Improved non-Abelian tensor multiplet action

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Introduction

Existence of a non-Abelian version of the $d = 6$ tensor multiplet was conjectured by Witten in 1995 in the context of low-energy description of multiple interacting $M5$ branes. Indeed, like the single D-brane, which effective description is given by the Born-Infeld-type action of the vector multiplet, single $M5$-brane is effectively described by a Born-Infeld-type tensor multiplet action. Moreover multiple coinciding D-branes are described by non-Abelian Born-Infeld action, and in lowest energy limit, by the supersymmetric Yang-Mills theory.
Separate but related point of interest in such theories is related to the fact that 2-tensor field action in six dimensions is conformal. Therefore, the maximal possible superconformal theory, $N = (2, 0)$, $d = 6$ one, is also described by non-Abelian tensor multiplets.
Description of non-Abelian tensor is mysterious, however. It was shown in [Henneaux, Knaepen, 1997] that it is not possible to deform characteristic tensor gauge symmetry $\delta B_{MN} = \partial_M a_N - \partial_N a_M$ in non-Abelian way in theories that are local and contain no extra fields. Another problem is argument by Witten based on reduction to 5 dimensions that a $d = 6$ action could not be superconformal. As a technical complication, $d = 6$ tensor multiplets involve tensor which field strength is self-dual on-shell, which Lagrangian description is a task on its own.

Introduction

Construction of the theory that captures at least some properties of $M5$-brane system is still of interest, and many attempts on non-Abelian generalizations of tensor multiplets have been made. In some of these the tensor multiplet strength was constructed out Yang-Mills strength and constant vector, others explicitly introduced nonlocal terms or extended $d = 5$ Yang-Mills theory. The most "conventional" and conservative approach involves so-called tensor hierarchy, there antisymmetric 2-tensor is accompanied by vector and higher antisymmetric tensor fields.

Tensor hierarchies were introduced in [de Wit, Samtleben, 2005] and were combined with supersymmetry in [Samtleben, Sezgin, Wimmer, 2011] at the component level. Superfield version of tensor hierarchy constructed in [Bandos,2013], superfield action in [Buchbinder, Pletnev, 2014]. Pasti-Sorokin-Tonin mechanism of inducing self-duality equation was added in [Bandos,Samtleben, Sorokin,2013].

The purpose of the current talk is to describe the known supersymmetric tensor hierarchy systems, focussing on their weaknesses, and to propose a manifestly supersymmetric $N = (1,0)$, $d = 6$ action that can at least partially solve their problems.

The tensor hierarchy

Tensor hierarchies were introduced in [de Wit, Samtleben, 2005] and were combined with supersymmetry in [Samtleben, Sezgin, Wimmer, 2011]. Tensor hierarchy consists of $1, 2, 3$-form fields subjected to a set of coupled gauge transformations

$$
\begin{aligned}
\delta A^r_M &= \mathcal{D}_M \Lambda^r - h^r_l \Lambda^l_M, \\
\delta B^l_{[MN]} &= 2\mathcal{D}_{[M}\Lambda^l_{N]} - 2d^l_{rs}\Lambda^r F^s_{MN} + 2d^l_{rs} A^r_{[M}\delta A^s_{N]} - g^{lr}\Lambda_{MNr}, \\
\delta C_{[MNK]r} &= 3\mathcal{D}_{[M}\Lambda_{NK]r} + \ldots, \\
F^r_{MN} &= 2\partial_{[M}A^r_{N]} - f^r_{st} A^s_M A^t_N + h^r_l B^l_{MN}, \quad F^l_{MNP} = 3\mathcal{D}_{[M}B^l_{NP]} + g^{lr} C_{MNPr} + \ldots .
\end{aligned}
$$

Coupling constants are dimensionless and satisfy a system of nonlinear equations which ensures closure of the algebra.
In [Samtleben, Sezgin, Wimmer, 2011] dynamical supersymmetric systems that have such gauge symmetry were constructed. However,

- Careful analysis shows that Stückelberg parts of A^r_M and B^l_{MN} can be used to compensate terms with derivatives in transformations of non-Stückelberg parts of B^l_{MN} and C_{MNPr}. Thus no non-trivial generalization of Abelian shift symmetry.
- Conditions of existence of the action lead to indefinite metric in the scalar sector, with additional scalar being superpartner of the C_{MNPr} Lagrange multiplier that induces the self-dual equation motion for F^l_{MNP}.

PST and tensor hierarchy

One can expect that the problem can be solved if one uses another method to induce self-dual equation of motion, such as PST approach.

The original Pasti-Sorokin-Tonin action (1996) produces self-dual equation of motion of the two-form in six dimensions without adding a new degrees of freedom:

$$S_{PST} = \int d^6x \Big(\frac{1}{6} F_{MNP} F^{MNP} - \frac{1}{2\partial^K z \partial_K z} \mathcal{F}_{ABC} \mathcal{F}^{ABD} \partial^C z \partial_D z \Big),$$

$$F_{ABC} = \partial_A B_{BC} - \partial_B B_{AC} + \partial_C B_{AB}, \quad \mathcal{F}^{ABC} = F^{ABC} - \frac{1}{6} \epsilon^{ABCMNP} F_{MNP}.$$

It was combined with tensor hierarchies in [Bandos,Samtleben,Sorokin,2013]. However,

- All relations of the tensor hierarchy were not altered. In particular, internal metric is indefinite.
- 3-form C_{MNPr}, which acted as Lagrange multiplier, is still present and is required to check the gauge symmetries
- Parameter of another symmetry $\delta B^I_{MN} = \partial_{[M} z a^I_{N]}$ is constrained by $g^r_I a^I_N = 0$, trivializing for non-Stčkelberg part.

Composite Lagrangian multiplier

Though not directly usable, the PST action still gives an idea how to avoid non-compactness of the gauge group.
Consider the polynomial form of the PST action [Mkrtchyan 2019], which reads

$$S_{polyPST} = \int d^6x \Big(\frac{1}{6} F_{MNP} F^{MNP} - \mathcal{F}_{MNP} \partial^M z R^{NP} + \frac{3}{2} \partial_{[M} z R_{NP]} \partial^{[M} z R^{NP]} \Big).$$

Excluding $R_{[MN]}$ with its equation of motion, one recovers the PST action.

Let us truncate the action by removing the last term in $S_{polyPST}$ and make it non-Abelian:

$$S_{polyPST} = \int d^6x \Big(\frac{1}{6} F^I_{MNP} F^{MNP}_I - \mathcal{F}^{MNP\,I} \partial_M z R_{NP\,I} \Big).$$

Note that $S_{polyPST}$ is still capable of producing a self-dual equation of motion. Varying w.r.t. R_{MN}

$$\partial_P z \mathcal{F}^{MNP\,I} = -\frac{1}{6} \epsilon^{MNPABC} \partial_P z\, \mathcal{F}^I_{ABC} = 0 \;\;\Rightarrow\;\; \mathcal{F}^I_{ABC} = \partial_{[A} z S^I_{BC]} \;\;\Rightarrow$$

$$(\partial z \partial z) S^{AB\,I} + 2 \partial^{[A} z S^{B]C\,I} \partial_C z = 0 \;\;\Rightarrow\;\; S^{AB\,I} = \partial^{[A} z\, U^{B]\,I} \;\;\Rightarrow\;\; \mathcal{F}^I_{ABC} = 0.$$

It is obvious in spinor notation, where $\partial_P z \mathcal{F}^{MNP\,I}$ is equivalent to

$$\partial_{[\alpha\gamma]} z \mathcal{F}^{(\beta\gamma)\,I} = 0 \;\text{ and }\; \partial_{\alpha\gamma} z \partial^{\beta\gamma} z = \frac{1}{4} \delta^\beta_\alpha \partial_{\mu\nu} z \partial^{\mu\nu} z.$$

Composite Lagrangian multiplier

Now $\partial_{[M} z R_{NP]I}$ functions as a composite Lagrangian multiplier. It is algebraically equivalent to the standard one, as any anti-self-dual tensor C_{MNP} can be presented as

$$C_{MNP} = 3\partial_{[M} z \widetilde{R}_{NP]} + \frac{1}{2}\epsilon_{MNPIJK}\partial^{K} z \widetilde{R}^{IJ}.$$

(Shown by multiplying by $\partial^P z$, extracting $\widetilde{R}^{IJ}$ and substituting it back).
Therefore, we obtain an action with following properties:

- It is equivalent to the standard one with natural minimal non-Abelian generalization;
- Its supersymmetric version can be obtained by truncating known superfield PST action;
- New supersymmetric version is substantially different from standard one, as it does not add a new physical scalar;
- A requirement to take the gauge group non-compact does not appear.

Note that the Lagrange multiplier, composite or not, is in general a new degree of freedom. It leads to ghost in the Abelian case, but it is harmless [A.Sen, 2015,2019]. In the non-Abelian case it is required to make it non-dynamical to avoid instabilities.

Abelian tensor multiplet action

The superfield action of the Abelian tensor multiplet reads

$$
\begin{aligned}
-8S[X,Y,Z] \;=\; & \int d^6x d^4\theta^- du \Big[D^+_\beta \Phi[X]\, D^{++} X^{+\beta} + \frac{1}{4}\Phi[X]\, D^{++} D^+_\beta X^{+\beta} - \\
& -2\Big(D^+_\beta H[Z,Y]\, D^{++} X^{+\beta} + \frac{1}{4} H[Z,Y]\, D^{++} D^+_\beta X^{+\beta} \Big) + \\
& + D^{++} Z \Big(D^+_\beta H[Z,Y]\, Y^{+\beta} + \frac{1}{4} H[Z,Y]\, D^+_\beta Y^{+\beta} \Big) + \\
& + M^{--} (D^{++})^3 Z + N^{+6} \Big(D^{--} Z + \mathrm{i} \frac{D^-_\alpha Z\, D^-_\beta Z\, \partial^{\alpha\beta} Z}{\partial_{\mu\nu} Z \partial^{\mu\nu} Z} \Big) \Big],
\end{aligned}
$$

where

$$
\Phi[X] \;=\; D^{--} D^+_\gamma X^{+\gamma} - 2 D^-_\gamma X^{+\gamma}, \quad H[Z,Y] = D^{--} Z\, D^+_\alpha Y^{+\alpha} - 2 D^-_\alpha Z\, Y^{+\alpha}.
$$

It is given by an integral over analytic harmonic superspace $(x^{[\alpha\beta]}, \theta^{+\alpha}, u)$. Superfields Z, M^{--}, N^{+6} are analytic, while tensor superfields $X^{+\alpha}$ and $Y^{+\alpha}$ also depend on $\theta^{-\alpha}$. X and Y contain physical and auxiliary tensor fields, while Z contains a PST scalar

$$
X^{+\alpha} = \theta^\alpha q + \theta^\beta B_\beta{}^\alpha + \ldots, \quad Y^{+\alpha} = \theta^\alpha c + \theta^\beta R_\beta{}^\alpha + \ldots, \quad Z = z + \ldots
$$

The $N=(1,0)$, $d=6$ harmonic superspace

The $N=(1,0)$, $d=6$ harmonic superspace in the analytic basis described by the coordinates $\{x^{\alpha\beta}, \theta^{+\alpha}, \theta^{-\alpha}, u^{+i}, u^{-i}\}$, $u^{+i}u_i^- = 1$.
In analytic basis, the derivatives with respect to θ's and harmonics read

$$D^{++} = \partial^{++} + \mathrm{i}\theta^{+\alpha}\theta^{+\beta} + \theta^{+\gamma}\frac{\partial}{\partial\theta^{-\gamma}},\ \partial^{++} = u^{+i}\frac{\partial}{\partial u^{-i}},\ \ D^+_\alpha = \frac{\partial}{\partial\theta^{-\alpha}},$$

$$D^{--} = \partial^{--} + \mathrm{i}\theta^{-\alpha}\theta^{-\beta} + \theta^{-\gamma}\frac{\partial}{\partial\theta^{+\gamma}},\ \partial^{--} = u^{-i}\frac{\partial}{\partial u^{+i}},\ \ D^-_\alpha = -\frac{\partial}{\partial\theta^{+\alpha}} - 2\mathrm{i}\theta^{-\gamma}\partial_{\alpha\gamma},$$

$$D_0 = u^{+i}\frac{\partial}{\partial u^{+i}} - u^{-i}\frac{\partial}{\partial u^{-i}} + \theta^{+\gamma}\frac{\partial}{\partial\theta^{+\gamma}} - \theta^{-\gamma}\frac{\partial}{\partial\theta^{-\gamma}}.$$

The (anti)commutators of these derivatives are

$$\{D^+_\alpha, D^-_\beta\} = 2\mathrm{i}\partial_{\alpha\beta},\ \ \{D^+_\alpha, D^+_\beta\} = 0,\ \ \{D^-_\alpha, D^-_\beta\} = 0,$$
$$[D^{++}, D^{--}] = D_0,\ \ [D_0, D^{++}] = 2D^{++},\ \ [D_0, D^{--}] = -2D^{--},$$
$$[D^{++}, D^+_\alpha] = 0,\ \ [D^{--}, D^+_\alpha] = D^-_\alpha,\ \ [D_0, D^+_\alpha] = D^+_\alpha,$$
$$[D^{++}, D^-_\alpha] = D^+_\alpha,\ \ [D^{--}, D^-_\alpha] = 0,\ \ [D_0, D^-_\alpha] = -D^-_\alpha.$$

The fields satisfy

$$D^+_\alpha X^{+\beta} = \frac{1}{4}\delta^\beta_\alpha D^+_\gamma X^{+\gamma},\ \ D^+_\alpha Y^{+\beta} = \frac{1}{4}\delta^\beta_\alpha D^+_\gamma Y^{+\gamma},\ \ D^+_\alpha Z = 0.$$

Adding Yang-Mills multiplet

As the truncation of the bosonic action involved removing R^2 term, one should truncate the supersymmetric action by removing

$$\int d^6x d^4\theta^- du \Big[D^{++} Z \Big(D^+_\beta H[Z,Y]\, Y^{+\beta} + \frac{1}{4} H[Z,Y]\, D^+_\beta Y^{+\beta} \Big) \Big].$$

For simplicity, let us consider standard minimal coupling to the Yang-Mills multiplet and assume that X and Y superfields transform as

$$\delta X^{+\alpha I} = -g\Lambda^r (T_r)^I{}_J X^{+\alpha J}, \;\; \delta Y^{+\alpha I} = -g\Lambda^r (T_r)^I{}_J Y^{+\alpha J}$$

with g being a dimensionless coupling constant, Λ^r is analytic parameter and indices I, J can be raised and lowered by non-degenerate positive-definite η_{IJ}. Then the derivatives should be covariantized by analytic V^{++r} and its relatives

$$\nabla^{++} = D^{++} - gV^{++r} T_r \cdot, \;\; \nabla^{--} = D^{--} - gV^{--r} T_r \cdot, \;\; \nabla^+_\alpha = D^+_\alpha,$$
$$\nabla^-_\alpha = D^-_\alpha - gV^{-r}_\alpha T_r \cdot, \;\; \nabla_{\alpha\beta} = \partial_{\alpha\beta} - gV^r_{\alpha\beta} T_r \cdot,$$
$$V^{-r}_\alpha = -D^+_\alpha V^{--r}, \;\; V^r_{\alpha\beta} = \frac{\mathrm{i}}{2} D^+_\alpha D^+_\beta V^{--r}, \quad \text{and}$$
$$D^{++} V^{--r} - D^{--} V^{++r} - g f^r_{st} V^{++s} V^{--t} = 0, \;\; \delta V^{++r} = \nabla^{++}\Lambda^r, \;\; \delta V^{--r} = \nabla^{--}\Lambda^r.$$

For non-simple gauge groups, $\delta X^{+\alpha I} = -g\Lambda^r (T_r)^I{}_J X^{+\alpha J} - g\Lambda^r k^I_{rs} V^{+\alpha s}$.

Harmonic Yang-Mills

Covariatized derivatives nontrivially (anti)commute:

$$[\nabla^{++},\nabla^-_\alpha]=D^+_\alpha,\;\; [\nabla^{--},D^+_\alpha]=\nabla^-_\alpha,\;\; \{D^+_\alpha,\nabla^-_\beta\}=2\mathrm{i}\nabla_{\alpha\beta},$$

$$[\nabla_{\alpha\beta},D^+_\gamma]=\frac{ig}{2}\epsilon_{\alpha\beta\gamma\mu}V^{+\mu r}T_r\cdot,\;\; [\nabla_{\alpha\beta},\nabla^-_\gamma]=\frac{ig}{2}\epsilon_{\alpha\beta\gamma\mu}V^{-\mu r}T_r\cdot=\frac{ig}{2}\epsilon_{\alpha\beta\gamma\mu}\nabla^{--}V^{+\mu r}T_r\cdot$$

where $V^{+\alpha r}$ is the covariant field strength

$$V^{+\alpha r}=\left(D^{+3}\right)^\alpha V^{--r},\;\; \nabla^{++}V^{+\alpha r}=0,\;\; \nabla^{--}D^+_\alpha V^{+\alpha r}-2\nabla^-_\alpha V^{+\alpha r}=0.$$

This leads to complications in generalizing the the Abelian action: simple replacing all the derivatives to covariant ones does not result in analytic object, as

$$D^+_\alpha D^+_\beta\Phi[X]=D^+_\alpha D^+_\beta\left(D^{--}D^+_\gamma X^{+\gamma}-2D^-_\gamma X^{+\gamma}\right)=0\;\text{ but}$$
$$D^+_\alpha D^+_\beta\left(\nabla^{--}D^+_\gamma X^{+\gamma I}-2\nabla^-_\gamma X^{+\gamma I}\right)=-2g\epsilon_{\alpha\beta\mu\nu}V^{+\mu r}\left(T_r\right)^I{}_J X^{+\nu J}\neq 0.$$

The correct generalization is $\Phi_{cov}[X^I]$, $D^+_\alpha D^+_\beta\Phi_{cov}[X^I]=0$, involves prepotential X^{--I}, $X^{+\alpha I}=\left(D^{+3}\right)^\alpha X^{--I}$:

$$\Phi_{cov}[X^I]=\nabla^{--}D^+_\gamma X^{+\gamma I}-2\nabla^-_\gamma X^{+\gamma I}+2gV^{+\beta r}\left(T_r\right)^I{}_J D^+_\beta X^{--J}-gD^+_\beta V^{+\beta r}\left(T_r\right)^I{}_J X^{--J},$$

Let us note that such generalization is impossible if shift by $V^{+\alpha r}$ is added.

Proposed action

Therefore, we suspect that the core of action of the non-Abelian tensor multiplet should read

$$-8S_{NAcore} = \int d^6xd^4\theta^- du\Big[D^+_\alpha\left(\Phi_{cov}[X_I] + H[Z, Y_I]\right)\nabla^{++}X^{+\alpha I} + \\ +\frac{1}{4}\left(\Phi_{cov}[X_I] + H[Z, Y_I]\right)\nabla^{++}D^+_\alpha X^{+\alpha I} + \\ +M^{--}\left(D^{++}\right)^3 Z + N^{+6}\left(D^{--}Z + \mathrm{i}\frac{D^-_\alpha Z\, D^-_\beta Z\, \partial^{\alpha\beta}Z}{\partial_{\mu\nu}Z\partial^{\mu\nu}Z}\right)\Big].$$

Some analysis shows that it is not satisfactory on its own and should be supplemented with additional constraints on superfields, coming with new Lagrange multipliers:

$$\begin{aligned} -8S_{NA} \quad = \quad & -8S_{NAcore} + \int d^6xd^4\theta^- duL^{--}_{rst}\, D^+_\alpha V^{+\alpha r}\, D^+_\beta V^{+\beta s}\, D^+_\gamma V^{+\gamma t} + \\ & + \int d^6xd^8\theta du\, L^{-4}_{IJ}\nabla^{--}\nabla^{++}D^+_\alpha X^{+\alpha I}\,\nabla^{--}\nabla^{++}D^+_\beta X^{+\beta J}. \end{aligned}$$

Here, L^{--}_{rst} is analytic and L^{-4}_{IJ} is a general superfield.

Added constraints

The necessity to add constraints to the action is that it contains a lot more components than one can expect for a vector-tensor system, and equations of motion coming from S_{NAcore} alone are not strong enough to suppress all of them. These components are contained in the prepotential $X^{--\prime}$, superfield Z and Yang-Mills potential (auxiliary field). They interfere with other components of the tensor multiplet, producing terms in the eom's of physical fields and making equations for auxiliaries practically unsolvable. With the new terms added the action, it is possible to remove these fields from the action and show that it is sensible in the bosonic limit. Actually, due to nonlinearity of the constraints, solution is simple in the bosonic limit only, when equations like $a^2 = 0$ have only solution $a = 0$.

The M^{--} and N^{+6} equations of motion produce constraints on Z

$$\left(D^{++}\right)^3 Z = 0, \quad D^{--}Z + \mathrm{i}\,\frac{D^{-}_{\alpha} Z\, D^{-}_{\beta} Z\, \partial^{\alpha\beta} Z}{\partial^{\mu\nu} Z \partial_{\mu\nu} Z} = 0$$

with known bosonic solution

$$Z = z + \theta^{+\mu}\theta^{+\nu} d^{--}_{\mu\nu}, \quad \partial^{--} z = 0, \quad \partial^{--} d^{--}_{\mu\nu} = 0,$$
$$\partial^{\mu\nu} z d^{--}_{\mu\nu} = 0, \quad d^{--\mu\nu} d^{--}_{\mu\nu} = 0, \quad \partial^{\mu\nu} d^{--}_{\mu\nu} = 0.$$

Solving constraints

Similarly, variation with respect to L^{--}_{rst} induces a constraint

$$D^+_\alpha V^{+\alpha r}\, D^+_\beta V^{+\beta s}\, D^+_\gamma V^{+\gamma t} = 0.$$

As

$$\begin{aligned} D^+_\alpha V^{+\alpha r} &= \frac{2}{3}(\partial^{++})^2 C^{--r} + \theta^{+\alpha}\theta^{+\beta}\Big(-\frac{2\mathrm{i}}{3}\partial^{++}\mathcal{D}_{\alpha\beta}C^{--r} - 8\mathcal{D}_{[\alpha\nu}F_{\beta]}{}^{\nu r}\Big) + \\ &+\theta^{+4}\Big(\frac{4}{3}\mathcal{D}^{\mu\nu}\mathcal{D}_{\mu\nu}C^{--r} + \frac{2g}{3}C^{--s}\partial^{++}C^{--t}f^r_{st}\Big), \end{aligned}$$

the constraint starts from

$$(\partial^{++})^2 C^{--r}(\partial^{++})^2 C^{--s}(\partial^{++})^2 C^{--t} = 0 \quad \Rightarrow C^{--r} = 0$$

and whole constraint becomes trivial.
Similarly, constraint $\nabla^{--}\nabla^{++}D^+_\alpha X^{+\alpha I}\,\nabla^{--}\nabla^{++}D^+_\beta X^{+\beta J} = 0$ implies

$$\begin{aligned} &\partial^{--}\partial^{++}f^{++I}\partial^{--}\partial^{++}f^{++J} = 0 \quad \Rightarrow \quad \partial^{++}f^{++I} = 0, \\ &\theta^{-\mu}\theta^{+\nu}\theta^{-\rho}\theta^{+\sigma}\big(\mathrm{i}\mathcal{D}_{\mu\nu}f^{++I} + \partial^{++}a^I_{\mu\nu}\big)\big(\mathrm{i}\mathcal{D}_{\rho\sigma}f^{++J} + \partial^{++}a^J_{\rho\sigma}\big) = 0 \quad \Rightarrow \\ &\mathrm{i}\mathcal{D}_{\mu\nu}f^{++I} + \partial^{++}a^I_{\mu\nu} = 0. \end{aligned}$$

Both do not induce equations of motion of physical fields.

Added constraints

With these results taken into account, Y equation of motion

$$H[Z, \nabla^{++} X^{+\alpha I}] = 0.$$

becomes tractable, and one can find $a^I_{\mu\nu}$ in terms of other fields and show that physical scalar and tensor do not depend on harmonics. Then one can perform integration of the action to find

$$\begin{aligned} -8S_{NA} &= \int d^6x \Big[-8\mathcal{D}^{\gamma(\beta} B_\gamma{}^{\alpha)}{}_I \big(\mathcal{D}_{\alpha\gamma} B_\beta{}^{\gamma I} - 2gF_\beta{}^{\gamma r} (T_r)^I{}_J w^J_{\alpha\gamma} + \partial_{\alpha\gamma} z\, R_\beta{}^{\gamma I}\big) + \\ &\quad + 16\mathcal{D}_{\mu\nu} \tilde{q}^I \mathcal{D}^{\mu\nu} \tilde{q}_I + 8\mathrm{i} g \tilde{q}_I F_\alpha{}^{\beta r} (T_r)^I{}_J B_\beta{}^{\alpha J} \Big], \end{aligned}$$

Here $w^J_{\alpha\beta}$ is the only extra component coming from the prepotential X^{--I}. Even it is unimportant, as one can define

$$\begin{aligned} C^I_{(\alpha\beta)} &= 2\mathcal{D}_{(\alpha\gamma} B_{\beta)}{}^{\gamma I} - 4gF_{(\beta}{}^{\gamma r} (T_r)^I{}_J w^J_{\alpha)\gamma} + 2\partial_{(\alpha\gamma} z\, R_{\beta)}{}^{\gamma I} \quad \Rightarrow \\ -8S_{NAT} &= \int d^6x \Big[-4\mathcal{D}^{\gamma(\beta} B_\gamma{}^{\alpha)}{}_I C^I_{(\alpha\beta)} + 16\mathcal{D}_{\mu\nu} \tilde{q}^I \mathcal{D}^{\mu\nu} \tilde{q}_I + 8\mathrm{i} g \tilde{q}_I F_\alpha{}^{\beta r} (T_r)^I{}_J B_\beta{}^{\alpha J} \Big] \end{aligned}$$

to obtain sensible bosonic action without demanding gauge group to be non-compact.

Constraints on gauge group

To avoid presence of ghosts, the Lagrange multiplier $C^{I}_{(\alpha\beta)}$ has to be non-dynamical. It can be shown to satisfy two algebraic equations

$$-B_{[\alpha}{}^{\gamma J} C_{\beta]\gamma I} (T_r)^I{}_J = -8g (T_r)^I{}_J \tilde{q}^J \mathcal{D}_{\alpha\beta} \tilde{q}_I + 4 (T_r)^I{}_J \mathcal{D}_{\gamma[\alpha} (\tilde{q}_I B_{\beta]}{}^{\gamma J}),$$
$$F_{[\alpha}{}^{\rho r} (T_r)^J{}_I C_{\beta]\rho J} = 4\mathcal{D}_{[\alpha\gamma} (F_{\beta]}{}^{\rho r} (T_r)^J{}_I \tilde{q}_J).$$

First is $A^r_{\mu\nu}$ equation, second derivative of $B_\alpha{}^{\beta I}$ equation.
If $r = 1\ \ldots, r_m$ and $I = 1, \ldots, I_m$, $C_{(\alpha\beta)I}$ has $10 I_m$ components, and there are at most $6 r_m + 6 I_m$ equations. Therefore,

$$3 r_m \geq 2 I_m$$

For $SU(N)$ or $SO(N)$ representations, r.h.s. of equations is not constrained only if exactly $3 r_m = 2 I_m$. Two such cases were identified

- The gauge group is $SU(3)$, $T_r \to T_\alpha{}^\beta$, $T_\alpha{}^\alpha = 0$, $\alpha = 1, 2, 3$, and C is a pair of symmetric conjugated bispinors with upper and lower indices $C^{(\alpha\beta)}_{(\alpha\beta)}$, $\overline{C}_{(\alpha\beta)(\alpha\beta)}$.
- The gauge group is $SO(4)$, $T_r \to T_{[\alpha\beta]}$, $\alpha = 1, \ldots, 4$. C is a symmetric traceless tensor, $C_{(\alpha\beta)(\alpha\beta)}$, $C_{(\alpha\beta)(\alpha\beta)} \delta_{\alpha\beta} = 0$.

Conclusion

We discussed a possibility to construct an action of the non-Abelian tensor theory which would possess a positive-definite metric in internal space. Our conclusion is that

- Tensor hierarchy does not contain non-Abelian deformations of tensor gauge shift symmetry, only local rotations of the tensor potential.
- Present combination of tensor hierarchy with Pasti-Sorokin-Tonin approach to self-dual tensor field do not solve problem of indefiniteness of the metric in the scalar sector, and do not possess the necessary symmetries.
- Construction of supersymmetric action with non-Abelian symmetries and proper scalar sector is possible by truncation of known superfield PST action.
- The constructed action still involves the Lagrangian multiplier, which is not dynamical if gauge groups are $SU(3)$ and $SO(4)$, with the tensor field belonging to specific representations.