# Generation of a scalar vortex in a rotating frame

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based on

M. Bordag and D. N. Voskresensky. Generation of a scalar vortex in a rotational frame, 2025. arXiv 2507.10791

M. Bordag and I. G. Pirozhenko. Casimir effect for scalar field rotating on a disk. *Europhys. Lett.*, 150:52001, 2025

- Rotation
- Rotating frames
- Fields in a rotating frame
- Lagrangian with rotation
- Gross-Pitaevskii-like equation for a scalar field
- Formation of a condensate

# Rotation is ubiquitous

- we rotate (together with the earth)
- planets rotate
- contracting matter tends to rotation (e.g. Kerr black hole)
- in heavy ion collisions: plasma rotates
- superfluids rotate through the formation of quantized vortices
- on lattice - rotation is a mean of exploring

There are many more examples

# in QFT: rotating fields

What is a rotating field?
it is to some extend similar to a rotating rigid body
Consider the formalism: transformation of coordinates

$$t = t_R, \qquad r = r_R, \qquad \varphi = \varphi_R - \Omega t, \qquad z = z_R.$$

speed of a rotating point: $\vec{v} = \vec{\omega} \times \vec{x}$, where $\vec{\omega} = \vec{e}_z \Omega$, $\vec{x} = \vec{e}_r r$
so, $v = \Omega r$, but what happens for large $r$? $\quad \rightarrow$ transformation is non-relativistic

way out: restrict space by boundary condition, e.g., $\phi(R) = 0$ with $\Omega R < 1$.

no need to do that in non-relativistic physics, e.g. in laboratory on earth but if considering a relativistic field?

better way out [1], (1922)

$$t = t_R \cosh\left(\frac{\Omega r}{c}\right) - \frac{r\varphi_R}{c} \sinh\left(\frac{\Omega r}{c}\right), \quad r = r_R, \quad \varphi = -\frac{ct}{r} \sinh\left(\frac{\Omega r}{c}\right) + \varphi_R \cosh\left(\frac{\Omega r}{c}\right)$$

we have a Gallilei-like transform vs. a Lorentz-like transform of the angle
However, that is technically more involved and not yet well explored

[1] Philip Franklin. The Meaning of Rotation in the Special Theory of Relativity.
*Proceedings of the National Academy of Sciences of the United States of America*, 8(9):265–268, 1922

# Transformation to a rotating frame

transformation of the differentials:

$$\frac{\partial x^\mu}{\partial x_R^\nu} = T^\mu{}_\nu = \begin{pmatrix} 1 & 0 \\ -\vec{v} & \boldsymbol{D}(\Omega t) \end{pmatrix}, \qquad \frac{\partial x_R^\mu}{\partial x^\nu} = (T^{-1})^\mu{}_\nu = \begin{pmatrix} 1 & 0 \\ \vec{v}_R & \boldsymbol{D}^\top(\Omega t) \end{pmatrix}.$$

The derivatives of the spatial coordinates,

$$\frac{\partial \vec{x}}{\partial t} = \partial_t \boldsymbol{D}^\top \vec{x}_R = \partial_t r \vec{e}_r(\varphi_R - \Omega t) = -\Omega r \vec{e}_\varphi(\varphi) \equiv -\vec{v},$$

defines the speed $\vec{v}$ of a rotating point in the rotating frame,

The interval is invariant under the transformation,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \eta_{\mu\nu} dx_R^\mu dx_R^\nu$$

where $\eta_{\mu\nu} = diag(1, -1, -1, -1)$ is the metric of the resting system. This relation defines the metric in the rotating frame,

$$g_{\mu\nu} T^\mu{}_{\mu'} T^\nu{}_{\nu'} = \eta_{\mu'\nu'}, \quad g_{\mu\nu} = (T^{-1})^{\mu'}{}_\mu (T^{-1})^{\nu'}{}_\nu \eta_{\mu'\nu'},$$

and we get the metric tensor in the rotating frame:

$$g_{\mu\nu} = \begin{pmatrix} 1 - \vec{v}^2 & -\vec{v} \\ -\vec{v} & -\boldsymbol{1} \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} 1 & -\vec{v} \\ -\vec{v} & -\boldsymbol{1} + \vec{v} \circ \vec{v} \end{pmatrix}, \quad \sqrt{-g} = 1.$$

This metric is flat: $R_{\dots} = 0$

from here, we represent the co- and contravariant background fields in the form,

$$A_R^\mu = \begin{pmatrix} A_{R\,0}(x_R) \\ \vec{A}_R(x_R) \end{pmatrix}, \; A_\mu = \begin{pmatrix} A_{R\,0}(x) - \vec{v}\vec{A}_R(x) \\ -\vec{A}_R(x) \end{pmatrix}, \; A^\mu = \begin{pmatrix} A_{R\,0(x)} \\ -\vec{v}A_{R\,0}(x) + \vec{A}_R(x) \end{pmatrix}$$

We consider only the rotating frame (and drop the index 'R'). The derivatives in the rotating frame are

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \begin{pmatrix} \partial_0 \\ \vec{\nabla} \end{pmatrix}, \qquad \partial^\mu = g^{\mu\nu}\partial_\mu = \begin{pmatrix} \partial_0 - \vec{v}\vec{\nabla} \\ -\vec{v}(\partial_0 - \vec{v}\vec{\nabla}) - \vec{\nabla} \end{pmatrix}.$$

This way, we applied a passive transformation (*alias*) with a resting observer (field) and a rotating frame. This is in distinction from an active transform (*alibi*), where the frame is resting and the observer moving.

The covariant ('long') derivative is defines by $D_\mu = \partial_\mu + iA_\mu = \begin{pmatrix} \partial_0 + iA_0 \\ \vec{\nabla} - i\vec{A} \end{pmatrix}$,

In the rotating frame, we arrive at

$$D_\mu = \begin{pmatrix} D_0 \\ \vec{D} \end{pmatrix}, \qquad \begin{matrix} D_0 = \partial_0 + iA_0 \\ \vec{D} = \vec{\nabla} - i\vec{A} \end{matrix}, \qquad D^\mu = \begin{pmatrix} D_0 - \vec{v}\vec{D} \\ -\vec{v}(D_0 - \vec{v}\vec{D}) - \vec{D} \end{pmatrix},$$

and $D_\mu D^\mu = (D_0 - \vec{v}\vec{D})^2 - \vec{D}^2 = |\partial_0 - \vec{v}\vec{\nabla} + iA_0|^2 - |\vec{\nabla} - i\vec{A}|^2$.

# The Lagrangian in the rotating frame

We consider a scalar field, $\mathcal{L}_\phi = (D_\mu\phi)^* D^\mu\phi - m^2|\phi|^2 - \frac{\lambda}{2}|\phi|^4$.
Without background field, Klein-Gordon equation (without self-interaction) results,

$$((\partial_0 - \vec{v}\,\vec{\nabla})^2 - \Delta + m^2)\phi(x) = 0.$$

Making the ansatz $\phi(x) = \sum_{l=-\infty}^{\infty} e^{-i\omega_{l,n}t+il\varphi} J_l(\alpha r)$,

we get the one-particle energies in the form $\omega_{l,n} = -\Omega l + \sqrt{m^2 + \left(\frac{j_{l,n}}{R}\right)^2}$,

where $j_{l,n}$ are the zeros of the Bessel function $J_l(z)$.
Note: $\Omega l$ reduces this energy, from $j_{j,n} \gtrsim l + l^{1/3}$ this is overcompensated (for $\Omega r < 1$) and the $\omega_{l,n}$ stay real, allowing for the calculation of the vacuum energy of the scalar field.

However, with a magnetic background $\vec{A}(x) = \vec{e}_\varphi \frac{\mu(r)}{r}$, $\mu(r) = \frac{Br^2}{2}$ we get

$$\mathcal{L}_\phi = \phi(r)\left((\omega + \Omega l)^2 + \Delta_r - \left(\frac{l-\mu(r)}{r}\right)^2 - m^2 - \frac{\lambda}{2}\phi(r)^2\right)\phi(r).$$

Here, the magnetic field may compensate the centrifugal term and an instability is possible, see below the formation of a condensate (ne needs to keep the self-interaction to get a stable condensate)

# The vacuum energy (Casimir effect) in a rotating frame

With the energies $\omega_{l,n} = -\Omega l + \sqrt{m^2 + \left(\frac{j_{l,n}}{R}\right)^2}$, we get the vacuum (ground state) energy,

$$E_0 = \frac{\mu^{2s}}{2} \sum_{l=-\infty}^{\infty} \sum_{n=1}^{\infty} \omega_{l,n}^{1-2s},$$

where $s$ is the regularization parameter, $s \to 0$ at the end, and $\mu$ is the dimensional parameter associated with this regularization.

The problem to be solved is the analytic continuation of $E_0$ to $s = 0$. We proceed by transforming the sum over $n$ into an integral using

$$\Phi(\lambda) = \lambda^{-l} J_l(\lambda)$$

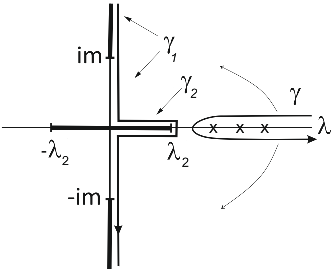
as the mode generating function, $\Phi(j_{l,n}) = 0$, which is regular at $\lambda = 0$. We get

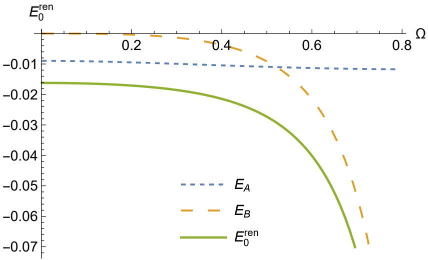
$$E_0 = \frac{1}{2} \sum_{l=-\infty}^{\infty} \int_\gamma \frac{d\lambda}{2\pi i} g\left(\lambda/R\right)^{1-2s} \frac{\partial}{\partial\lambda} \ln \Phi(\lambda), \quad g(\lambda) = -\Omega l + \sqrt{m^2 + \lambda^2},$$

where the integration path $\gamma$ surrounds the real positive zeros of the mode generating function $\Phi(\lambda)$.

# Calculation of the vacuum energy

Proceed by deformation of the integration contour to avoid oscillations,




then perform the renormalization.
Note: Since the background is flat, we have only the 'usual' divergences from the boundary (cylinder).
In the result, the rotation makes the vacuum energy more negative.

# Generation of a scalar vortex

Return to the case with the magnetic field, make the ansatz $\phi(\vec{x}) = e^{il\varphi + ik_z z}\phi(r)$, get

$$\left((\Omega l)^2 + \partial_r^2 + \frac{1}{r}\partial_r - \frac{(l-\mu(r))^2}{r^2} - m^2 - \lambda\phi^2(r)\right)\phi(r) = 0, \quad \phi(R) = 0.$$

This is a relativistic generalization of the Gross-Pitaevskii equation. Energetically more favorable is $\omega = 0$ (static). This equation describes a condensate, in the sense

$$\hat{\phi} = \phi + \delta\hat{\phi}$$

The energy of the condensate is $E_{GP} = -\frac{\lambda}{2}\int d^2x\, |\phi|^4$, provided a non-zero solution exists.

There is an approximation (linearization) by substituting $\lambda\phi(r)^2 \to \epsilon^2$,

$\left((\Omega l)^2 + \partial_r^2 + \frac{1}{r}\partial_r - \frac{(l-\mu(r))^2}{r^2} - m^2 - \epsilon^2\right)\phi(r) = 0.$

The existence of non-zero solution for $\phi$ depends on the parameters

We consider two examples,

a thin flux tube, $\mu(r) = \delta_\phi$, $\vec{B} = \lim\limits_{R_s \to 0} \vec{e}_z \frac{\mu'(r)}{r} = \vec{e}_z \Phi\, \delta^2(\vec{x}_{(2)})$,

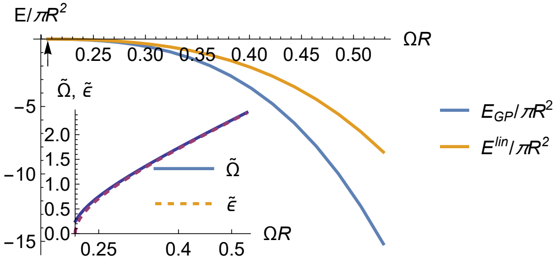
and a homogeneous field in the cylinder, $\mu(r) = \frac{Br^2}{2}$, $\vec{B} = \vec{e}_z B$

# Thin flux tube

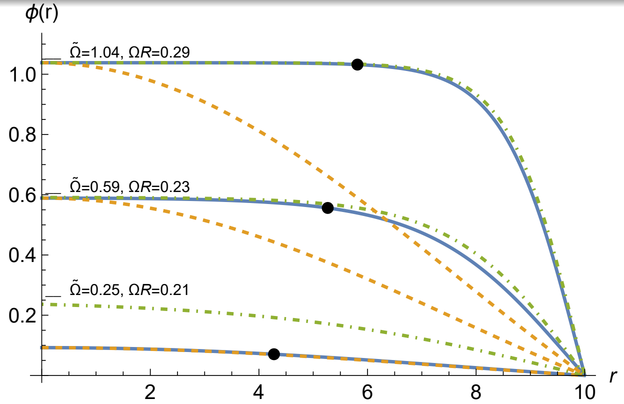
A 'thin' flux tube: $\mu(r) = \delta_\phi$, $B = \vec{e}_z \Phi \delta^2(\vec{x})$, this is a flux tube in the limit of vanishing radius

$$\left(-\partial_r^2 - \frac{1}{r}\partial_r + \frac{\nu^2}{r^2} - \tilde{\Omega}^2 + \lambda\phi^2(r)\right)\phi(r) = 0, \qquad \nu = |\delta_\Phi - l|,$$

energy for the thin flux tube



# Solution for thin flux tube




For the thin flux tube, at $\nu = 0$, the exact solutions $\phi(r)$ of the GP-like equation (solid lines) and the solutions of the linearized equation (dashed lines) for several values of the parameter $\Omega R$, $I = \delta_\Phi = 50$, and $R = 10, m = 1, \lambda = 1$. The dash-dotted line shows the approximate solution $\phi_{appr}(r) = \frac{\tilde{\Omega}}{\sqrt{\lambda}} \tanh\left(\frac{R-r}{\sqrt{2}}\tilde{\Omega}\right)$. of the GP-like equation. The dots indicate the mean radius.

In this example, the mean radius is quite far from the boundary and its influence may be small.

# Another example

formation of a plateau

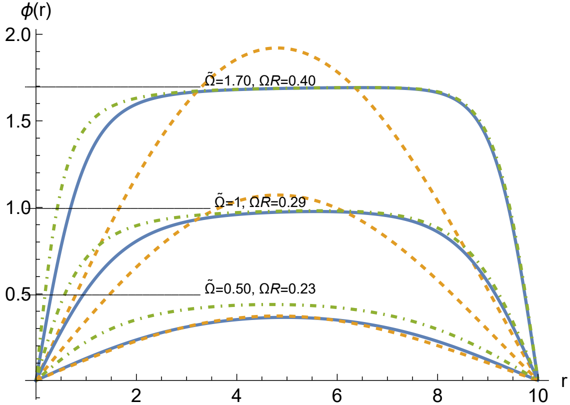



Figure: For the thin flux tube, at $\nu = 1$, the exact solutions $\phi(r)$ of the GP-like equation (solid lines), the solutions of the linearized equation (dashed lines) and the interpolating solution (dot-dashed lines) for several values of the parameter $\Omega R$, $l = 49$, $\delta_\Phi = 50$, and $R = 10, m = 1, \lambda = 1$.

# An example with the homogeneous field inside the cylinder

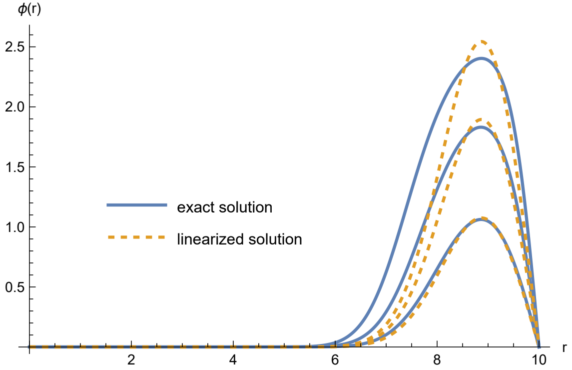



Figure: The case of the magnetic field in the whole cylinder $r \leq R$. The solutions $\phi(r)$ of the exact equation (solid lines) and of the linearized equation, (dashed lines) for $l = 45$, $\delta_\Phi = 50$ and $\Omega R = 0.45, 0.53, 0.62$, correspondingly $\epsilon = 0.07, 1.33, 1.97$, from bottom to top. Other parameters are $m = 1$, $R = 10$ (in units of the pion mass).

Here the condensate function is concentrated near the surface, so that its influence may be not small.

# The energy of the condensate

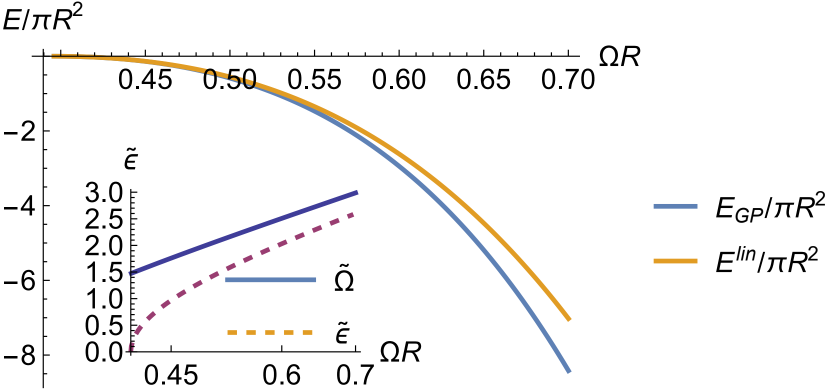
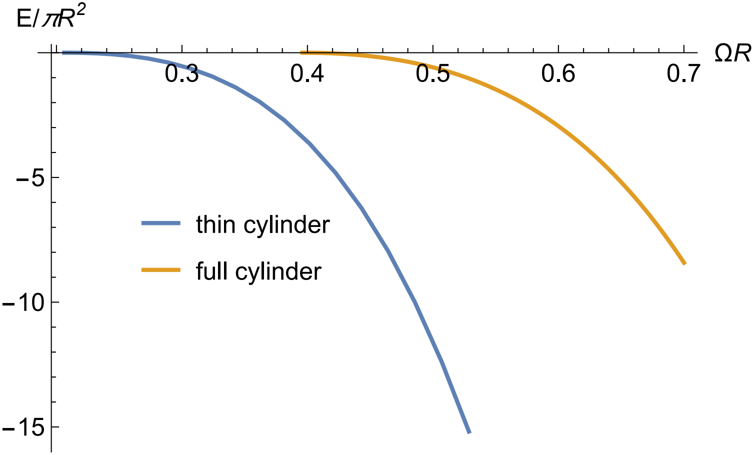



Figure: For the magnetic field in the whole cylinder, the condensate energy density, $\frac{E^{GP}}{\pi R^2}$, and $\frac{E^{lin}}{\pi R^2}$, as functions of $\Omega R$. The parameters are $m = 1$, $R = 10$, $\delta_\phi = 50$ and $l = 45$. The inset shows the dependence of $\tilde{\Omega}$ and $\epsilon$ on $\Omega R$.

# Different magnetic fields

A 'thin' flux tube: $\mu(r) = \delta_\phi$, $B = \vec{e}_z \Phi \delta^2(\vec{x})$, this is a flux tube in the limit of vanishing radius

$$\left( -\partial_r^2 - \frac{1}{r}\partial_r + \frac{\nu^2}{r^2} - \tilde{\Omega}^2 + \lambda\phi^2(r) \right) \phi(r) = 0, \qquad \nu = |\delta_\Phi - l|,$$



# Conclusions

- We studied rotation of a scalar field, having in mind possible applications to heavy ion collisions and took parameters in terms of $m_\pi$.
- We considered a passive transform to a rigidly rotating frame, taking the magnetic field, considered as external, from a rest frame
- To obey the causality condition $\Omega r \leq 1$ we introduced Dirichlet boundary conditions at $r = R$ with $\Omega R < 1$. We discussed the influence of the boundary on the results.
- We calculated the vacuum energy resulting from rotation, it is negative.
- For the solution of the (nonlinear) Gross-Pitaevskii equation we used a linearized version, an approximate solution and direct numerical methods.
- The condensate energy shows a minimum at orbital momenta close to the flux.
- Smaller flux tube has larger condensate (at equal flux).
- To do: account for the backreaction
- To do: consider relativistic rotation

Thank you for attention.