On the Behavior of Correlation Functions at Large Time-Like Separation in Curved Space-Time

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Introduction

- In a large number of physical systems, correlation functions decay exponentially in time:

$$\langle O(t)O(0)\rangle \sim e^{-\frac{t}{\tau}},$$

  where τ is the relaxation time.
- In a thermal equilibrium system, τ is a function of temperature:

$$\tau = \tau(T).$$

- Example: The velocity correlation function in Brownian motion:

$$\langle v(t)v(0)\rangle \sim e^{-\frac{t}{\tau}}, \quad \text{where} \quad \tau(T) = \frac{m}{6\pi R\eta(T)}.$$

- For a gas, viscosity increases proportionally with the square root of temperature:

$$\eta(T) \sim \sqrt{T}.$$

- In liquids, viscosity usually decreases with increasing temperature.

Flat Space-Time

- The thermal two-point function is defined as follows:

$$W_\beta(x,y) = \frac{\mathrm{Tr}\left[e^{-\beta H}\varphi(x)\varphi(y)\right]}{\mathrm{Tr} e^{-\beta H}}.$$

where β is the inverse temperature.

- In four-dimensional Minkowski space-time, in the high-temperature limit or in the massless case, the late-time behavior of the two-point function at coincident spatial points is given by:

$$W_\beta(t) \sim \sum_{n=-\infty}^{\infty} \frac{1}{(t+i\beta n)^2} \sim T^2 e^{-\frac{2\pi}{\beta}t}.$$

- The relaxation time decreases with increasing temperature, which is common for gases:

$$\tau(T) = \frac{1}{2\pi T}.$$

Viscosity in Weakly Coupled Field Theories

- The relaxation time decreases with increasing temperature, which is common for gases:
$$\tau(T) = \frac{1}{2\pi T}.$$
- This behavior is ***not*** explained by viscosity since we consider non-interacting field theory.
- In interacting field theory, viscosity can be estimated from kinetic theory:
$$\eta \sim \epsilon l \sim \frac{\epsilon}{n\sigma v}$$
using: $\epsilon \sim T^3$, $n \sim T^3$, $\sigma \sim \lambda^2/T^2$ and $v \sim 1$:
$$\eta \sim \frac{T^3}{\lambda^2}.$$
- Viscosity measures the rate of momentum diffusion. Hence, the smaller λ is, the longer a particle travels before colliding with another one, and the easier the momentum transfer.

CFT

- The two-dimensional CFT correlator at finite temperature can be completely fixed using conformal invariance:

$$\langle O(t,x)O(0)\rangle_\beta = \left(\frac{2\pi}{\beta}\right)^{2\triangle}\left(2\cosh\left(\frac{2\pi x}{\beta}\right) - 2\cosh\left(\frac{2\pi t}{\beta}\right)\right)^{-\triangle},$$

  where $\triangle$ is the conformal dimension.
- The late-time correlation function decays exponentially in time:

$$\langle O(t)O(0)\rangle \sim e^{-\frac{2\pi}{\beta}\triangle t}.$$

- In this case, the relaxation time depends on the conformal dimension and also decreases with increasing temperature:

$$\tau(T) = \frac{1}{2\pi T\triangle}.$$

Thermal QFT in Space-Time with Killing Horizons

- Due to Hawking-type radiation, space-times with Killing horizons are usually endowed with a natural (canonical) temperature:

$$\beta_c = \frac{2\pi}{\kappa}.$$

- The requirement of regularity of the Euclidean metric (i.e., the absence of a conical singularity) imposes that τ is compact with period β_c.
- We can also consider a thermal gas with the Planckian density matrix at an arbitrary temperature different from the canonical one.
- If the temperature differs from the canonical one, correlation functions do not possess Hadamard properties on Killing horizons, and the back-reaction on the background geometry becomes strong.
- There is a problem with renormalization of the divergent contribution, since it depends on the state of the system.

dS

- The metric of the de Sitter static patch is given by:

$$ds^2 = -\left(1 - r^2\right) dt^2 + \left(1 - r^2\right)^{-1} dr^2 + r^2 d\Omega_{d-2}^2,$$

where we set the de Sitter radius to $R = 1$.
- The static patch is bordered by the Killing horizon at $r = 1$, where the metric degenerates.
- The thermal two-point function is given by:

$$W_\beta(t) \sim$$

$$\sim \int_{-\infty}^{\infty} d\omega e^{i\omega t} \frac{\sinh(\pi\omega)}{e^{\beta\omega} - 1} \left| \Gamma\left(\frac{1}{4}(d - 1 - 2i\nu - 2i\omega)\right) \Gamma\left(\frac{1}{4}(d - 1 + 2i\nu - 2i\omega)\right) \right|^2,$$

where:

$$\nu = \sqrt{m^2 - \left(\frac{d-1}{2}\right)^2}.$$

dS

- The integrand has poles at:

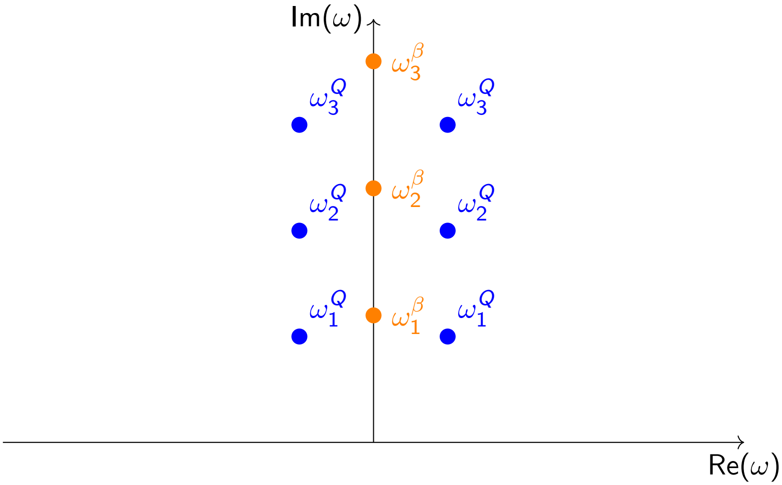
$$\omega_n^Q = \pm\frac{n+d-1}{2} \pm i\nu, \quad n \in \mathbf{Z}^+, \quad \omega_k^\beta = \frac{2\pi ik}{\beta} \quad k \in \mathbf{Z},\ k \neq 0.$$

- In the limit $t \to \infty$, the leading contributions come from the poles closest to the real axis:

$$W_\beta(t) \approx \begin{cases} C_+ e^{-t(\frac{d-1}{2}-i\nu)} + C_- e^{-t(\frac{d-1}{2}+i\nu)} & \text{, if } \beta < \frac{4\pi}{d-1} \\ C_\beta e^{-t\frac{2\pi}{\beta}} & \text{, if } \beta > \frac{4\pi}{d-1} \end{cases}.$$

- The asymptotic behavior of the propagator changes at $\beta = \frac{4\pi}{d-1}$.
- ω_n^Q represents quasi-normal modes (QNM).

Pole Structure



Planar BTZ

- As another example, consider the BTZ black hole:

$$ds^2 = -\left(r^2 - 1\right) dt^2 + \frac{dr^2}{(r^2-1)} + r^2 d\varphi^2.$$

- The thermal two-point function at coincident points on the conformal boundary $(r_1 = r_2 = r \to +\infty,\ \varphi_1 = \varphi_2)$ has the form:

$$W_\beta\left(t, r \to \infty\right) = \frac{r^{-2\Delta_+}}{\Gamma(2\Delta_+)} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\, e^{i\omega t} \frac{\sinh\left(\pi\omega\right)}{e^{\beta\omega} - 1} \Gamma(\Delta_+ + i\omega)\Gamma(\Delta_+ - i\omega),$$

- The integrand has two sets of poles:

$$\omega_n^Q = \pm i\left(n + \Delta_+\right), \quad n \in \mathbb{N} \cup \{0\}; \qquad \omega_k^\beta = \frac{2\pi i}{\beta} k, \quad k \in \mathbb{Z} \setminus \{0\}.$$

Planar BTZ

- In the limit $t \to +\infty$, the leading contributions are:

$$W_\beta(t \to +\infty, r \to \infty) \simeq r^{-2\Delta_+} \begin{cases} C_\beta \, e^{-\frac{2\pi}{\beta} t}, & \beta > \frac{2\pi}{\Delta_+} \\ C_\Delta \, e^{-\Delta_+ t}, & \beta < \frac{2\pi}{\Delta_+} \end{cases} .$$

Relaxation Time

- For systems with a Killing horizon, we have two possibilities for the characteristic relaxation time.
- From the Heisenberg energy-time uncertainty principle: $\triangle E \triangle t \gtrsim 1$ with $\tau = \triangle t$ and $T = \triangle E$, it follows that the relaxation time has a bound:

$$\tau \gtrsim \tau_T = \frac{1}{T},$$

- In space-time with a Killing horizon, one can define quasinormal modes that are purely ingoing near the horizon. These quasinormal modes describe the decay of perturbations and have complex frequencies such that the field decays exponentially at late times:

$$\phi(t) \sim e^{-t\omega_I},$$

where ω_I is the imaginary part of the lowest quasinormal frequency, and

$$\tau_Q \sim \frac{1}{\omega_I}.$$

Results

- We show through several examples that the late-time behavior of thermal correlation functions for a Bose gas in curved space-time with a Killing horizon can change at some critical temperature (T_c):

$$\boxed{W_\beta(t \to +\infty) \sim \int_{-\infty}^{\infty} d\omega \; e^{i\omega t} \frac{\sinh(\pi\omega)}{e^{\beta\omega} - 1} \prod_{\omega_i} \frac{1}{\omega - \omega_i} \sim \begin{cases} e^{-2\pi T t}, & T < T_c \\ e^{-2\pi T_c t}, & T > T_c \end{cases}}.$$

- In the first case, the relaxation time saturates the thermal bound $\tau \sim \tau_T \sim \frac{1}{T}$.
- In the second case, the relaxation time is determined by the imaginary part of quasi-normal modes $\tau \sim \tau_Q$.
- At temperature $T = T_c$, the relaxation time changes, resembling a phase transition.
- At high temperatures, the relaxation time does not decrease to zero as one might expect.

The End