta_3^2 m_4^2),$$

where θ_4,θ_5,θ_6 are

$$\theta_4 = r_1 b_2 - r_2 b_1, \qquad \theta_5 = r_1 a_2 - r_2 a_1, \qquad \theta_6 = r_1 h_2 - r_2 h_1,$$

By changing integration variables in the integral on the left hand side

$$k_1 = (\theta_1\theta_2\theta_3)^{\frac{1}{2}} \widetilde{k}_1, \qquad k_2 = (\theta_1\theta_2\theta_3)^{\frac{1}{2}} \widetilde{k}_2,$$

and performing analogous changes for the integrals on the right hand side we obtain the relation

$$\frac{1}{\theta_1\theta_2\theta_3} J_0\left(\frac{\theta_1}{\theta_2\theta_3} m_1^2, \frac{\theta_2}{\theta_1\theta_3} m_2^2, \frac{\theta_3}{\theta_1\theta_2} m_3^2\right) = \frac{x_1}{\theta_1\theta_4\theta_6} J_0\left(\frac{\theta_1}{\theta_4\theta_6} m_4^2, \frac{\theta_6}{\theta_1\theta_4} m_2^2, \frac{\theta_4}{\theta_1\theta_6} m_3^2\right)$$
$$+\frac{x_2}{\theta_2\theta_5\theta_6} J_0\left(\frac{\theta_6}{\theta_2\theta_5} m_1^2, \frac{\theta_2}{\theta_6\theta_5} m_4^2, \frac{\theta_5}{\theta_2\theta_6} m_3^2\right)$$
$$+\frac{x_3}{\theta_3\theta_4\theta_5} J_0\left(\frac{\theta_4}{\theta_3\theta_5} m_1^2, \frac{\theta_5}{\theta_3\theta_4} m_2^2, \frac{\theta_3}{\theta_5\theta_4} m_4^2\right).$$

FR by deforming propagators

In terms of redefined masses M_1, M_2, M_3 related to original masses m_1, m_2, m_3 as

$$m_1^2 = \frac{\theta_2\theta_3}{\theta_1} M_1^2, \quad m_2^2 = \frac{\theta_1\theta_3}{\theta_2} M_2^2, \quad m_3^2 = \frac{\theta_1\theta_2}{\theta_3} M_3^2,$$

we get

$$J_0(M_1^2, M_2^2, M_3^2) = \frac{\theta_2\theta_3}{\widetilde{\theta}_4\widetilde{\theta}_6} \widetilde{x}_1 J_0\left(\frac{\theta_1}{\widetilde{\theta}_4\widetilde{\theta}_6} m_4^2, \frac{\theta_3\widetilde{\theta}_6}{\theta_2\widetilde{\theta}_4}\ M_2^2, \frac{\theta_2\widetilde{\theta}_4}{\theta_3\widetilde{\theta}_6}\ M_3^2\right)$$
$$+ \frac{\theta_1\theta_3}{\widetilde{\theta}_5\widetilde{\theta}_6}\ \widetilde{x}_2 J_0\left(\frac{\theta_3\widetilde{\theta}_6}{\theta_1\widetilde{\theta}_5} M_1^2, \frac{\theta_2}{\widetilde{\theta}_5\widetilde{\theta}_6} m_4^2, \frac{\theta_1\widetilde{\theta}_5}{\theta_3\widetilde{\theta}_6}\ M_3^2\right)$$
$$+ \frac{\theta_1\theta_2}{\widetilde{\theta}_4\widetilde{\theta}_5}\ \widetilde{x}_3 J_0\left(\frac{\theta_2\widetilde{\theta}_4}{\theta_1\widetilde{\theta}_5} M_1^2, \frac{\theta_1\widetilde{\theta}_5}{\theta_2\widetilde{\theta}_4}\ M_2^2, \frac{\theta_3}{\widetilde{\theta}_4\widetilde{\theta}_5} m_4^2\right),$$

where m_4^2 is an arbitrary mass.

FR by deforming propagators

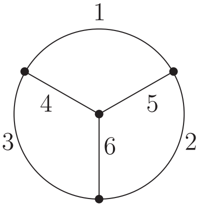
At $m_4 = 0$ dependence on all parameters a_i, b_j, h_k drops out and we get

$$
\begin{aligned}
&J_0(M_1^2, M_2^2, M_3^2) \\
&= J_0\left(0, \frac{-M_1^2 + M_2^2 + M_3^2 + \sqrt{\Delta_2}}{2}, \frac{-M_1^2 + M_2^2 + M_3^2 - \sqrt{\Delta_2}}{2}\right) \\
&+ J_0\left(\frac{M_1^2 - M_2^2 + M_3^2 + \sqrt{\Delta_2}}{2}, 0, \frac{M_1^2 - M_2^2 + M_3^2 - \sqrt{\Delta_2}}{2}\right) \\
&- J_0\left(\frac{-M_1^2 - M_2^2 + M_3^2 + \sqrt{\Delta_2}}{2}, \frac{-M_1^2 - M_2^2 + M_3^2 - \sqrt{\Delta_2}}{2}, 0\right).
\end{aligned}
$$

where

$$\Delta_2 = M_1^4 + M_2^4 + M_3^4 - 2M_1^2 M_2^2 - 2M_1^2 M_3^2 - 2M_2^2 M_3^2.$$

FR for the three-loop vacuum type integral



Consider derivation of FR for the three -loop vacuum type integral with arbitrary masses:

$$U_6^{(d)}(m_1^2, m_2^2, m_3^2, m_4^2, m_5^2, m_6^2) \equiv \int\int \frac{\mathrm{d}^d k_1 \mathrm{d}^d k_2 \mathrm{d}^d k_3}{(i\pi^{d/2})^3} \frac{1}{D_1 D_2 D_3 D_4 D_5 D_6},$$

where

$$D_1 = k_1^2 - m_1^2 + i\epsilon, \qquad D_2 = k_2^2 - m_2^2 + i\epsilon,$$
$$D_3 = k_3^2 - m_3^2 + i\epsilon, \qquad D_4 = (k_1 - k_3)^2 - m_4^2 + i\epsilon,$$
$$D_5 = (k_1 - k_2)^2 - m_5^2 + i\epsilon, \quad D_6 = (k_2 - k_3)^2 - m_6^2 + i\epsilon.$$

FR for the three-loop vacuum type integral

Parametric representation of this integral reads

$$U_6^{(d)}(m_1^2,m_2^2,m_3^2,m_4^2,m_5^2,m_6^2) = \int_0^\infty \dots \int_0^\infty \frac{d\alpha_1 \dots d\alpha_6}{[D(\alpha)]^{\frac{d}{2}}} \exp\left[-iM^2\right],$$

where

$$M^2 = \sum_{l=1}^{6} \alpha_l (m_l^2 - i\epsilon),$$

$$D(\alpha) = \alpha_2(\alpha_1+\alpha_3)\alpha_4 + \alpha_3(\alpha_1+\alpha_2)\alpha_5 + \alpha_1(\alpha_2+\alpha_3)\alpha_6 + \alpha_1\alpha_2\alpha_3 + (\alpha_4\alpha_5+\alpha_4\alpha_6+\alpha_5\alpha_6)(\alpha_1+\alpha_2+\alpha_3).$$

Instead of original propagators we will use deformed propagators

$$\widetilde{D}_1 = (a_1k_1+a_2k_2+a_3k_3)^2 - m_1^2 + i\epsilon, \quad \widetilde{D}_2 = (b_1k_1+b_2k_2+b_3k_3)^2 - m_2^2 + i\epsilon,$$
$$\widetilde{D}_3 = (c_1k_1+c_2k_2+c_3k_3)^2 - m_3^2 + i\epsilon, \quad \widetilde{D}_4 = (d_1k_1+d_2k_2+d_3k_3)^2 - m_4^2 + i\epsilon,$$
$$\widetilde{D}_5 = (e_1k_1+e_2k_2+e_3k_3)^2 - m_5^2 + i\epsilon, \quad \widetilde{D}_6 = (h_1k_1+h_2k_2+h_3k_3)^2 - m_6^2 + i\epsilon,$$
$$\widetilde{D}_7 = (r_1k_1+r_2k_2+r_3k_3)^2 - m_7^2 + i\epsilon,$$

FR for the three-loop vacuum type integral

For the product of six propagators one can try to find an algebraic relation of the form:

$$\frac{1}{\widetilde{D}_1\widetilde{D}_2\widetilde{D}_3\widetilde{D}_4\widetilde{D}_5\widetilde{D}_6} = \frac{x_1}{\widetilde{D}_7\widetilde{D}_2\widetilde{D}_3\widetilde{D}_4\widetilde{D}_5\widetilde{D}_6} + \frac{x_2}{\widetilde{D}_7\widetilde{D}_1\widetilde{D}_3\widetilde{D}_4\widetilde{D}_5\widetilde{D}_6} + \frac{x_3}{\widetilde{D}_7\widetilde{D}_1\widetilde{D}_2\widetilde{D}_4\widetilde{D}_5\widetilde{D}_6}$$
$$+ \frac{x_4}{\widetilde{D}_7\widetilde{D}_1\widetilde{D}_2\widetilde{D}_3\widetilde{D}_5\widetilde{D}_6} + \frac{x_5}{\widetilde{D}_7\widetilde{D}_1\widetilde{D}_2\widetilde{D}_3\widetilde{D}_4\widetilde{D}_6} + \frac{x_6}{\widetilde{D}_7\widetilde{D}_1\widetilde{D}_2\widetilde{D}_3\widetilde{D}_4\widetilde{D}_5},$$

where m_k are arbitrary masses, x_k are undetermined unknowns. Bringing all terms to the common denominator and setting coefficients in front of $k_1^2,\ k_2^2, k_3^2,\ k_1k_2,\ k_1k_3,\ k_2k_3$ as well as free term to zero, leads to a nonlinear system of 7 equations:

$$\begin{aligned}
&a_1^2x_1 + b_1^2x_2 + c_1^2x_3 + d_1^2x_4 + e_1^2x_5 + h_1^2x_6 - r_1^2 = 0,\\
&a_2^2x_1 + b_2^2x_2 + c_2^2x_3 + d_2^2x_4 + e_2^2x_5 + h_2^2x_6 - r_2^2 = 0,\\
&a_3^2x_1 + b_3^2x_2 + c_3^2x_3 + d_3^2x_4 + e_3^2x_5 + h_3^2x_6 - r_3^2 = 0,\\
&a_1a_2x_1 + b_1b_2x_2 + c_1c_2x_3 + d_1d_2x_4 + e_1e_2x_5 + h_1h_2x_6 - r_1r_2 = 0,\\
&a_1a_3x_1 + b_1b_3x_2 + c_1c_3x_3 + d_1d_3x_4 + e_1e_3x_5 + h_1h_3x_6 - r_1r_3 = 0,\\
&a_2a_3x_1 + b_2b_3x_2 + c_2c_3x_3 + d_2d_3x_4 + e_2e_3x_5 + h_2h_3x_6 - r_2r_3 = 0,\\
&-m_1^2x_1 - m_2^2x_2 - m_3^2x_3 - m_4^2x_4 - m_5^2x_5 - m_6^2x_6 + m_7^2 = 0.
\end{aligned}$$

FR for the three-loop vacuum type integral

Integrating this relation with respect to k_1, k_2, k_3 leads to the relation

$$P^{(a,b,c,d,e,h)}(m_1^2, m_2^2, m_3^2, m_4^2, m_5^2, m_6^2)$$
$$= x_1 P^{(r,b,c,d,e,h)}(m_7^2, m_2^2, m_3^2, m_4^2, m_5^2, m_6^2) + x_2 P^{(a,r,c,d,e,h)}(m_1^2, m_7^2, m_3^2, m_4^2, m_5^2, m_6^2)$$
$$+x_3 P^{(a,b,r,d,e,h)}(m_1^2, m_2^2, m_7^2, m_4^2, m_5^2, m_6^2) + x_4 P^{(a,b,c,r,e,h)}(m_1^2, m_2^2, m_3^2, m_7^2, m_5^2, m_6^2)$$
$$+x_5 P^{(a,b,c,d,r,h)}(m_1^2, m_2^2, m_3^2, m_4^2, m_7^2, m_6^2)$$
$$+x_6 P^{(a,b,c,d,e,r)}(m_1^2, m_2^2, m_3^2, m_4^2, m_5^2, m_7^2),$$

where

$$P^{(a,b,c,d,e,h)}(m_1^2, m_2^2, m_3^2, m_4^2, m_5^2, m_6^2) = \int\int\int \frac{d^d k_1 d^d k_2 d^d k_3}{\widetilde{D}_1 \widetilde{D}_2 \widetilde{D}_3 \widetilde{D}_4 \widetilde{D}_5 \widetilde{D}_6},$$

This integral can be written in parametric form as it was done for the two-loop case:

$$P^{(a,b,c,d,e,h)}(m_1^2, m_2^2, m_3^2, m_4^2, m_5^2, m_6^2)$$
$$= \frac{1}{i^3}\left(\frac{\pi}{i}\right)^{3d/2} \prod_{j=1}^{6} \int_0^\infty \ldots \int_0^\infty \frac{d\alpha_j}{[\widetilde{D}(\alpha)]^{\frac{d}{2}}} \exp\left[-i\widetilde{M}^2\right],$$

where

$$\widetilde{M}^2 = \sum_{l=1}^{6} \alpha_l (m_l^2 - i\epsilon),$$

FR for the three-loop vacuum type integral

and $\widetilde{D}(\alpha)$ is 20 terms polynomial in α with coefficients depending on 18 parameters a, b, c, d, e, h

$$\widetilde{D}(\alpha) = \sum_{jkl} \tau_{jkl}(a, b, c, d, e, h) \ \ \alpha_j \alpha_k \alpha_l.$$

The polynomial $\widetilde{D}(\alpha)$ differs from the $D(\alpha)$ for the original 3-loop vacuum integral. $D(\alpha)$ has 16 terms. Four extra terms proportional to products $\alpha_1\alpha_2\alpha_5, \alpha_1\alpha_3\alpha_4, \alpha_2\alpha_3\alpha_6, \alpha_4\alpha_5\alpha_6$, which are absent in $D(\alpha)$, can be eliminated from $\widetilde{D}(\alpha)$ by appropriate choice of parameters a, b, c, d, e, h.
To eliminate the four terms one should solve 4 nonlinear equations wrt $a, \ b, \ \ldots$

$$a_1 b_2 e_3 - a_1 b_3 e_2 - a_2 b_1 e_3 + a_2 b_3 e_1 + a_3 b_1 e_2 - a_3 b_2 e_1 = 0,$$
$$a_1 c_2 d_3 - a_1 c_3 d_2 - a_2 c_1 d_3 + a_2 c_3 d_1 + a_3 c_1 d_2 - a_3 c_2 d_1 = 0,$$
$$b_1 c_2 h_3 - b_1 c_3 h_2 - b_2 c_1 h_3 + b_2 c_3 h_1 + b_3 c_1 h_2 - b_3 c_2 h_1 = 0,$$
$$d_1 e_2 h_3 - d_1 e_3 h_2 - d_2 e_1 h_3 + d_2 e_3 h_1 + d_3 e_1 h_2 - d_3 e_2 h_1 = 0.$$

Using Maple we obtained 86 solutions of this system. Only 14 of them lead to $\widetilde{D}(\alpha)$ with 16 terms. It turns out that by the rescaling $\alpha_k \to \beta_k \alpha_k$ one can choose β_k so that all these 14 polynomials $\widetilde{D}(\alpha)$ will be proportional to $D(\alpha)$ multiplied by some factor depending on the parameters $a_j, \ldots$.

FR for the three-loop vacuum type integral

For all 14 solutions, we have obtained x_k, $(k = 1..6)$ and m_7^2 as functions of the remaining parameters and m_k^2 .
Substitution of 14 solutions into the equation will give FR for the original integral in terms of P integrals corresponding to $\widetilde{D}(\alpha)$ with 17 and 18 terms. These integrals are a new type of integrals.

By choosing parameters of the additional propagator in the r.h.s of the FR we can eliminate 3 terms from the FR.

At the next steps one can derive FR for the new integrals with 17 and 18 terms in $\widetilde{D}(\alpha)$.
At this stage we fixed only 4 parameters thus keeping maximal number of free parameters $a, b, \ldots$.

Repeating this process we discovered quite a remarkable property.

FR for the three-loop vacuum type integral

To illustrate this property we consider the case when parameters a, b, ... correspond to the original integral and the additional propagator $1/D_7$ depends on 3 arbitrary parameters.

$$\frac{1}{D_1D_2D_3D_4D_5D_6} = \frac{x_1}{\widetilde{D}_7D_2D_3D_4D_5D_6} + \frac{x_2}{\widetilde{D}_7D_1D_3D_4D_5D_6} + \frac{x_3}{\widetilde{D}_7D_1D_2D_4D_5D_6} + \frac{x_4}{\widetilde{D}_7D_1D_2D_3D_5D_6} + \frac{x_5}{\widetilde{D}_7D_1D_2D_3D_4D_6} + \frac{x_6}{\widetilde{D}_7D_1D_2D_3D_4D_5},$$

Integrating this equation wrt k_1, k_2, k_3 we get FR for our original integral in terms of new integrals. By choosing in D_7 one parameter, three terms in FR can be eliminated.

At the next step we write FR for the terms with one additional propagator in terms of new integrals with one more additional propagator

$$\frac{1}{\widetilde{D}_7D_2D_3D_4D_5D_6} \to \frac{y_1}{\widetilde{D}_7\widetilde{D}_8D_3D_4D_5D_6} + \dots$$

We repeated this procedure two more times.

FR for the three-loop vacuum type integral

In terms of integrals it will correspond to the following steps:

First step

$$U_6^{(d)}(m_1^2, m_2^2, m_3^2, m_4^2, m_5^2, m_6^2) \to \text{ terms like } P_1(\{\widetilde{m}\}, k_2 + h_3 k_3).$$

Second step

$$P_1(\{\widetilde{m}\}, k_2 + h_3 k_3) \to \text{ terms like } P_2(\{\widetilde{m}\}, k_2 + h_3 k_3,\ k_2 + c_3 k_3) \text{ and } P_1.$$

Third step

$$P_2(\{\widetilde{m}\}, k_2 + h_3 k_3, k_2 + c_3 k_3) \to \text{ terms like } P_3(\{\widetilde{m}\}, k_2 + h_3 k_3, k_2 + c_3 k_3, k_1 + b_2 k_2) \text{ and } P_2.$$

Fourth step

$$P_3(\{\widetilde{m}\}, k_2 + h_3 k_3, k_2 + c_3 k_3, k_1 + b_2 k_2) \to \text{ terms like } P_4(\{\widetilde{m}\}, k_2 + h_3 k_3, k_2 + c_3 k_3, k_1 + b_2 k_2, k_1 + u_2 k_2) \text{ and } P_3.$$

FR for the three-loop vacuum type integral

It turns out that by permuting and scaling α's the $\widetilde{D}(\alpha)$ polynomial for P_4 can be expressed in terms of $\widetilde{D}(\alpha)$ polynomial for P_3. It means that the integral with 4 modified propagators is reducible to the integral with 3 modified propagators.

$$P_4(\{\widetilde{m}\}, k_2 + h_3 k_3, k_2 + c_3 k_3, k_1 + b_2 k_2, k_1 + u_2 k_2) \rightarrow$$
$$P_3(\{\widetilde{\widetilde{m}}\}, k_2 + \widetilde{h}_3 k_3, k_2 + \widetilde{c}_3 k_3, k_1 + \widetilde{b}_2 k_2).$$

As a result, we obtain a FR which includes only the function P_3 depending on 3 arbitrary parameters and arbitrary masses.

One can formulate for the P_3 a functional reduction procedure similar to one-loop integrals and to simplify it as much as possible by reducing to P_3 with simple combinations of masses. Then substitute it to the equation for P_2. Simplify P_2 and substitute it into the equation for P_1. Then simplify P_1 and substitute it into the equation for $U_6^{(d)}$.

Summary

Summary

- Applying presented methods one can get sets of FR for various types of multiloop integrals.
- The most general method is based on the method of deformed propagators
- Further modifications of all methods are needed
- Careful investigation of the solutions of nonlinear algebraic systems are needed
- Probably derivations based on other parametric representations of integrals will produce more functional relations