Off-diagonal expansions for some functions of minimal operators: basis kernels and Mellin-Barnes representations

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Heat kernel method

$$\hat{K}_F(\tau|x,x') \equiv e^{-\tau\hat{F}}\delta(x,x') \tag{1}$$

$$\hat{F}^{-1}(\nabla) = -\int_0^\infty d\tau \hat{K}(\tau|x,x'), \quad \log\,\mathrm{Det}\,\hat{F}(\nabla) = -\int_0^\infty \frac{d\tau}{\tau}\,\mathrm{Tr}\,\hat{K}_F(\tau|x,x') \tag{2}$$

Schwinger–DeWitt expansion for Laplace-type operator

$$\hat{F}(\nabla) = -\hat{1}\Box + \hat{P}(x), \qquad \Box \equiv g_{ab}\nabla^a\nabla^b \tag{3}$$

Asymptotic expansion at $\tau \to 0$

$$e^{-\tau\hat{F}(\nabla)}\delta(x,x') = \frac{\Delta^{1/2}(x,x')}{(4\pi\tau)^{d/2}} e^{-\frac{\sigma(x,x')}{2\tau}} \sum_{n=0}^\infty \tau^n\,\hat{a}_n(F|x,x'), \tag{4}$$

- $\sigma(x,x') \equiv \frac{1}{2}[\mathrm{dist}(x,x')]^2$ is the Synge world function,
- $\hat{a}_n(F|x,x')$ are Schwinger-DeWitt coefficients – contain all information about $\hat{F}$.

Nonminimal heat kernel

Proca field operator:

$$H \equiv H^a_b(\nabla) = -\Box\delta^a_b + \nabla^a\nabla_b + R^a_b \equiv F^a_b(\nabla) + \nabla^a\nabla_b. \tag{5}$$

Exact heat kernel [Barvinsky, A.K. 2408.16174]

$$\big[e^{-\tau H}\delta(x,x')\big]^a_b = \big[e^{-\tau F}\delta(x,x')\big]^a_b + \nabla^a\nabla_c\Big[\,\frac{e^{-\tau F}-1}{F}\,\delta(x,x')\Big]^c_b, \tag{6}$$

Second term contains a derivative of **hybrid kernel** of the type $F^{-\mu}e^{-\tau F^\nu}$ — typical for nonminimal heat kernels for operators of any order [Barvinsky, A.K., Wachowski 2508.06439]:

$$e^{-\tau\hat{H}} \simeq \sum_i \hat{\pi}_i^{a_1\ldots a_{2s}}\nabla_{a_1}\ldots\nabla_{a_{2s}}\frac{e^{-\tau\lambda_i F^N}}{F^s}. \tag{7}$$

$$f(\hat{F})\delta(x,x') = ?$$

Basis kernels $\mathbb{B}_\alpha$

$$e^{-\tau\hat{F}}\delta(x,x') = \sum_{n=0}^{\infty} \frac{1}{(4\pi)^{d/2}} \frac{e^{-\frac{\sigma(x,x')}{2\tau}}}{\tau^{d/2-n}} \Delta^{1/2}\hat{a}_n(F|x,x') \tag{8}$$

$$f(\hat{F})\delta(x,x') = \sum_{n=0}^{\infty} \mathbb{B}_{d/2-n}\big(f\big|\sigma(x,x')\big)\,\Delta^{1/2}\hat{a}_n(F|x,x') \tag{9}$$

Same coefficients $\hat{a}_n(\hat{F}|x,x')$ for any* function $f(\hat{F})$ —

—**"Off-diagonal functoriality."**

Representation via inverse Laplace transform

$$f(\hat{F}) = \mathfrak{L}_f e^{-\tau\hat{F}} \equiv \int_0^\infty d\tau\, \bar{f}(\tau)\, e^{-\tau\hat{F}} \tag{10}$$

Applying $\mathfrak{L}_f$ to the DeWitt series term-by-term we get

$$\mathbb{B}_\alpha(f|\sigma) = \mathfrak{L}_f\, \mathbb{B}_\alpha\big(e^{-\tau F}|\sigma\big) = \int_0^\infty d\tau\, \bar{f}(\tau) \frac{\tau^{-\alpha}\, e^{-\frac{\sigma}{2\tau}}}{(4\pi)^{d/2}}, \qquad \alpha \equiv d/2 - n. \tag{11}$$

Treat $\alpha = d/2 - n$ as an arbitrary complex parameter (for now).

Green function of the power

"Schwinger representation:"

$$\hat{F}^{-\mu}\delta(x,x') = \frac{1}{\Gamma(\mu)}\int_0^\infty d\tau\, \tau^{\mu-1}\, e^{-\tau\hat{F}}\delta(x,x'), \tag{12}$$

$$\mathbb{B}_\alpha\left(F^{-\mu}|\sigma\right) = \frac{1}{(4\pi)^{d/2}}\frac{\Gamma(\alpha-\mu)}{\Gamma(\mu)}\left(\frac{\sigma}{2}\right)^{\mu-\alpha} \quad \text{if} \quad \mathrm{Re}(\alpha-\mu)<0. \tag{13}$$

Heat kernel of the power

$$e^{-\tau\hat{F}}\delta(x,x') = \int_{\mathcal{C}}\frac{d\rho}{2\pi i}\tau^{-\rho}\,\Gamma(\rho)\hat{F}^{-\rho}\delta(x,x') \tag{14}$$

$$\mathbb{B}_\alpha\left(e^{-\tau F^\nu}\middle|\sigma\right) = \frac{\tau^{-\alpha/\nu}}{(4\pi)^{d/2}}\mathcal{E}_{\nu,\alpha}\left(-\frac{\sigma}{2\tau^{1/\nu}}\right), \tag{15}$$

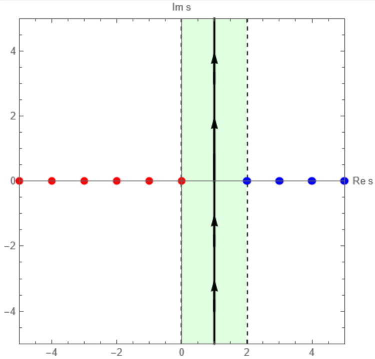
where $\mathcal{E}_{\nu,\alpha}(-z)$ is the Generalized Exponential Function defined via the **Mellin-Barnes integral** [Barvinsky, Pronin, Wachowski 1908.02161].

$$\mathcal{E}_{\nu,\alpha}(-z) = \int_{\mathcal{C}}\frac{ds}{2\pi i}z^{-s}\frac{\Gamma(s)\Gamma\left(\frac{\alpha-s}{\nu}\right)}{\nu\Gamma(\alpha-s)}. \tag{16}$$

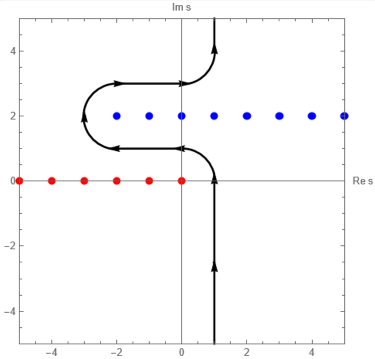
Mellin transform

$$f^*(s) = \mathfrak{M}[f](s) = \int_0^\infty dz\, z^{s-1}\, f(z), \quad c_1 < \operatorname{Re} s < c_2, \tag{17}$$

$$f(s) = \mathfrak{M}^{-1}[f^*](z) = \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{dz}{2\pi i} z^{-s} f^*(s), \quad c_1 < \gamma < c_2. \tag{18}$$



(a) $\Gamma(s)\Gamma(a-s)$, $a > 0$. Inverse Mellin transform contour is straight and lies inside holomorphy strip.



(b) $\Gamma(s)\Gamma(a-s)$, $a \in \mathbb{C} \setminus \{0, -1, ...\}$. Inverse Mellin transform contour is **non-splitting**.

Mellin-Barnes integrals

MB integral is the inverse Mellin transform of the ratio of Γ-functions,

$$\mathfrak{F}(z) = \int_{\mathcal{C}} \frac{ds}{2\pi i}\, z^{-s} \prod_{i,j} \frac{\Gamma(a_i s + b_i)}{\Gamma(c_j s + d_j)} = \mathfrak{M}^{-1}\Big[\prod_{i,j} \frac{\Gamma(a_i s + b_i)}{\Gamma(c_j s + d_j)}\Big](z) \tag{19}$$

along a vertical non-splitting contour $\mathcal{C}$. It defines a wide class of special functions of hypergeometric type [Marichev 1983].

Closing the contour to the left (right) yields asymptotics at $z = 0$ ($z = \infty$),

$$\oint_{\mathcal{C}\cup\{-\infty\}} \frac{ds}{2\pi i} z^{-s} \frac{\Gamma(\dots)}{\Gamma(\dots)} \simeq \sum_n A_n z^n, \qquad \oint_{\mathcal{C}\cup\{+\infty\}} \frac{ds}{2\pi i} z^{-s} \frac{\Gamma(\dots)}{\Gamma(\dots)} \simeq \sum_m B_m \left(\frac{1}{z}\right)^m .$$

IR issues ⟷ contour pinch

For some functions $f(\hat{F})$ the integral

$$\mathbb{B}_\alpha(f|\sigma) = \int_0^{\infty} d\tau\, \bar{f}(\tau) \frac{\tau^{-\alpha}\, e^{-\frac{\sigma}{2\tau}}}{(4\pi)^{d/2}} \tag{20}$$

may be divergent in the IR ($\tau \to \infty$). At the MB level this manifests as a **pinch** as $\alpha \to d/2 - n \in \mathbb{Z}$. This necessitates introduction of mass.

Basis kernels for massive functions

Massive basis kernels — basis kernels for $f(F+m^2)$ are obtained similarly to massless ones,

$$\mathbb{B}_\alpha\left(f(F+m^2)|\sigma\right) = \int_0^\infty d\tau\,\bar{f}(\tau)\frac{\tau^{-\alpha}e^{-\frac{\sigma}{2\tau}-\tau m^2}}{(4\pi)^{d/2}}, \tag{21}$$

however, introduction of a dimensional parameter leads to an additional complex integration, which allows one to **avoid the pinch.**

Example: Massive power Green function

$$\begin{aligned}\mathbb{B}_\alpha\left((F+m^2)^{-\mu}|\sigma\right) &= \frac{2m^{2(\alpha-\mu)}}{(4\pi)^{d/2}\Gamma(\mu)}\mathcal{K}_{\alpha-\mu}(\sigma m^2/2),\\ \mathcal{K}_\nu(z) &= \int_{\mathcal{C}}\frac{ds}{2\pi i}z^{-s}\frac{\Gamma(s)\Gamma(s-\nu)}{2},\end{aligned} \tag{22}$$

where $\mathcal{K}_\nu(z)$ is the Bessel-Clifford function of second kind.

Heat kernel of the power of the massive operator

$$\mathbb{B}_\alpha\left(e^{-\tau(F+m^2)^\nu}\Big|\sigma\right) = \frac{m^{2\alpha}}{(4\pi)^{d/2}} H_1\left(\frac{\sigma m^2}{2}, m^2\tau^{1/\nu}\right),$$
$$H_1(z_1,z_2) = \int_{\mathcal{C}_1} \frac{ds_1 ds_2}{(2\pi i)^2}\, z_1^{-s_1} z_2^{-s_2} \frac{\Gamma(s_1)\Gamma(s_2/\nu)\Gamma(s_1+s_2-\alpha)}{\Gamma(s_2)}. \quad (23)$$

Massive hybrid kernel

$$\mathbb{B}_\alpha\left(\frac{e^{-\tau(F+m^2)^\nu}}{(F+m^2)^\mu}\Bigg|\sigma\right) = \frac{m^{2\alpha}\tau^{\mu/\nu}}{(4\pi)^{d/2}} H_2\left(\frac{\sigma}{2\tau^{1/\nu}}, m^2\tau^{1/\nu}\right),$$
$$H_2(z_1,z_2) = \int_{\mathcal{C}_2} \frac{ds_1 ds_2}{(2\pi i)^2}\, z_1^{-s_1} z_2^{-s_2} \frac{\Gamma(s_1)\Gamma(s_2-\alpha)\Gamma\left(\frac{s_2-s_1-\mu}{\nu}\right)}{\nu\Gamma(s_2-s_1)}. \quad (24)$$

No pinch!

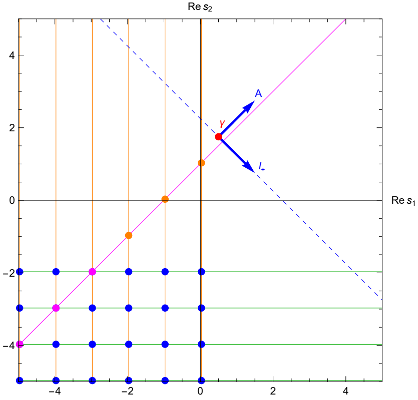
Multidimensional residues ⇒ series representations.

Double MB integrals: mathematically strict theory [Zhdanov, Tsikh '98]
Higher-order MB integrals: Conic hulls method `MBConicHulls.wl` [Ananthanarayan et al 2012.15108], Triangulation method [Banik, Friot 2309.00409].

Example: massive hybrid kernel for $\mu = \nu = 1$

$$\mathbb{B}_\alpha \left(\frac{e^{-\tau(F+m^2)}}{F+m^2} \middle| \sigma \right) = \frac{m^{2\alpha}\tau}{(4\pi)^{d/2}} H_2^{(1,1)}\left(\frac{\sigma}{2\tau}, m^2\tau\right), \qquad (25)$$

$$H_2^{(1,1)}(z_1, z_2) = \int_{\mathcal{C}_2} \frac{ds_1 ds_2}{(2\pi i)^2} \; z_1^{-s_1} z_2^{-s_2} \frac{\Gamma(s_1)\Gamma(s_2-\alpha)\Gamma(s_2-s_1-1)}{\Gamma(s_2-s_1)}.$$



(a) Integrand polar structure for $\alpha \in \mathbb{Z}_{<1}$ (b) Integrand polar structure for $\alpha \in \mathbb{Z}_{>1}$

Example: massive hybrid kernel for $\mu = \nu = 1$

Final answer:

$$\mathbb{B}_\alpha \left(\frac{e^{-\tau(F+m^2)}}{F+m^2} \middle| \sigma \right) = \frac{m^{2\alpha}\tau}{(4\pi)^{d/2}} H_2^{(1,1)}\left(\frac{\sigma}{2\tau}, m^2\tau\right), \tag{26}$$

$$\begin{aligned} H_2^{(1,1)}(z_1, z_2) = & \sum_{\substack{n_1,n_2=0 \\ n_1-n_2+\alpha\neq 1}}^{\infty} \frac{z_2^{-\alpha}}{\alpha+n_1-n_2-1} \frac{(-z_1)^{n_1}}{n_1!} \frac{(-z_2)^{n_2}}{n_2!} \\ & + \frac{(-1)^\alpha}{z_2} \sum_{n=0}^{\infty} \frac{(z_1 z_2)^n}{n!(n-1+\alpha)!} \left[\log z_2 - \psi(n+\alpha)\right] \mathbb{1}\{\alpha \geq 1\} \\ & + \frac{(-1)^\alpha}{z_2} \sum_{n=0}^{\infty} \frac{(z_1 z_2)^{n+1-\alpha}}{n!(n+1-\alpha)!} \left[\log z_2 - \psi(n+1)\right] \mathbb{1}\{\alpha < 1\} \\ & + \frac{1}{z_2} \sum_{n=0}^{-\alpha} \frac{(-\alpha-n)!}{n!} (-z_1 z_2)^n \mathbb{1}\{\alpha < 1\}, \end{aligned} \tag{27}$$

$$\mathbb{1}\{\alpha < 1\} = \begin{cases} 1, & \text{if} \quad \alpha = d/2 - n < 1 \\ 0, & \text{if} \quad \alpha = d/2 - n \geq 1. \end{cases} \tag{28}$$

Summary

- A new functional method for building **off-diagonal** expansions for kernels of functions of minimal operators:

$$f(\hat{F})\delta(x,x') = \sum_{n=0}^{\infty} \mathbb{B}_{d/2-n}\big(f\big|\sigma(x,x')\big)\Delta^{1/2}\hat{a}_n(F|x,x')$$

- Basis kernels $\mathbb{B}_\alpha(f|\sigma)$ for various f are obtained via analytic continuation in $\alpha = d/2 - n$ in terms of multiple Mellin-Barnes integrals.
- With the help of existing techniques these Mellin-Barnes integrals can be converted to multiple series of hypergeometric type.

Acknowledgments

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