The modern status of heavy quark physics in the covariant confined quark model

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Covariant Constituent Quark Model

The CCQM is based on a phenomenological relativistic Lagrangian describing the coupling of a hadron H to its constituents:

$$\mathcal{L}_{\rm int} = g_H H(x) J_H(x) + \mathrm{H.c.}$$

The coupling constant g_H is determined from the so-called *compositeness condition*, which was proposed by Salam and Weinberg.

The quark currents $J_H(x)$ have the nonlocal shapes as

$$J_M(x) = \int dx_1 \int dx_2\, F_M(x; x_1, x_2) \cdot \bar{q}^a_{f_1}(x_1)\, \Gamma_M\, q^a_{f_2}(x_2) \qquad \textbf{Meson}$$

$$J_B(x) = \int dx_1 \int dx_2 \int dx_3\, F_B(x; x_1, x_2, x_3) \qquad \textbf{Baryon}$$
$$\times\ \Gamma_1\, q^{a_1}_{f_1}(x_1) \left[\varepsilon^{a_1 a_2 a_3} q^{T\, a_2}_{f_2}(x_2) C\, \Gamma_2\, q^{a_3}_{f_3}(x_3) \right]$$

$$J_T(x) = \int dx_1 \ldots \int dx_4\, F_T(x; x_1, \ldots, x_4) \qquad \textbf{Tetraquark}$$
$$\times\ \left[\varepsilon^{a_1 a_2 c} q^{T\, a_1}_{f_1}(x_1)\, C\Gamma_1\, q^{a_2}_{f_2}(x_2) \right] \cdot \left[\varepsilon^{a_3 a_4 c} \bar{q}^{T\, a_3}_{f_3}(x_3)\, \Gamma_2 C\, \bar{q}^{a_4}_{f_4}(x_4) \right]$$

Vertex functions

Translational invariance of the vertex functions:

$$F_H(x+a, x_1+a, \ldots, x_n+a) = F_H(x, x_1, \ldots, x_n)\,, \qquad \forall a\,.$$

Our choice:

$$F_H(x, x_1, \ldots, x_n) \;=\; \delta^{(4)}\Big(x - \sum_{i=1}^{n} w_i x_i\Big)\; \Phi_H\Big(\sum_{i<j}(x_i - x_j)^2\Big)$$

where $w_i = m_i / (\sum_{j=1}^{n} m_j)$ and m_i is the mass of the quark at point x_i.

The vertex function Φ_H is written as

$$\Phi_H\Big(\sum_{i<j}(x_i - x_j)^2\Big) = \prod_{i=1}^{n-1}\Big[\int \frac{dq_i}{(2\pi)^4}\Big] e^{-i\sum_{j=1}^{n-1} q_j (x_j - x_n)}\, \widetilde{\Phi}_H\Big(-\vec{\Omega}_q^2\Big)\,,$$

$$\widetilde{\Phi}_H\Big(-\vec{\Omega}_q^2\Big) = \exp\Big(\vec{\Omega}_q^2/\Lambda_H^2\Big)\,, \qquad \vec{\Omega}_q^2 = \frac{1}{2}\sum_{i\le j} q_i q_j\,.$$

Evaluation of the diagrams

By integrating the loop momenta one can arrive at the following representation for the Feynman diagram with n-propagators.

$$\Pi = \int_0^\infty d^n\alpha\, F(\alpha_1, \ldots, \alpha_n)\,,$$

where F stands for the whole structure of a given diagram.
The set of Schwinger parameters α_i can be turned into a simplex by introducing an additional t–integration via the identity

$$1 = \int_0^\infty dt\, \delta(t - \sum_{i=1}^{n} \alpha_i)$$

leading to

$$\Pi = \int_0^\infty dt t^{n-1} \int_0^1 d^n\alpha\, \delta\Big(1 - \sum_{i=1}^{n} \alpha_i\Big)\, F(t\alpha_1, \ldots, t\alpha_n)\,.$$

The variable t is analogous to the Fock-Schwinger proper time.

Infrared confinement

- Cut off the upper integration at $1/\lambda^2$

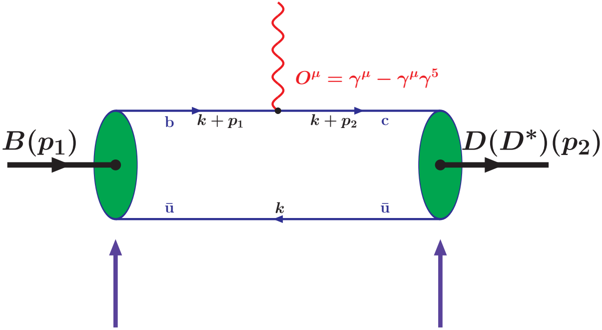
$$\Pi^c = \int\limits_0^{1/\lambda^2} dt t^{n-1} \int\limits_0^1 d^n\alpha\, \delta\Big(1 - \sum_{i=1}^{n} \alpha_i\Big)\, F(t\alpha_1, \ldots, t\alpha_n)$$

- The infrared cut-off has removed all possible thresholds in the quark loop diagram.

- We take the cut-off parameter λ to be the same in all physical processes.

Heavy Quark Limit

Heavy quark limit in $B-D$ transition



$$\phi_B\left(-(k+w_u\,p_1)^2\right) \quad \phi_{D(D^*)}\left(-(k+w'_u\,p_2)^2\right)$$

$$w_u = \frac{m_u}{m_u+m_b} \qquad w'_u = \frac{m_u}{m_u+m_c}$$

Heavy quark limit: $m_H = m_Q + E, \quad m_Q \to \infty; \quad \Lambda_B = \Lambda_D = \Lambda_{D^*}.$

Isgur-Wise function

$$\frac{1}{m_i - \not{k} - \not{p}_i} \to -\frac{1+\not{v}_i}{2} \cdot \frac{1}{kv_i + E}, \qquad v_i = \frac{p_i}{m_i}$$

$$M^{\mu}_{BD}(p_1, p_2) = f_+(q^2)(p_1 + p_2)^{\mu} + f_-(q^2)(p_1 - p_2)^{\mu},$$

$$f_{\pm} = \pm\frac{m_1 \pm m_2}{2\sqrt{m_1 m_2}} \cdot \xi(w), \qquad w = v_1 \cdot v_2.$$

As a result we get the Isgur-Wise function

$$\xi(w) = \frac{J_3(E,w)}{J_3(E,1)}, \qquad J_3(E,w) = \int\limits_0^1 \frac{d\tau}{W} \int\limits_0^{\infty} du\, \widetilde{\Phi}^2(z) \frac{m_u + \sqrt{u/W}}{m_u^2 + z}$$

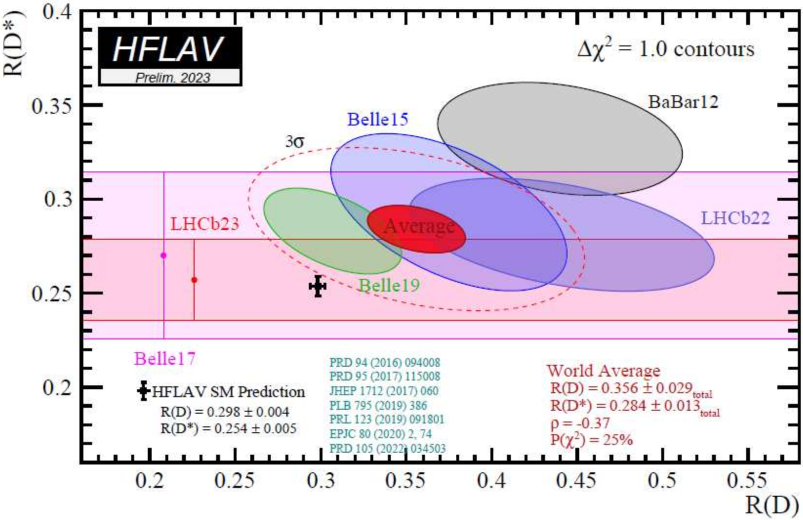
where $W = 1 + 2\tau(1-\tau)(w-1)$, $z = u - 2E\sqrt{u/W}$.

Some applications

- Semileptonic, nonleptonic and rare (B, B_s, B_c) and (D, D_s)– decays
- Heavy-to-light semileptonic decays of Λ_b and Λ_c
- Rare decays $\Lambda_b \to \Lambda + \ell^+\ell^-$
- Polarization effects in the cascade decay
  $\Lambda_b \to \Lambda(\to p\pi^-) + J/\psi(\to \ell^+\ell^-)$
- Analyzing new physics in the decays $\bar{B}^0 \to D^{(*)}\tau^-\bar{\nu}_\tau$ and
  $\Lambda_b \to \Lambda_c + \tau^- + \bar{\nu_\tau}$.
- Strong and radiative decays of the tetraquark state $X(3872)$.
  Four-quark structure of $Z_c(3900)$, $Z_b(10610)$, $Z_b'(10650)$ exotic states.

The state-of-art related to $R(D^{(*)})$ ratios. [arXiv:2305.08133 [hep-ex]]

$$R(D^{(*)}) = \frac{\mathcal{B}(B \to D^{(*)}\tau\nu_\tau)}{\mathcal{B}(B \to D^{(*)}\ell\nu_\ell)}, \quad D^{(*)} = D \text{ or } D^*, \quad \ell = e \text{ or } \mu$$



The world average shows a 3.2σ tension with the SM prediction.

Analyzing New Physics in the decays $B \to D^{(*)}\tau\nu_\tau$

SM+NP effective Hamiltonian for the quark-level transition $b \to c\tau^-\bar{\nu}_\tau$:

$$\mathcal{H}_{eff} \propto G_F V_{cb}\left[(1+V_L)\mathcal{O}_{V_L} + V_R\mathcal{O}_{V_R} + S_L\mathcal{O}_{S_L} + S_R\mathcal{O}_{S_R} + T_L\mathcal{O}_{T_L}\right]$$

where the four-fermion operators are written as

$$\mathcal{O}_{V_L} = (\bar{c}\gamma^\mu P_L b)(\bar{\tau}\gamma_\mu P_L\nu_\tau) \qquad \mathcal{O}_{V_R} = (\bar{c}\gamma^\mu P_R b)(\bar{\tau}\gamma_\mu P_L\nu_\tau)$$

$$\mathcal{O}_{S_L} = (\bar{c}P_L b)(\bar{\tau}P_L\nu_\tau) \qquad \mathcal{O}_{S_R} = (\bar{c}P_R b)(\bar{\tau}P_L\nu_\tau)$$

$$\mathcal{O}_{T_L} = (\bar{c}\sigma^{\mu\nu}P_L b)(\bar{\tau}\sigma_{\mu\nu}P_L\nu_\tau)$$

Here, $\sigma_{\mu\nu} = i\left[\gamma_\mu,\gamma_\nu\right]/2$, $\quad P_{L,R} = (1\mp\gamma_5)/2$.
$V_{L,R}$, $S_{L,R}$, and T_L are complex Wilson coefficients governing NP.

In the SM: $V_{L,R} = S_{L,R} = T_L = 0$.
NP only affects leptons of the third generation.
The best value for each NP coupling to fit the data for the ratios $R(D)$ and $R(D^*)$:

$$V_L = -1.33 + i\,1.11, \qquad V_R = 0.03 - i\,0.60,$$

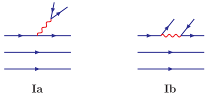
$$S_L = -1.79 - i\,0.22, \qquad T_L = 0.38 - i\,0.06.$$

Ab initio calculation of the W-exchange contribution to nonleptonic decays of double charm baryons

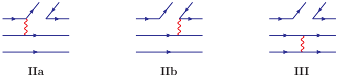
Two-body weak decays of baryons have five different quark topologies. It was considered the decays that belong to the same topological class:

$$\Xi_{cc}^{++} \quad \to \quad \Xi_c^+ \, (\Xi_c'^+) + \pi^+ (\rho^+) \qquad \textbf{T-Ia and W-IIb}$$

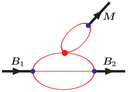
$$\Omega_{cc}^{+} \quad \to \quad \Xi_c^+ \, (\Xi_c'^+) + \bar{K}^0 (K^{*\,0}) \qquad \textbf{T-Ib and W-IIb}$$



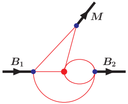
Tree diagrams



W-exchange diagrams



tree diagrams Ia, Ib



W-exchange diagram IIb

Standard Model **Covariant Quark Model**

$\Omega_{cc}^{+} \to \Xi_c'^{+} + \bar{K}^0$

Helicity	Tree diagram	W diagram	total
$H^V_{\frac{1}{2}\,t}$	0.20	-0.01	0.19
$H^A_{\frac{1}{2}\,t}$	0.25	-0.01	0.24

$$\Gamma(\Omega_{cc}^{+} \to \Xi_c'^{+} + \bar{K}^0) = 0.15 \cdot 10^{-13}\ \mathbf{GeV}$$

$\Omega_{cc}^{+} \to \Xi_c^{+} + \bar{K}^0$

Helicity	Tree diagram	W diagram	total
$H^V_{\frac{1}{2}\,t}$	-0.35	1.06	0.71
$H^A_{\frac{1}{2}\,t}$	-0.10	0.31	0.21

$$\Gamma(\Omega_{cc}^{+} \to \Xi_c^{+} + \bar{K}^0) = 0.95 \cdot 10^{-13}\ \mathbf{GeV}$$

$\Xi_c'^{+}$ contains symmetric $\{us\}$ diquark. The W–exchange contribution is strongly suppressed due to the Körner-Pati-Woo theorem.

Ξ_c^{+} contains antisymmetric $[us]$ diquark. In this case the W–exchange contribution is not a priori suppressed.

Charmonium states below $D\bar{D}$-threshold

J^{PC}	${}^{2S+1}L_J$	quark current	M_{cc} (MeV)	Γ_{cc}^{tot} (MeV)
0^{-+}	${}^1S_0 = \eta_c$	$\bar{q}\, i\gamma^5\, q$	2984.1(4)	30.5(5)
1^{--}	${}^3S_1 = J/\psi$	$\bar{q}\, \gamma^\mu\, q$	3096.900(6)	0.0926(17)
0^{++}	${}^3P_0 = \chi_{c0}$	$\bar{q}\, q$	3414.71(30)	10.5(8)
1^{++}	${}^3P_1 = \chi_{c1}$	$\bar{q}\, \gamma^\mu \gamma^5\, q$	3510.67(5)	0.88(5)
1^{+-}	${}^1P_1 = h_c(1P)$	$\bar{q}\ \overleftrightarrow{\partial}^\mu\ \gamma^5\, q$	3525.37(14)	0.78(28)
2^{++}	${}^3P_2 = \chi_{c2}$	$\frac{i}{2}\bar{q}\left(\gamma^\mu\ \overleftrightarrow{\partial}^\nu + \gamma^\nu\ \overleftrightarrow{\partial}^\mu\right) q$	3556.17(7)	2.00(11)
1^{--}	${}^3S_1 = \psi(2S)$	$\bar{q}\, \gamma^\mu\, q$	3686.097(11)	0.293(9)

Gauging of nonlocal quark current

Gauge invariance of the nonlocal strong interaction Lagrangian is provided by multiplying each quark field $q(x_i)$ by a gauge field exponentional according to

$$q(x_i) \to Q(x_i) = e^{-ie_q I(x_i,x,P)}\, q(x_i), \qquad I(x_i,x,P) = \int\limits_x^{x_i} dz_\mu A^\mu(z),$$

where P is the path taken from x to x_i.

It is readily seen that the neutral nonlocal quark current defined by

$$J^{\rm em}(x) = \iint dx_1 dx_2 \delta(x - wx_1 - wx_2)\Phi\left((x_1 - x_2)^2\right) \bar{Q}(x_1)\Gamma Q(x_2), \ (w = 1/2)$$

is invariant under the local gauge transformations

$$\begin{aligned} q(x_i) &\to e^{ie_q f(x_i)} q(x_i), \qquad \bar{q}(x_i) \to e^{-ie_q f(x_i)} \bar{q}(x_i), \\ A^\mu(z) &\to A^\mu(z) + \partial^\mu f(z) \Rightarrow I(x_i,x,P) \to I(x_i,x,P) + f(x_i) - f(x), \end{aligned}$$

if the matix Γ has no derivative.

Gauging of nonlocal quark current

Superficially the results appear to depend on the path P when one expands the gauge exponential in powers of $I(x_i, x, P)$. However, one needs to know only derivatives of the path integrals when doing the perturbative expansion. One can make use of the formalism developed in

S. Mandelstam, Annals Phys. 19, 1-24 (1962), J. Terning, Phys. Rev. D 44, no.3, 887-897 (1991)

and based on the path-independent definition of derivative of $I(x, y, P)$:

$$\frac{\partial}{\partial x^\mu} I(x, y, P) = A_\mu(x)$$

which states that the derivative of the path integral $I(x, y, P)$ does not depend on the path P originally used in the definition.

It is easy to check that such procedure of gauging the free quark lagrangian leads to the standard form of $e_q\, \bar{q}((x)\, \hat{A}(x) q(x)$.

Gauging of nonlocal quark current

The evaluation of the Feynman diagrams involving the strong quark vertex with emitting photon leads to the typical integral:

$$R(x;k_1,k_2) = \iint dx_1 dx_2\, \delta(x - w\,x_1 - w\,x_2)\Phi\left((x_1 - x_2)^2\right) I(x_1,x_2)\, e^{ik_1x_1 - ik_2x_2}$$

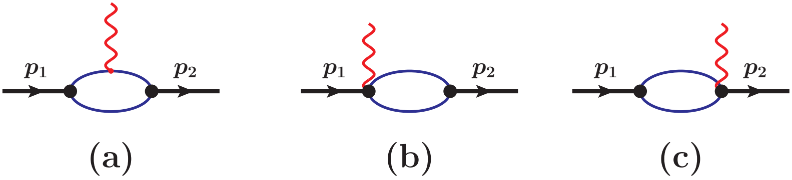
By using the definition of derivative of $I(x,y,P)$ and choosing the free electromagnetic field as $A_\alpha(x) = \epsilon^*_\alpha e^{iqx}$ one has

$$\begin{aligned} R(x;k_1,k_2) &= i\,\epsilon^*_\alpha\, e^{i(k_1-k_2+q)x} \\ &\times \int_0^1 d\tau \left\{\widetilde{\Phi}'\left(-z^+_\tau\right)(k+w^2q)^\alpha + \widetilde{\Phi}'\left(-z^-_\tau\right)(k-w^2q)^\alpha\right\} \end{aligned}$$

where $k = \frac{1}{2}(k_1+k_2)$ and $z^\pm_\tau = (k \pm w\,q)^2\,\tau + k^2\,1 - \tau)$.

Radiative decays $(\bar{c}c)_1 \to (\bar{c}c)_2 + \gamma$: Feynman diagrams

Diagrams describing the $(\bar{c}c_1) \to (\bar{c}c)_2 + \gamma$ transition are shown below.



We start with the group of decays $\chi_{cJ}(p_1) \to J/\psi(p_2) + \gamma(q)$ $(J = 0, 1, 2)$. The invariant matrix element describing these decays are written as

$$M_{\chi_{cJ} \to J/\psi + \gamma} = 6\, g_{\chi_{cJ}}\, g_{J/\psi}\, e_q\, \epsilon^{*}_{2\,\beta}(p_2)\, \epsilon^{*}_{\gamma\,\alpha}(q) \left(M^{\beta\alpha}_{\triangle\, a} + M^{\beta\alpha}_{\bigcirc\, b} + M^{\beta\alpha}_{\bigcirc\, c} \right)$$

The amplitudes $M^{\beta\alpha}$ is written via the loop integrals corresponding to the Feynman diagrams.

Radiative decays $\chi_{cJ}(p_1) \to J/\psi(p_2) + \gamma(q)$: loop integrals

$$
\begin{aligned}
M^{\beta\alpha}_{\triangle\, a} &= -\int \frac{d^4\ell}{(2\pi)^4 i} \widetilde{\Phi}_{cJ}\big(-(\ell + wp_1)^2\big)\widetilde{\Phi}_{J/\psi}\big(-(\ell+wp_2)^2\big) \\
&\quad \times \mathrm{tr}\big[\gamma^\beta S(\ell+p_2)\gamma^\alpha S(\ell+p_1)\tilde{\Gamma}_{cJ} S(\ell)\big] \\
M^{\beta\alpha}_{\bigcirc\, b} &= +\int \frac{d^4\ell}{(2\pi)^4 i} \int\limits_0^1 d\tau\, \widetilde{\Phi}'_{cJ}(-z^+_\tau)\widetilde{\Phi}_{J/\psi}\big(-\ell^2\big)\,\ell^\alpha \\
&\quad \times \mathrm{tr}\big[\gamma^\beta S(\ell+wp_2)\tilde{\Gamma}_{cJ} S(\ell - wp_2)\big] \\
M^{\beta\alpha}_{\bigcirc\, c} &= +\int \frac{d^4\ell}{(2\pi)^4 i} \widetilde{\Phi}_{cJ}\big(-\ell^2\big) \int\limits_0^1 d\tau\, \widetilde{\Phi}'_{J/\psi}(-z^+_\tau)\,\ell^\alpha \\
&\quad \times \mathrm{tr}\big[\gamma^\beta S(\ell+wp_1)\tilde{\Gamma}_{cJ} S(\ell - wp_1)\big]
\end{aligned}
$$

Here $\tilde{\Gamma}_{c0} = I$, $\tilde{\Gamma}_{c1} = \epsilon_\mu(p_1)\gamma^\mu\gamma_5$, $\tilde{\Gamma}_{c2} = 2\,\epsilon_{\mu\nu}(p_1)\ell^\mu\gamma^\nu$

and $z^+_\tau = (\ell + w\,q)^2\,\tau + \ell^2\,(1-\tau)$

Check the gauge invariance

The first step is to check the gauge invariance before the loop integration:

$$M^{\beta\alpha} q_\alpha = 0.$$

It maybe done by using two identities:

$$\begin{aligned} S(\ell + p_2)\, \not{q}\, S(\ell + p_1) &= S(\ell + p_1) - S(\ell + p_2)\,, \\ \int_0^1 d\tau \widetilde{\Phi}'(-z_\tau)(\ell + w^2 q)^\alpha q_\alpha &= \widetilde{\Phi}(-\ell^2) - \widetilde{\Phi}(-(\ell + wq)^2) \end{aligned}$$

The second step is to reduce the loop integrals to the three-fold integrals which are evaluated numerically.

The calculation of the matrix elements of the decays $\psi(2S) \to \chi_{cJ} + \gamma$ and $J/\psi \to \eta_c + \gamma$, $h_c \to \eta_c + \gamma$ is performed in a similar manner.

Matrix element and decay width: $\chi_{c0} \to J/\psi + \gamma$ transition

$$
\begin{aligned}
M_{\chi_{c0}\to J/\psi+\gamma} &= e\,\epsilon^*_{2\,\beta}(p_2)\,\epsilon^*_{\gamma\,\alpha}(q)\,M^{\beta\alpha}_{\chi_{c0}}\,, \\
M^{\beta\alpha}_{\chi_{c0}} &= (m_1 A_1)\big(g^{\beta\alpha} - \frac{q^\beta p_2^\alpha}{p_2 q}\big), \qquad p_2 q = \frac{m_1^2 - m_2^2}{2}\,,
\end{aligned}
$$

$$
\Gamma(\chi_{c0} \to J/\psi + \gamma) = \alpha\,|\mathrm{q}|\,A_1^2, \qquad |\mathrm{q}| = \frac{m_1^2 - m_2^2}{2m_1}.
$$

Matrix element and decay width: $\chi_{c1} \to J/\psi + \gamma$ transition

The invariant matrix element is written in the form:

$$\begin{aligned} M_{\chi_{c1} \to J/\psi + \gamma} &= e\, \epsilon_{1\,\mu}(p_1)\epsilon^*_{2\,\beta}(p_2)\, \epsilon^*_{\gamma\,\alpha}(q)\, M^{\mu\beta\alpha}_{\chi_{c1}} \\ M^{\mu\beta\alpha}_{\chi_{c1}} &= \left(\epsilon^{p_2 q \mu\alpha} q^\beta\, D_E + \epsilon^{p_2 q \beta\alpha} p_2^\mu\, D_M\right) \end{aligned}$$

It is convenient to present the decay width via helicity amplitude.

$$\begin{aligned} H_L &= H_{+;0,-} = -H_{-;0,+} = \varepsilon_{1\,\mu}(+)\varepsilon^\dagger_{2\,\beta}(0)\bar{\varepsilon}^\dagger_\alpha(-)M^{\mu\beta\alpha}_{\chi_{c1}} = i\frac{m_1^2}{m_2}|\mathrm{q}|^2 D_E \\ H_T &= H_{0;+,+} = -H_{0;-,-} = \varepsilon_{1\,\mu}(0)\varepsilon^\dagger_{2\,\beta}(+)\bar{\varepsilon}^\dagger_\alpha(+)M^{\mu\beta\alpha}_{\chi_{c1}} = -i m_1|\mathrm{q}|^2 D_M \end{aligned}$$

Then the decay width is written as

$$\begin{aligned} \Gamma(\chi_{c1} \to J/\psi + \gamma) &= \frac{\alpha}{3}\frac{|\mathrm{q}|}{m_1^2}\left(|H_L|^2 + |H_T|^2\right) \\ &= \frac{\alpha}{3}|\mathrm{q}|^5\left(\frac{m_1^2}{m_2^2} D_E^2 + D_M^2\right) \end{aligned}$$

Matrix element: $\chi_{c2} \to J/\psi + \gamma$ transition

The invariant matrix element $M^{\mu\nu\beta\alpha}_{\chi_{c2}}$ is represented in terms of the five form factors. By using the gauge invariance the number of the form factors is reduced to three.

$$\begin{aligned} M_{\chi_{c2}\to J/\psi+\gamma} &= e\,\epsilon_{1\,\mu\nu}(p_1)\epsilon^*_{2\,\beta}(p_2)\,\epsilon^*_{\gamma\,\alpha}(q)\,M^{\mu\nu\beta\alpha}_{\chi_{c2}} \\ M^{\mu\nu\beta\alpha}_{\chi_{c2}} &= F_1\,(p_2^\alpha q^\beta - p_2 q\, g^{\alpha\beta})q^\mu q^\nu + F_2\,(p_2^\alpha q^\nu - p_2 q\, g^{\nu\alpha})g^{\mu\beta} \\ &+ F_3\,(g^{\mu\alpha} q^\nu q^\beta - g^{\alpha\beta} q^\mu q^\nu) \end{aligned}$$

There are three independent helicity amplitudes $H_{\lambda_1;\lambda_2\lambda}$ characterizing the decay $\chi_{c2} \to J/\psi + \gamma$.

$$\begin{aligned} H_{+2;+1-1} &= H_{-2;-1+1} = \epsilon_{1\,\mu\nu}(+2)\epsilon^*_{2\,\beta}(+)\,\bar{\epsilon}^*_\alpha(-)\,M^{\mu\nu\beta\alpha}_{\chi_{c2}}\,, \\ H_{+1;0-1} &= H_{-1;0+1} = \epsilon_{1\,\mu\nu}(+1)\epsilon^*_{2\,\beta}(0)\,\bar{\epsilon}^*_\alpha(-)\,M^{\mu\nu\beta\alpha}_{\chi_{c2}}\,, \\ H_{0;+1+1} &= H_{0;-1-1} = \epsilon_{1\,\mu\nu}(0)\epsilon^*_{2\,\beta}(+)\,\bar{\epsilon}^*_\alpha(+)\,M^{\mu\nu\beta\alpha}_{\chi_{c2}}\,. \end{aligned}$$

Decay width: $\chi_{c2} \to J/\psi + \gamma$ transition

The decay width via helicity amplitudes:

$$\Gamma(\chi_{c2} \to J/\psi + \gamma) = \frac{\alpha}{5}\frac{|\mathrm{q}|}{m_1^2}\left(|H_{+2;+1-1}|^2 + |H_{+1;0-1}|^2 + |H_{0;+1+1}|^2\right),$$

$$\begin{aligned}
H_{+2;+1-1} &= -m_1|\mathrm{q}|\,F_2\,,\\
H_{+1;0-1} &= -\frac{1}{\sqrt{2}}\frac{m_1}{m_2}|\mathrm{q}|\left(E_2\,F_2 + |\mathrm{q}|\,F_3\right),\\
H_{0;+1+1} &= -\sqrt{\frac{2}{3}}m_1\,|\mathrm{q}|\left(|\mathrm{q}|^2 F_1 + \frac{1}{2}\,F_2 + \frac{|\mathrm{q}|}{m_1}\,F_3\right).
\end{aligned}$$

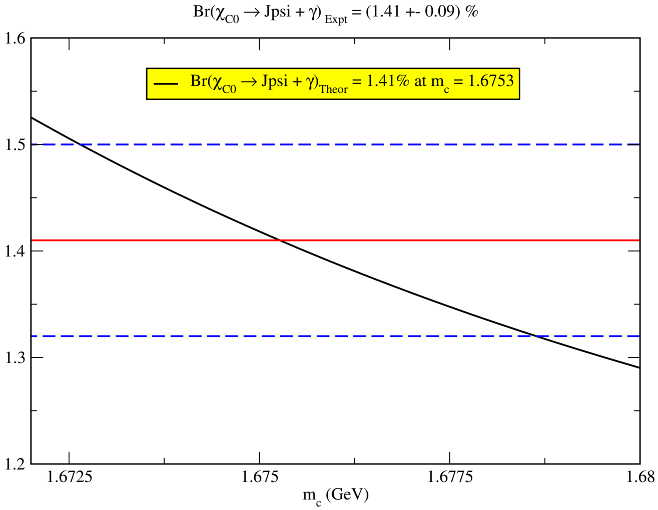
Numerical results

We observed that the charmonium radiative decay widths depend rather slowly on their size parameters Λ_{cc}. So we parametrize them in the following way:

$$\Lambda_{cc} = M_{cc} \cdot \rho$$

Therefore, we have two adgustable parameters: ρ and m_c.

m_c fit, $\rho = 1$ fixed



Fitting branching ratios (in %)

m_c fit, $\rho = 1$-fixed. $< m_c >= \frac{1}{2}(m_1/2 + m_2/2)$

Mode	m_c (GeV)	CCQM	Expt.	$< m_c >$
$\chi_{c0} \to J/\psi + \gamma$	1.6753(27)	1.41(9)	1.41(9)	1.628
$\chi_{c1} \to J/\psi + \gamma$	1.6935(6)	34.4(1.4)	34.3(1.3)	1.652
$\chi_{c2} \to J/\psi + \gamma$	1.6996(7)	19.4(7)	19.5(8)	1.663
$h_c \to \eta_c + \gamma$	1.6785(11)	60(4)	60(4)	1.627
$J/\psi \to \eta_c + \gamma$	1.845(97)	1.41(14)	1.41(14)	1.520
$\psi(2S) \to \chi_{c0} + \gamma$	1.8120(6)	9.75(21)	9.77(23)	1.775
$\psi(2S) \to \chi_{c1} + \gamma$	1.8197(13)	9.76(26)	9.75(27)	1.799
$\psi(2S) \to \chi_{c2} + \gamma$	1.8073(5)	9.36(23)	9.36(23)	1.811

Fitting procedure

To determine uncertainties we have used iminuit which is a Python frontend to the Minuit2 library in C++, an integrated software that combines a local minimizer (called MIGRAD) and two error calculators (called HESSE and MINOS).

We were aiming to identify precision of our calculation with respect of the charm-quark mass. For proper describing of the data, we used three different fitting function

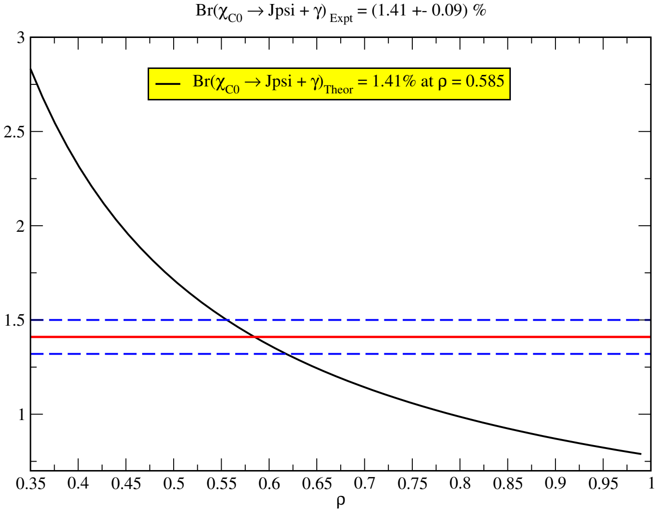
$$f(x) = \begin{cases} a/(b + c(x - h) + d(x - h)^2) \\ a/(b + c(x - h)) \\ a + \exp(b(x - h)) \end{cases} .$$

It was used the assumption that our analytical calculation has uncertainties 3 %.

Fitting results for branching ratios (in %) with uncertainties.

Decay mode	CCQM	Exp.
$\chi_{c0} \to J/\psi + \gamma$	1.41 ± 0.15	1.41 ± 0.09
$\chi_{c1} \to J/\psi + \gamma$	34.4 ± 3.8	34.3 ± 1.3
$\chi_{c2} \to J/\psi + \gamma$	19.5 ± 1.8	19.5 ± 0.8
$J/\psi \to \eta_c + \gamma$	1.41 ± 0.11	1.41 ± 0.14
$\Psi(2S) \to \chi_{c0} + \gamma$	9.75 ± 1.09	9.77 ± 0.23
$\Psi(2S) \to \chi_{c1} + \gamma$	9.77 ± 1.10	9.75 ± 0.27
$\Psi(2S) \to \chi_{c2} + \gamma$	9.36 ± 0.79	9.36 ± 0.23
$h_c \to \eta_c(1S) + \gamma$	60.2 ± 6.4	60 ± 4

$m_c = 1.85$ fixed, ρ fit



Fitting results for branching ratios (in %).

$m_c = 1.85$ fixed, ρ fit.

Mode	ρ	CCQM	Expt.
$\chi_{c0} \to J/\psi + \gamma$	$0.585^{+0.035}_{-0.031}$	$1.41^{+0.10}_{-0.10}$	1.41(9)
$\chi_{c1} \to J/\psi + \gamma$	$0.615^{+0.027}_{-0.050}$	$34.6^{+0.3}_{-1.6}$	34.3(1.3)
$\chi_{c2} \to J/\psi + \gamma$	0.425 ± 0.015	$19.4^{+0.9}_{-0.8}$	19.5(8)
$h_c \to \eta_c + \gamma$	$0.485^{+0.024}_{-0.020}$	60(4)	60(4)
$J/\psi \to \eta_c + \gamma$	$0.985^{+0.235}_{-0.193}$	$1.41^{+0.14}_{-0.14}$	1.41(14)
$\psi(2S) \to \chi_{c0} + \gamma$	$0.505^{+0.010}_{-0.013}$	$9.75^{+0.25}_{-0.18}$	9.77(23)
$\psi(2S) \to \chi_{c1} + \gamma$	$0.695^{+0.025}_{-0.022}$	$9.77^{+0.25}_{-0.28}$	9.75(27)
$\psi(2S) \to \chi_{c2} + \gamma$	$0.355^{+0.006}_{-0.010}$	$9.31^{+0.29}_{-0.17}$	9.36(23)

Fitting results for branching ratios (in %).

$m_c = 1.83$, $\rho = 0.60$.

Decay mode	CCQM	Expt.
$\chi_{c0} \to J/\psi + \gamma$	1.39	1.41 ± 0.09
$\chi_{c1} \to J/\psi + \gamma$	35.1	34.3 ± 1.3
$\chi_{c2} \to J/\psi + \gamma$	13.5	19.5 ± 0.8
$h_c \to \eta_c(1S) + \gamma$	45.3	60 ± 4
$J/\psi \to \eta_c + \gamma$	1.75	1.41 ± 0.14
$\Psi(2S) \to \chi_{c0} + \gamma$	9.80	9.77 ± 0.23
$\Psi(2S) \to \chi_{c1} + \gamma$	12.6	9.75 ± 0.27
$\Psi(2S) \to \chi_{c2} + \gamma$	7.31	9.36 ± 0.23

Summary

We have calculated the amplitudes and branching ratios of radiative decays of charmonium states: $\chi_{c0,c1,c2} \to J/\psi\gamma$, $\psi(2S) \to \chi_{c0,c1,c2}\gamma$, $h_c \to \eta_c\gamma$ and $J/\psi \to \eta_c\gamma$ in the framework of covariant confined quark model (CCQM).

We have applied the method of electromagnetic gauging of the nonlocal Lagrangian by using a gauge field exponential and the path-independent definition of its derivative.

We have assumed that the values of the size parameters are proportional to the charmonium masses, i.e. $\Lambda_{cc} = \rho M_{cc}$.

We have performed the two-parameter ρ and m_c to the available experimental data for eight decay modes.