# Thermal holographic RG flows and bulk-boundary propagators in truncated supergravities

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# Motivation

**Holography states that the on-shell value of the supergravity action in AdS$_{(d+1)}$ is equated with the generating functional of composite operators in CFT$_d$**

$$e^{-S_{AdS}(\phi)}|_{\lim_{z\to 0}(\phi(x,z)z^{\Delta-d})=\phi_0(x)} = \langle e^{\int d^dx\phi_{(0)}\mathcal{O}(x)}\rangle_{CFT}\,,$$

where $\phi_{(0)}$ is a $d$-dim. field is a boundary value of a $(d+1)$-dim. field $\phi$, an operator $\mathcal{O}$ on the field theory side with the conformal dimension $\Delta$.

Maldacena'97,Witten'98, Gubser,Klebanov,Polyakov'98

- Holographic renormalization, RG flows: systematic removing the divergences and identifying the finite expressions, implies a careful analysis near the boundary.

Akhmedov'98; de Boer et.al'98; Skenderis'99, de Haro et.al.'99

Papadimitriou&Skenderis'04

The asymptotically AdS/dS metric (the domain wall) Skenderis'99, de Haro et.al.'99

$$ds^2 = e^{2\mathcal{A}(w)}\eta_{ij}dx^i dx^j + dw^2, \quad \phi = \phi(w)$$

- Holographic QGP, holographic RG flows Policastro et. al.'15 Aref'eva&Rannu'18,Aref'eva'19
- Irrelevant deformations, in particular , $T\bar{T}$-deformations Chang,Ferko&Sethi'23
- Thermal holography Witten'98
- Black hole interior Hartnoll et.al.'20, Caceres et al.'23
- de Sitter holography Witten'01,Strominger'01, Maldacena'03

# Thermal holographic RG flow

- Thermal states correspond to asymptotically AdS black hole geometries

**The ansatz for the metric and the scalar field**

$$ds^2 = e^{2\mathcal{A}(w)}\left(-f(w)dt^2 + d\vec{x}^2\right) + \frac{dw^2}{f(w)}, \quad \phi = \phi(w)$$

The Hawking temperature $T_H$ is (dual to $T$ of a dual field theory `Witten'98`)

$$T_H = \frac{e^{A(w_h)}}{4\pi}\left|\frac{df}{dw}\right||_{w=w_h}.$$

- The conformal symmetry restores near asymptotical regions, which correspond to fixed points (at the same time, the asymptotic regions correspond to extrema of the scalar potential)
- Imposing boundary conditions on the field content ( for example, Dirichlet b.c. indicate that the derivative with respect to the radial variable is asymptotically identified with the dilatation operator of the dual field theory)
- Assuming appropriate b.c. Hamiltonian flow equations, which follow from the gravity action, can be brought to the form of the Callan-Symanzik equation for the generating function (`de Boer, Verlinde,Verlinde'98`)
- $e^{\mathcal{A}}$ – measures the field theory energy scale; $\phi(w)$ identifies with the running coupling along the flow; the $\beta$-function of the operator

$$\beta = \frac{d\lambda}{d\log E}|_{QFT} = \frac{d\phi}{dA}_{Holo}$$

# Trunced supergravity model

## $3d\ \mathcal{N}=2$ supergravity action

The supergravity model includes a graviton $e_\mu^{\ a}$, a gravitini $\psi_\mu$, a gauge field $A_\mu$ and $\mathcal{N}=2$ multiplet ($n$ scalar fields $\phi^\alpha$ and $n$ fermions $\lambda^r$)

Deger, Kaya, Sezgin, Sundell (2000)

The bosonic part of the Lagrangian with a complex scalar $\Phi$ (modulus $|\phi|$, phase $\theta$), it parametrizes the coset space $\frac{SU(1,1)}{U(1)} = \mathbb{H}^2$

$$e^{-1}\mathcal{L} = \frac{1}{4}R - \frac{e^{-1}}{16m\,a^4}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho - \frac{|D_\mu\Phi|^2}{a^2(1-|\Phi|^2)} - V(\Phi),$$

where $e = \det e_\mu^{\ a}$, $D_\mu\Phi = (\partial_\mu + iA_\mu)\Phi$, $-4m^2$ is the AdS$_3$ cosmological constant, $a$ the curvature of the scalar manifold. $V(\Phi)$ is given by

$$V(\Phi) = 2m^2C^2\left(2a^2|S|^2 - C^2\right) \qquad C = \frac{1+|\Phi|^2}{1-|\Phi|^2}, \qquad S = \frac{2\Phi}{1-|\Phi|^2}.$$

Introducing the following redefinition of the scalar field

$$C \equiv \cosh\phi\ ,\ |S| \equiv \sinh\phi$$

allows us to come to

$$e^{-1}\mathcal{L} = \frac{1}{4}R - \frac{e^{-1}}{a^4}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho - \frac{1}{4a^2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{4a^2}|S|^2(\partial_\mu\theta + A_\mu)(\partial^\mu\theta + A^\mu) - V(\phi).$$

# The 3d truncated supergravity action with $\mathbb{H}^2$

The truncated action ($\theta = 0$, $A^\mu = 0$) is given by

$$S = \frac{1}{16\pi G_3}\int d^3x\sqrt{|g|}\left(R - \frac{1}{a^2}(\partial\phi)^2 - V(\phi)\right) + \text{G.H.}$$

The potential of the scalar field $V(\phi)$ is

$$V(\phi) = 2\Lambda_{uv}\cosh^2\phi\left[(1-2a^2)\cosh^2\phi + 2a^2\right],$$

where $\Lambda_{uv} < 0$ is a cosmological constant, $a$ is a constant (the curvature of the scalar target space $\mathcal{M}$), $0 < a^2 < 1$.

3d $\mathcal{N} = 2$ gauged supregravity with $\mathbb{H}^2$: Deger'02, AG&Usova'22, Arkhipova et al.'24, AG,Nikolaev&Podoinitsyn'24, AG,Gourgoulhon&Podoinitsyn'24, Gutperle&Hultgreen-Mena'24 (Janus flows)

# The behaviour of the dilaton potential with respect to $a^2$

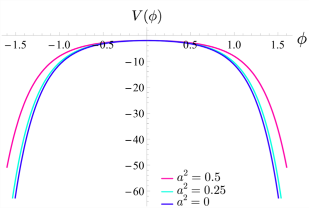


**Figure 1:** The dependence of the dilaton potential $V(\phi)$ for different $a^2$: light blue curve – $a^2 = 0.25$, rose curve - $a^2 = 0.5$

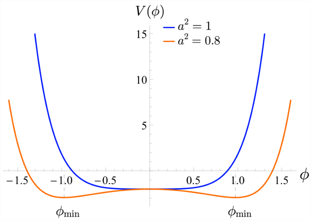


**Figure 2:** The dependence of the dilaton potential $V(\phi)$ for different $a^2$ orange curve – $a^2 = 0.8$; blue curve - $a^2$=1

$$\phi_1 = 0, \quad \phi_{2,3} = \frac{1}{2} \ln \left( \frac{1 \pm 2|a|\sqrt{1 - a^2}}{2a^2 - 1} \right).$$

$a^2 > \frac{1}{2}$ the scalar potential has also zeroes:

$$\phi_{\pm} = \pm \cosh^{-1} \left( \frac{a}{\sqrt{a^2 - \frac{1}{2}}} \right).$$

Extrema of $V(\phi)$ – UV/IR fixed points (CFT) of RG flows, AdS geometries.

$V(\phi) \to \infty$ – scaling fixed points, scaling geometries, if Gubser's bound is ok.

# The conformal dimension of the operator

Holographic RG flows can be associated with deformations of CFT induced in two ways: either by a relevant operator or by non-zero VEV of a scalar operator.
Near $\phi_1 = 0$ (Usova&AG'23, Musaev et. al.'24)

$$V = -2m^2 + 4m^2(a^2 - 1)\phi^2 + \mathcal{O}(\phi^3),$$

while near the other extrema $\phi = \phi_{2,3}$

$$V = -\frac{2a^4 m^2}{2a^2 - 1} - \frac{8a^2(a^2 - 1)m^2}{2a^2 - 1}(\phi - \phi_{2,3})^2 + \mathcal{O}(\phi^3).$$

General solution for the scalar field near extrema

$$\boxed{\phi = \phi_- e^{-\Delta_- w} + \phi_+ e^{-\Delta_+ w}, \quad w \to +\infty,}$$

$\phi_-$ and $\phi_+$ are related to the source and to the VEV of the dual operator $\langle \mathcal{O} \rangle$.

$$\phi_1 : \qquad \Delta_\pm = 1 \pm |1 - 2a^2|, \quad \phi_{2,3} : \qquad \Delta_\pm = 1 \pm \sqrt{1 + \frac{8a^4(1 - a^2)}{2a^2 - 1}},$$

For $\phi_1 = 0$ with $0 < a^2 < 1$ we have $1 \le \Delta_+ < 2$ and $0 < \Delta_- < 1$. The scale factor $A(w) \sim w$ (as $w \to \infty$) near the boundary.

# 3d autonomous dynamical systems,thermal flows $f \neq 1$

# The autonomous dynamical system

- Gukov (2016), Kuipers,Gursoy,Kuznetsov'(2018) E.Kiritsis et.al (2024)

New variables (Kiritsis et.al.'08'14-'19,Aref'eva,Policastro,AG'19):

$$X = \frac{d\phi}{dA} = \frac{\dot{\phi}}{\dot{A}}, \qquad Y = \frac{dg}{dA} = \frac{\dot{g}}{\dot{A}}, \quad z = \frac{1}{1+e^{\phi}}, \quad z \in [0,1] \quad \text{as} \quad \phi \in (-\infty;\infty).$$

Then the equations of motion are brought to the dynamical system on $\mathbf{R}^3$

$$\begin{aligned}
\frac{dz}{dA} &= z(z-1)X, \\
\frac{dX}{dA} &= \left(\frac{X^2}{a^2} - Y - 2\right)\left(X + \mathcal{C}_{(z,a)}\right), \\
\frac{dY}{dA} &= Y\left(\frac{X^2}{a^2} - Y - 2\right),
\end{aligned}$$

where

$$\mathcal{C}_{(z,a)} := \frac{a^2}{2}\frac{V_\phi}{V}, \quad \frac{V_\phi}{V} = \frac{4\left(\left(1-2a^2\right)\left((z-1)^8 - z^8\right) - 2z^6(z-1)^2 + 2z^2(z-1)^6\right)}{(2(z-1)z+1)^2\left((2(z-1)z+1)^2 - 2a^2(1-2z)^2\right)}.$$

**a)** $T = 0 \quad \Leftrightarrow \quad Y = 0,$ **b)** $T \neq 0 \quad \Leftrightarrow \quad Y \to \infty.$

The initial conditions

$$z = [z_1 - \delta, z_1 + \delta] \quad x = 0, \quad y = 1 - \varepsilon,$$

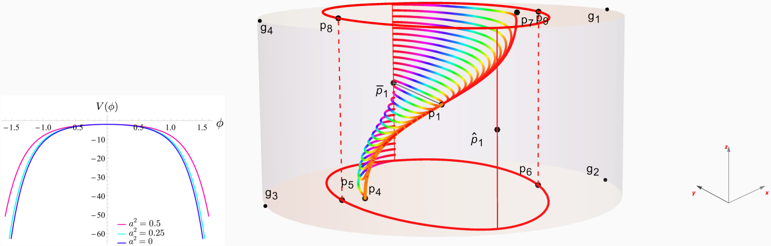


**Figure 3:** AG,Nikolaev&Podoinitsyn (2024), AG,Gougoulhon&Podoinitsyn (2024)

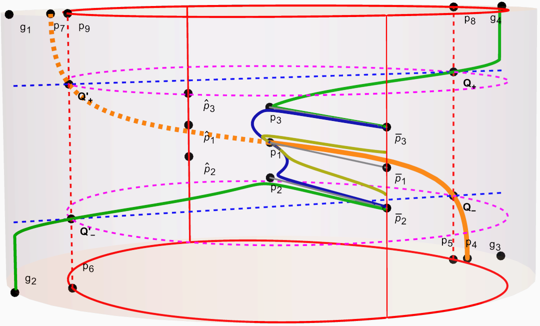


**Figure 4:** Numerical solutions, exact solutions, critical sets, regular and singular points of the $3d$ dynamical system with $a^2 = 0.8$ in the unit cylinder. Numerical trajectories are: $\bar{p}_{1,2,3} - p_{1,2,3}$ (shown by gray), $\bar{p}_{2,3} - p_1$ (shown by blue), $\bar{p}_2 - g_2$, $\bar{p}_3 - g_4$ (both shown by green), numerical solutions from 2(b) (both shown by olive). The exact solutions $p_1 - p_4$ and its mirror image $p_1 - p_7$ are shown by thick orange curves (solid and dashed, correspondingly).AG,Nikolaev&Podoinitsyn (2024), AG,Gougoulhon&Podoinitsyn (2024)

# Solutions from near the horizon to the boundary

## The additional condition

$$X^2 \sim 0.$$

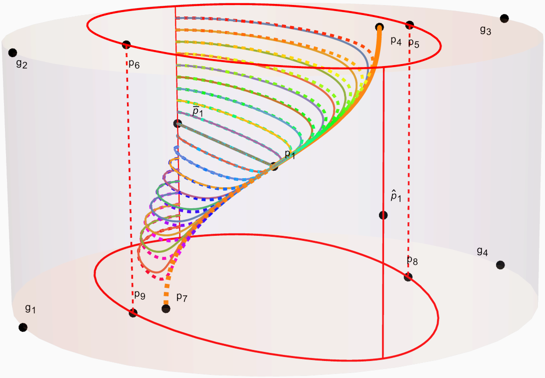


**Figure 5:** The numerical trajectories of the dynamical systems in the cylinder for $a^2 = 0.25$.

The solutions can be obtained using an additional condition

$$X^2 \sim 0.$$

The constraint corresponds to a slowly changing scalar field. EOMs come to the form

$$\frac{dz}{dA} = z(z-1)X, \tag{1}$$

$$\frac{dX}{dA} = -(Y+2)\left(X + \frac{a^2}{2}\frac{V_\phi}{V}\right), \tag{2}$$

$$\frac{dY}{dA} = -Y(Y+2)\,. \tag{3}$$

Near the horizon, $Y \to \infty$

$$Y(A) = \frac{2}{e^{2(A-A_h)} - 1}\,.$$

Expansion of the potential near an extremum

$$\frac{a^2}{2}\frac{V_\phi(\phi)}{V(\phi)}\Big|_{\phi_h} = \Lambda^{(h)} + \mathsf{K}^{(h)}(\phi - \phi_h),$$

$$\Lambda^{(h)} = \frac{a^2}{2}\frac{V_\phi(\phi_h)}{V(\phi_h)}, \quad \Delta^{(h)} = \frac{a^2}{2}\frac{V_{\phi\phi}(\phi_h)}{V(\phi_h)}, \quad \mathsf{K}^{(h)} = \left(\Delta^{(h)} - \frac{2}{a^2}\left(\Lambda^{(h)}\right)^2\right).$$

Returning from $z$ to $\phi$ the other two equations can be represented as

$$r(1-r)\frac{d^2\Phi(r)}{dr^2} + (1-2r)\frac{d\Phi(r)}{dr} - \frac{\mathsf{K}^{(h)}}{2}\Phi(r) = 0,$$

where

$$\Phi := \phi - \phi_h + \frac{\Lambda^{(h)}}{\mathsf{K}^{(h)}}, \quad r = \exp 2(A - A_h),$$

The regular singular points $r = 1$ and $r = \infty$ correspond to the near-horizon and boundary regions, respectively.

The solution for the scalar field near the horizon reads off from the fundamental solution near $r = 1$

$$\Phi(r) = {}_2F_1(a_h, 1 - a_h, 1, 1 - r)\ , a_h = \frac{1}{2}\left(1 - \sqrt{1 - 2\mathsf{K}^{(h)}}\right).$$

The solution is finite in the region from $r = 1$ to $r = \infty$.

The scale factor due to the constraint is

$$\ddot{A} = 0 \quad \Rightarrow \quad A = \mathfrak{c}_A w + \mathrm{c}_2, \quad \mathfrak{c}_A = \sqrt{-\frac{V(\phi_h)}{2}}, \quad \mathrm{c}_2 = 0.$$

The blackness function is $f = \left(1 - e^{-2\mathfrak{c}_A(w - w_h)}\right)$. The scalar field takes the form

$$\phi = \phi_h + {}_2F_1\left(\frac{1 - \sqrt{1 - 2\Delta^{(h)}}}{2}, \frac{1 + \sqrt{1 - 2\Delta^{(h)}}}{2}, 1, 1 - e^{\mathfrak{c}_A(w - w_h)}\right).$$

The constraint brings us that the metric of the solution matches with the metric of the non-rotating BTZ black hole . This is valid for solutions near $\phi_1 = 0$, with $1 \leq \Delta_+ < 2$. ("light operators") Near $\phi_{2,3}$, we have $\Delta_+ > 2$.

# The asymptotics near the boundary

$$\phi = \phi_h + {}_2F_1\left(\frac{1-\sqrt{1-2\Delta^{(h)}}}{2}, \frac{1+\sqrt{1-2\Delta^{(h)}}}{2}, 1, 1-e^{\mathfrak{c}_A(w-w_h)}\right).$$

near the boundary $r \to \infty$ with $\Lambda^{(h)} \sim 0$. The expansion $r \to +\infty$ leads to

$$\phi_{A\to\infty} \simeq \frac{\Gamma(\Delta_- - 1)}{\Gamma(\frac{\Delta_-}{2})^2} e^{-\Delta_+ w/\ell} + \ldots + \frac{\Gamma(\Delta_+ - 1)}{\Gamma(\frac{\Delta_+}{2})^2} e^{-\Delta_- w/\ell} + \ldots,$$

where we used that near $\phi_h = 0$, $\mathfrak{c}_A = 1/\ell$. The conformal dimensions $\Delta_\pm$ of the dual operator are defined by

$$\Delta_\pm = 1 \pm \sqrt{1+M^2\ell^2} = 1 \pm \sqrt{1+\frac{a^2}{2}V_{\phi\phi}(\phi_1)\ell^2}, \quad M^2 = V_{\phi\phi}(\phi_1)$$

the coefficients coincide with those from Balasubramanian, Kraus, and Lawrence'98. The scalar field near the boundary

$$\phi \simeq \phi_- e^{-\Delta_- w} + \ldots + \phi_+ e^{\Delta_+ w/\ell} + \ldots.$$

BTZ in the Poincare like coordinates can be related with AdS$_3$ . Coming to the Poincaré coordinates and doing the change $r_h^2 = -1$ we get

$$\Phi(r) = {}_2F_1\left(\frac{\Delta_-}{2}, \frac{\Delta_+}{2}, 1, 1+r^2\right)$$

in agreement with Freedman et.al.'98,Balasubramanian'98 ( $\tan^{-1} r = \rho$, $\tan\rho = \sinh\mu$.

# Near-horizon solutions and thermodynamics

# Asymptotically BTZ non-rotating BH

The black hole metric can be represented as

$$ds^2 \simeq \left(1 + \frac{\kappa}{\mathrm{c}_A} w\right)^{2/\kappa} \left(-f dt^2 + dx^2\right) + \frac{dw^2}{f},$$

where we introduced the quantity

$$\kappa = \frac{(\Lambda^{(h)})^2}{a^2}, \quad \Lambda^{(h)} = \frac{a^2}{2} \frac{V_\phi(\phi_h)}{V(\phi_h)}, \quad \mathrm{c_A} = \sqrt{-\frac{2}{V(\phi_h)}},$$

with the blackening function reads

$$f = \frac{e^{\mathrm{c}_g}}{\kappa} \ln\left(\frac{1 + \frac{\kappa}{\mathrm{c}_A} w}{1 + \frac{\kappa}{\mathrm{c}_A} w_h}\right), \quad \mathrm{c}_g = \ln\left(2\left(1 + \frac{\kappa}{\mathrm{c}_A} w_h\right)^2\right).$$

The scalar field of the solution is given by

$$\phi(A) = \phi_h - \frac{\Lambda^{(h)}}{\kappa} \ln\left(\frac{1 + \frac{\kappa}{\mathrm{c_A}} w}{1 + \frac{\kappa}{\mathrm{c_A}} w_h}\right).$$

For vanishing $\Lambda^{(h)}$, corresponding to $V_\phi = 0$, the near-horizon solution turns to be the BTZ black hole with a constant scalar field.

$a^2 \leq \frac{1}{2}$, the scalar field at the horizon $\phi_h$ can take any value on the range of $V$; $\frac{1}{2} < a^2 < 1$ we are restricted as $\phi_h \in (\phi_2, \phi_3)$.

The Hawking temperature reads

$$T_H = \frac{e^{A(w_h)}}{4\pi}\left|\frac{df}{dw}\right||_{w=w_h} = \frac{1}{2\pi c_A}\mathsf{B}^{\frac{\kappa+1}{\kappa}},$$

where we introduced the quantity

$$\mathsf{B} = 1 + \frac{\kappa}{c_A}w_h, \quad \kappa = \frac{(\Lambda^{(h)})^2}{a^2}, \quad \Lambda^{(h)} = \frac{a^2}{2}\frac{V_\phi(\phi_h)}{V(\phi_h)}, \quad c_A = \sqrt{-\frac{2}{V(\phi_h)}}.$$

At the extrema of the potentials , $\kappa \to 0$ , the thermodynamics becomes conformal. This can be checked expanding in series by small $\Lambda^{(h)}$, the temperature reads

$$T_{H,\Lambda^{(h)}\to 0} = \frac{e^{w_h/c_A}}{2\pi c_A} + \frac{e^{w_h/c_A}(2c_A - w_h)w_h(\Lambda^{(h)})^2}{4\pi a^2 c_A^3} + \ldots$$

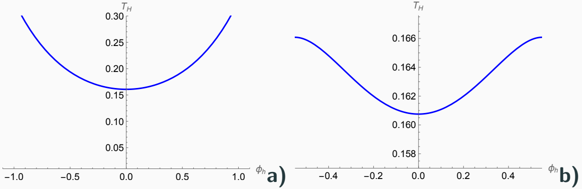


**Figure 6:** Hawking temperature as a function of $\phi_h$: a) $a^2 = 0.25$, b) $a^2 = 0.8$; $w_h = 0.01$.

The entropy of the black hole solution is given by $s = 4\pi M_p \mathrm{B}^{\frac{1}{\kappa}}$ .

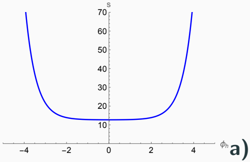


**Figure 7:** Entropy density as a function of $\phi_h$: a) $a^2 = 0.25$, b) $a^2 = 0.8$; for all $w_h = 0.01$.

The behaviour of $s$ as a function of $T_H$.

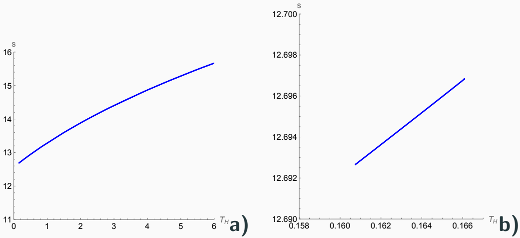


**Figure 8:** Entropy density vs Hawking temperature for a) $a^2 = 0.25$, b) $a^2 = 0.8$; for all $w_h = 0.01$ .

Near $\phi_h = 0$ (i.e. $\Lambda^{(h)} \simeq 0$) the entropy density can be represented as

$$s \sim 4\pi M_p e^{w_h/\mathrm{c}_A} + \dots .$$

Taking into account $\ell = \mathrm{c}_A$ and doing some algebra we can see that at first order of $\Lambda^{(h)}$ we have the conformal behavior of the entropy density $s \sim cT$, where $c = \frac{3\ell}{2G_3}$.

The free energy can be found as follows

$$\mathcal{F} = -\int s dT_H = -\frac{2M_p}{\mathrm{c}_A}\frac{\kappa+1}{\kappa+2}\left(\mathsf{B}^{1+\frac{2}{\kappa}} - 1\right).$$

For $\phi_h = \phi_1$ the thermodynamics is conformal, that can be seen from

$$\mathcal{F}_{\Lambda^{(h)}\to 0} = -\frac{e^{2\frac{w_h}{\mathrm{c}_A}}}{\mathrm{c}_A} - \frac{e^{\frac{2w_h}{\mathrm{c}_A}}(\mathrm{c}_A^2 - 2w_h^2 + 2\mathrm{c}_A w_h)}{2a^2\mathrm{c}_A^3}(\Lambda^{(h)})^2 + \dots,$$

i.e. we get as expected

$$\mathcal{F} \sim T^d, \quad d = 2.$$

The dependence of $\mathcal{F}$ on $T_H$

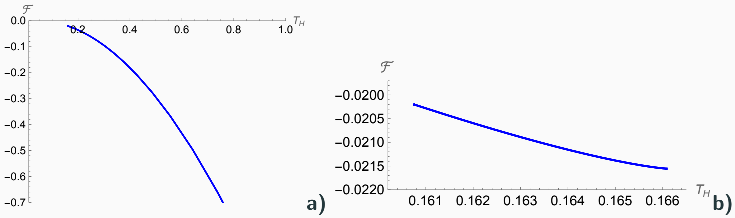


**Figure 9:** Free energy as a function of Hawking temperature for a) $a^2 = 0.25$, b) $a^2 = 0.8$; for all we set $w_h = 0.01$

# Irrelevant deformations, SUGRA with target space $\mathbb{S}^2$

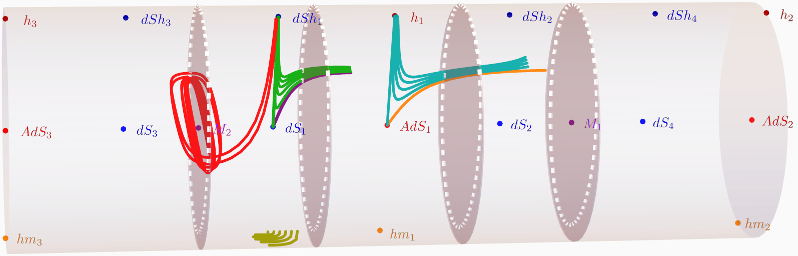


**Figure 10:** Initial conditions: green curves - $(\phi_0, x_0, y_0) = (\phi_{dS_1} - \delta, 0, 1 - \epsilon)$, $0 < \epsilon \ll 1$ , $\delta \in (0, \frac{\pi}{2} - |\phi_{dS_1}|)$; cyan curves - $(\phi_0, x_0, y_0) = (0 + \delta, 0, 1 - \epsilon)$, $0 < \epsilon \ll 1$, $\delta \in (-|\phi_{n1}|, \phi_{n2})$; red curves - $(\phi_0, x_0, y_0) = (\phi_{dS_1} - \delta, 0, 1 - \epsilon)$, $0 < \epsilon \ll 1$, $\delta \in (0, \frac{\pi}{2} - |\phi_{dS_1}|)$.

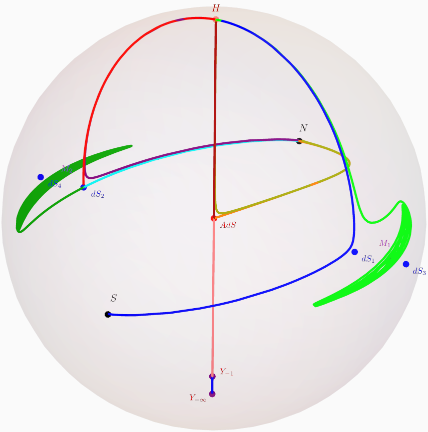


**Figure 11:** The flows: AdS-H (BTZ), AdS-N (AdS-Minkowski), H-dS (SdS), H-dS-S, H-AdS-N.

# Outlook

## Summary

- Numerical thermal RG flows with asymptotics AdS (BTZ)
- The dynamics of the flows can be described by stability analysis
- Analytic form of the solutions near horizon has found
- Special case $X^2 = 0$: the scalar field in BTZ background
- No flows between two dS or two AdS
- No flows between dS and AdS
- New solutions with horizons and singularities (AdS(dS)-strings?)

Thank you for attention!