Lorentz Covariance of $4d$ HS Equations in the BRST Formalism

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Motivation

Cartan gravity is based on two symmetry principles: diffeomorphisms and local Lorentz symmetry.

The formulation in terms of exterior algebra automatically respects diffeomorphisms.

From the very special form of HS Vasiliev equations, related to the deformed oscillator algebra, it follows that this system respects local Lorentz symmetry. MV 1992

Problem: to find the variables in which Lorentz covariance is manifest.

BRST Approach

Recently the BRST language was applied to formulate the $sp(2)$ invariance condition in the d-dimensional HS models. MV 2503.10967

We use BRST to Lorentz covariantize HS equations in AdS_4 within the framework of spinor coordinates. Our approach generalizes the construction of Didenko, Misuna, MV 1712.09272

In our approach in addition to the Lorentz connection ω^L a new Lorentz connection one-form $\widehat{\omega}$ is introduced

Additional Stückelberg shift symmetry acts on ω^L and on $\widehat{\omega}$

The gauge condition $\omega^L = 0$ leads to manifestly Lorentz covariant formulation of the theory while $\widehat{\omega} = 0$ leads to usual formulation of the HS theory

Nonlinear HS Equations

$$\begin{cases} \mathsf{d}\mathcal{W} + \mathcal{W} * \mathcal{W} = i\theta^A\theta_A + i\eta B * \gamma + i\bar{\eta} B * \bar{\gamma} \\ \mathsf{d}B + [\mathcal{W}, B]_* = 0 \end{cases}$$

Vasiliev 1992

Central elements $\gamma = k\kappa\theta^\alpha\theta_\alpha\,,\qquad \dot{\gamma} = \bar{k}\bar{\kappa}\bar{\theta}^{\dot{\alpha}}\bar{\theta}_{\dot{\alpha}}\,,\qquad \theta^A = \mathsf{d}Z^A$

$$\mathcal{W}(Z,Y;K|x) = W_{\underline{m}}(Z,Y;K|x)\mathsf{d}x^{\underline{m}} + S_A(Z,Y;K|x)\theta^A$$

η, $\bar{\eta}$ -free parameters , Space-time indices $\underline{m},\underline{n}$ take 4 values, $(AB) = (\alpha\beta) \cup (\dot{\alpha}\dot{\beta})$

Mutually commutative Klein operators κ, $\bar{\kappa}$, k, $\bar{k}$

$$\kappa = \exp i z_\alpha y^\alpha,\quad \bar{\kappa} = \exp i\bar{z}_{\dot{\alpha}}\bar{y}^{\alpha},\quad \kappa * f(y,\bar{y}) = f(-y,\bar{y}) * \kappa,\quad \kappa * \kappa = \bar{\kappa} * \bar{\kappa} = 1$$

$$ky^\alpha = -y^\alpha k\,,\qquad k\theta^\alpha = -\theta^\alpha k,\quad ,\qquad kk = \bar{k}\bar{k} = 1$$

HS star product

$$(f * g)(Z,Y) = \int dS dT f(Z+S, Y+S) g(Z-T, Y+T) \exp -iS_A T^A$$

$$[Y_A, Y_B]_* = -[Z_A, Z_B]_* = 2i\varepsilon_{AB}$$

Fields of the Nonlinear System

Infinite set of spins s

one-form ω **is a free of** Z **and** θ **part of** $\mathcal{W}$

$$\omega(Y;K\,|x) = \sum_s \omega_s(y,\bar{y};K\,|x)$$

$$\omega_s = \sum_{n+m=2s} \frac{1}{n!m!}\omega_{\alpha_1\ldots\alpha_n,\dot{\alpha}_1\ldots\dot{\alpha}_m} y^{\alpha_1}\ldots y^{\alpha_n}\bar{y}^{\dot{\alpha}_1}\ldots\bar{y}^{\dot{\alpha}_m}$$

zero-form C **is a free of** Z **part of** B

$$C(Y;K\,|x) = \sum_s C_s(y,\bar{y};K\,|x)$$

$$C_s = \sum_{|n-m|=2s}^{\infty} \frac{1}{n!m!}C_{\alpha_1\ldots\alpha_n,\dot{\alpha}_1\ldots\dot{\alpha}_m} y^{\alpha_1}\ldots y^{\alpha_n}\bar{y}^{\dot{\alpha}_1}\ldots\bar{y}^{\dot{\alpha}_m}$$

Additional Variables

Ghosts $c^{AB}\,, b^{AB}$, $\quad (AB) = (\alpha\beta) \cup (\dot\alpha\dot\beta)$

$$\{b_{MN}, c^{AB}\}_* = \delta^A_M \delta^B_N + \delta^B_M \delta^A_N$$

Ghosts anticommute with differentials $\mathsf{d}x$ **and** θ

$\mathbb{Z}$ **grading** $\quad \pi(c) = \pi(\mathsf{d}x) = \pi(\theta) = 1\,,\ \pi(b) = -1$

$$\pi(\mathcal{W}) = 1\,, \qquad \pi(B) = 0$$

HS equations are invariant under the HS gauge transformations

$$\delta\mathcal{W} = [\varepsilon, \mathcal{W}]_*,\ \ \delta B = [\varepsilon, B]_*$$

In addition

$$\delta B = \{\varphi, \mathcal{W}\}_*,\ \ \delta\mathcal{W} = \varphi * (i\eta\gamma + i\bar\eta\bar\gamma)$$

Mutually Commuting $\mathfrak{sl}_2(\mathbb{C})$-generators and BRST Operator

'inner'

$$L^Y_{AB} f = -\frac{i}{4}[Y_A Y_B\,, f]_*$$

$$L^Z_{AB} f = \frac{i}{4}[Z_A Z_B\,, f]_*$$

$$L^\theta_{AB} f = \frac{1}{2}\left(\theta_A \frac{\partial}{\partial\theta^B} + \theta_B \frac{\partial}{\partial\theta^A}\right) f$$

$$L^{ghost}_{AB} f = \frac{1}{2}\left\{c_A{}^C b_{CB} + c_B{}^C b_{CA}\,,\ f\right\}$$

'outer'

$$[\mathcal{T}_{AB}, \mathcal{T}_{DC}]_* = \frac{1}{2}(\epsilon_{AD}\mathcal{T}_{BC} + \epsilon_{AC}\mathcal{T}_{BD} + \epsilon_{BD}\mathcal{T}_{AC} + \epsilon_{BC}\mathcal{T}_{AD})$$

$$L^{Tot} = L^Y + L^Z + L^\theta + L^{ghost} + \mathcal{T}$$

BRST operator

$$Q = c^{AB}(\mathcal{T}_{AB} + L^\theta_{AB} + L^Z_{AB}) - \tfrac{1}{2} c^A{}_D c^{DB} b_{AB}$$

$$Q^2 = 0$$

Lorentz Covariant HS Equations

Additional one-form $\widehat{\omega}^{AB}(x)L_{AB}^{Tot}$ **and two-form** $U^{AB}(x)$

Decomposition $\mathcal{W} =: d_x + \mathcal{W}' + Q + \widehat{\omega}^{AB}(x)L_{AB}^{Tot} + \frac{1}{2}U^{AB}(x)b_{AB}$

$\mathcal{W}'$ **is free of terms like** $G^{AB}(x)b_{AB},\ \ G^{AB}(x)\mathcal{T}_{AB},\ \ G_{CD}^{AB}(x)c^{CD}\mathcal{T}_{AB}$

$\sim \mathcal{T}-$**term must be zero** $\Rightarrow$ **Substitution** $\mathcal{W}'$ **into HS equations implies LCHS equations:**

$$U^{AB} + d_x\widehat{\omega}^{AB} + \widehat{\omega}_C^A\widehat{\omega}^{BC} = 0$$

$$\mathsf{d}_x U^{AB} + 2\widehat{\omega}_D{}^A U^{DB} = 0$$

$$\mathcal{D}^L\mathcal{W}' + \mathcal{W}' * \mathcal{W}' + \{\mathcal{W}', Q\}_* + \frac{i}{4}U^{AB}Y_AY_B + \frac{1}{2}\{U^{AB}b_{AB}, \mathcal{W}'\}_* +$$
$$+ i\theta_\alpha \wedge \theta^\alpha(1 + \eta B * k\kappa) + i\bar{\theta}_{\dot{\alpha}} \wedge \bar{\theta}^{\dot{\alpha}}(1 + \bar{\eta}B * \bar{k}\bar{\kappa}) = 0$$

$$\mathcal{D}^L B + [U^{AB}b_{AB} + \mathcal{W}' + Q, B]_* = 0$$

where $\mathcal{D}^L f := \mathsf{d}_x f + \widehat{\omega}^{AB}\{L_{AB}^{Tot}, f]_*.$

Gauge Symmetries

$\varepsilon = \varepsilon' + \xi^{AB}(x) b_{AB}$, ε' **is free of terms like** $f^{AB}(x) b_{AB}$. **Hence LCHS equations are invariant under the following gauge transformations**

$$\delta\widehat{\omega}^{AB} = \xi^{AB}\,, \qquad \delta U^{AB} = -(\mathsf{d}_x \xi^{AB} + 2\widehat{\omega}_D{}^A \xi^{DB})\,,$$
$$\delta\mathcal{W}' = \varphi * \left(i\eta\gamma + i\bar{\eta}\bar{\gamma}\right) + \xi^{AB}\left(Y_A Y_B + \frac{1}{2}\{b_{AB}, \mathcal{W}'\}\right) +$$
$$+\left[\varepsilon',\, \mathsf{d}_x + \widehat{\omega}^{AB} L^{Tot}_{AB} + \mathcal{W}' + Q + \frac{1}{2} U^{AB} b_{AB}\right]_*,$$
$$\delta B = \left[\varepsilon' + \xi^{AB}(x) b_{AB}, B\right]_* + \{\varphi, \mathsf{d}_x + \widehat{\omega}^{AB} L^{Tot}_{AB} + \mathcal{W}' + Q + \frac{1}{2} U^{AB} b_{AB}\}_* .$$

The following gauge transformations

$$\delta\widehat{\omega}_{AB} = \xi_{AB}\,, \qquad \delta U^{AB} = -d_x \xi^{AB} - \widehat{\omega}^A_D \xi^{DB} - \widehat{\omega}^B_D \xi^{DA}$$
$$\delta\mathcal{W}' = -\xi^{AB}(Y_A Y_B + \{b_{AB}, \mathcal{W}'\})\,, \qquad \delta B = \left[\xi^{AB}(x) b_{AB}, B\right]_*$$

form a Stückelberg symmetry, allowing to gauge away either $\widehat{\omega}$ **or** ω^L

Vacuum Solution

$$B_0 = 0\,, \qquad \mathcal{W}'_0 = W_0 + S_0\,, \qquad W_0 = \omega(Y;K|x)\,, \qquad S_0 = Z_A\theta^A + Q\,,$$

$$\{Q,\theta^A Z_A\}_* = 0,\;\; S_0 * S_0 = i\theta^A\theta_A,\;\; [\theta^A Z_A\,,\cdot]_* = -2i\mathrm{d}_Z,\;\; \mathrm{d}_Z := \theta^A\frac{\partial}{\partial Z^A}$$

$$\Rightarrow \qquad \mathcal{D}^L_0\omega + \omega*\omega + \frac{i}{4}U_0{}^{AB}(x)Y_AY_B = 0.$$

Let ω **be free from the Lorentz connection.**

To find appropriate U_0 **we set to zero bilinear in** y **or in** $\bar{y}$ **parts of** $\omega*\omega$

$$\frac{\partial^2\Big(\omega*\omega\Big)}{\partial y^\nu\partial y^\mu}\Big|_{y=\bar{y}=0} + \frac{i}{2}U_{0\nu\mu} = 0, \qquad \frac{\partial^2\Big(\omega*\omega\Big)}{\partial\bar{y}^{\dot\alpha}\partial\bar{y}^{\dot\beta}}\Big|_{y=\bar{y}=0} + \frac{i}{2}U_{0\dot\alpha\dot\beta} = 0 \qquad \Rightarrow$$

For $\omega = -\frac{i}{2}\lambda h^{\alpha\dot\beta}y_\alpha\bar{y}_{\dot\beta}$ **one has** $U_0{}^{\alpha\beta} = \lambda^2\,h^{\alpha\dot\alpha}h^\beta{}_{\dot\alpha}\,, \qquad U_0{}^{\dot\alpha\dot\beta} = \lambda^2\,h^{\alpha\dot\alpha}h_\alpha{}^{\dot\beta}.$

Equations on $\widehat{\omega}_0$ **acquire the form**

$$d_x\widehat{\omega}_0^{\alpha\beta} + \widehat{\omega}_{0\gamma}{}^\alpha\widehat{\omega}_0{}^{\beta\gamma} + \lambda^2h^{\alpha\dot\alpha}h^\beta{}_{\dot\alpha} = 0\,, \qquad d_x\widehat{\omega}_0^{\dot\alpha\dot\beta} + \widehat{\omega}_{0\dot\gamma}{}^{\dot\alpha}\widehat{\omega}_0{}^{\dot\beta\dot\gamma} + \lambda^2h^{\alpha\dot\alpha}h_\alpha{}^{\dot\beta} = 0.$$

Hence $\widehat{\omega}_0$ **is the vacuum Lorentz connection.**

First-order Fluctuations

$$\mathcal{W} = \omega + S_0 + W_1 + S_1,\; U = U_0 + U_1,\; \widehat{\omega} = \widehat{\omega}_0 + \widehat{\omega}_1,\; \underline{B = C(x, Y)}$$

$$(-2i\mathrm{d}_Z + Q)S_1 + i\eta C * \gamma + i\bar{\eta} C * \bar{\gamma} = 0\,, \qquad \gamma = k \exp i z_\alpha y^\alpha \theta^\alpha \theta_\alpha$$

$$-2i\mathrm{d}_Z S' + i\eta C * \gamma + i\bar{\eta} C * \bar{\gamma} = 0$$

$S'(x, Y, Z)$ **the solution of** MV 1992

$\Rightarrow$ **Using** Didenko, Misuna, MV 1712.09272 $\qquad S_1 = S' + S''$

ghost-dependent $S'' = S''_{\alpha\beta} c^{\alpha\beta} + S''_{\dot{\alpha}\dot{\beta}} c^{\dot{\alpha}\dot{\beta}}\,, \qquad S''_{\alpha\beta} = \frac{i}{2} z_\alpha\, S'{}_\beta\,, \qquad S''_{\dot{\alpha}\dot{\beta}} = \bar{z}_{\dot{\alpha}}\, S'{}_{\dot{\beta}}$

$\Rightarrow W_1(x, Y, Z)$ **the solution of** MV 1992

$$\mathcal{D}^L \omega + \omega * \omega = -\Big(\mathcal{D}^L W_1 + \frac{i}{4}(U_0 + U_1)^{AB} Y_A Y_B + \{W_1\,, \omega\}_* + U_0{}^{AB} S''{}_{AB}\Big)\,.$$

If ω is demanded not to contain Lorentz connection $\Rightarrow$

$$\frac{\partial^2 R}{\partial y^\nu \partial y^\mu}\Big|_{y=\bar{y}=0} + \frac{i}{2} U_{1\,\nu\mu} = 0, \qquad \frac{\partial^2 R}{\partial \bar{y}^{\dot{\alpha}} \partial \bar{y}^{\dot{\beta}}}\Big|_{y=\bar{y}=0} + \frac{i}{2} U_{1\,\dot{\alpha}\dot{\beta}} = 0\,.$$

$$R := \mathcal{D}^L W_1 + \{W_1\,, \omega\}_* + U_0{}^{AB} S''{}_{AB}$$

Differential Homotopy

Various homotopy and shift parameters $\Rightarrow \quad \int dt_1 \int dt_2 \ldots . \quad 0 \leq t^i \leq 1$

Additional coordinates t^i

Differential Homotopy is based on the removal of the integrals

Total differential $\mathrm{d} = \mathrm{d}_Z + dt^i \frac{\partial}{\partial t^i}$ MV 2307.09331

At every perturbation order equations to be solved:

$$\mathrm{d}f \cong g \,, \quad \mathrm{d}g \cong 0$$

Functions like f **and** g **contain** $\theta(t^i)\,,\ \theta(1-t^i),\ \delta(t^j)\ \ldots$

Q**-dependent total differential** $\mathrm{d} = \mathrm{d}_Q = \mathrm{d}_Z + \frac{i}{2}Q + dt^i \frac{\partial}{\partial t^i}$

Ansatz for the η Deformations

$$f_\mu = \eta \int \mu \mathrm{d}\Omega^\alpha \mathrm{d}\Omega_\alpha \mathcal{E}(\Omega) G_l(g(r))\Big|_{r=0}, \qquad \mathrm{d}((\mathrm{d}\Omega)^2\mathcal{E}(\Omega)) = 0$$ MV 2307.09331

$$\mathcal{E}(\Omega) := \exp i\Big(\Omega_\beta(y^\beta + \sum_i p_i^\beta) - \sum_{j>i} p_{i\beta} p_j^\beta\Big)$$

$$\Omega_\alpha := \tau z_\alpha - (1-\tau)\Big(\sum_i p_{i\alpha}\sigma_i\Big)$$

$$G_l(g) := g_1(r_1)\dots g_l(r_l)k\,, \qquad g_i(y) = C(y) \quad \text{or} \quad \omega(y)\,, \qquad p_{j\alpha} = -i\frac{\partial}{\partial r^{j\alpha}}$$

Total differential $\quad \mathrm{d} = \mathrm{d}_Q = dz^\alpha \frac{\partial}{\partial z^\alpha} + d\tau \frac{\partial}{\partial \tau} + d\sigma_i \frac{\partial}{\partial \sigma_i} + \frac{i}{2}Q\,, \qquad [(\mathrm{d}_Q)^2, .] = 0$

If only Ω and μ can depend on ghosts and $\mathcal{T}$

$$\Rightarrow \qquad \mathrm{d}_Q\Big((\mathrm{d}_Q\Omega)^2\mathcal{E}(\Omega)\Big) = 0.$$

Lorentz Covariantization within Differential Homotopy

$$(\mathcal{D}^L - 2i\mathrm{d}_Q)\mathcal{W}' + \mathcal{W}' * \mathcal{W}' + \frac{i}{4}U^{AB}Y_AY_B + \frac{1}{2}\{U^{AB}b_{AB}\,,\mathcal{W}'\}_* +$$

$$+ i\theta_\alpha \wedge \theta^\alpha \eta B * k\kappa + i\bar{\theta}_{\dot{\alpha}} \wedge \bar{\theta}^{\dot{\alpha}}\bar{\eta}B * \bar{k}\bar{\kappa} \cong 0$$

$$(\mathcal{D}^L - 2i\mathrm{d}_Q)B + [\widehat{\omega}^{AB}L^{tot}_{AB}\,, B]_* + [\mathcal{W}\,, B]_* \cong 0$$

In this approach one has

$$S_1 = -\frac{\eta}{2}\int_\tau \delta(\tau)\delta(1-\tau)(\mathrm{d}_Q\Omega)^2\mathcal{E}(\Omega)C * k + cc$$

$$\Omega^\alpha = \tau z^\alpha\,,\ \ \bar{\Omega}^\alpha = \bar{\tau}\bar{z}^{\dot{\alpha}}$$

$$\mathrm{d}_Q\Omega^\alpha = d\tau z^\alpha + \tau\theta^\alpha - \frac{i}{2}\tau c^{\nu\alpha}z_\nu \quad\Rightarrow$$

$$S_1 = -\eta\int_\tau \delta(\tau)\delta(1-\tau)\tau d\tau(z^\alpha\theta_\alpha + i\frac{1}{2}c^{\nu\alpha}z_\alpha z_\nu)\mathcal{E}(\Omega)C * k + cc,$$

that coincides with S_1 obtained above.

Conclusion

A BRST extension with respect to local Lorentz symmetry of the standard $4d$ HS equations is proposed.

This leads to an additional Stückelberg symmetry allowing to exchange the Lorentz connection ω^L of the original HS theory by the new $\mathfrak{sl}(2|\mathbb{C})$ gauge field $\widehat{\omega}$.

The gauge condition $\omega^L = 0$ yields the proper Lorentz covariant derivative D^L with respect to $\widehat{\omega}$ at all stages of the computations making local Lorentz symmetry manifest.

The advantage of the proposed scheme, compared to that of Didenko, Misuna, MV 1712.09272, is that it applies to any homotopy solution scheme of the HS equations including the differential homotopy approach.