Critical Non-Abelian Strings and N=2 SUSY 2D Black Holes

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Main points:

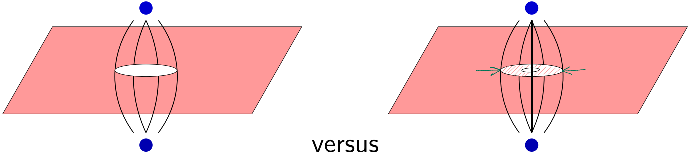
joint with A. Yung, E. Ievlev, G. Sumbatian;

- Vortex non-Abelian strings in conformal ($N_f = 2N$) 4D SQCD as *fundamental* at strong coupling;
- Critical superstring: sigma-model on conifold ($U(2)$ SQCD with $N_f = 4$);
- D-term in 4D $\tau = \frac{1}{2\pi\alpha'} = 2\pi\xi$ – "Planck scale" for 2D;
- D-term in 2D $\beta \sim 1/g^2 \to 0$ – Coulomb branch: $\mathcal{N} = 2$ Liouville (dual to 2D $\mathcal{N} = 2$ black hole);
- Mass deformation (up to $U(4)$ SQCD with $N_f = 8$);
  - T-dual to *the same* theory on world-sheet;
  - Deformation *only* by changing parameters of 2D black hole background;
  - Hadron (confined monopoles!) spectrum from string's effective action;
- Result:
  - Spectrum does not change, but;
  - Multiplicities grow up: (double)-divergence of the black hole entropy ...

Vortex/fundamental non-Abelian strings

Vortex strings in $\mathcal{N} = 2$ 4D SQCD:
confinement of monopoles in Higgs phase (up to EM duality).

Conformal: thick to thin (at $g \to \infty$)



- In 4D theory $b_g = 2N - N_f = 0$;
- 2D conformal theory on world-sheet: GLSM and/or $\mathcal{N} = 2$ Liouville theory;
- For mass deformation $\mathcal{M} \neq 0$ true string vacuum from background EOM;
- Hadron's spectrum in SQCD from world-sheet theory.

Non-Abelian strings

Effective world-sheet theory contains "non-Abelian" modes, fluctuations in flavor space $i, j = 1, \ldots, N = N_f/2$

Action (bosonic part of $\mathcal{N} = (2,2)$): GLSM

$$S = \int d^2z \left\{ \left|\nabla_\alpha n^i\right|^2 + \left|\widetilde{\nabla}_\alpha \rho^j\right|^2 - \frac{1}{4e_0^2} F_{\alpha\beta}^2 + \frac{1}{e_0^2} \, |\partial_\alpha \sigma|^2 + \frac{1}{2e_0^2} \, D^2 \right.$$
$$\left. - \, 2\, |\sigma + \mathcal{M}|^2 \left(\left|n^i\right|^2 + \left|\rho^j\right|^2 \right) + D \left(\left|n^i\right|^2 - \left|\rho^j\right|^2 - \mathrm{Re}\,\beta \right) - \frac{\vartheta}{2\pi} F_{01} \right\}$$

with $(\alpha = 1, 2)$

$$\nabla_\alpha = \partial_\alpha - iA_\alpha \,, \qquad \widetilde{\nabla}_\alpha = \partial_\alpha + iA_\alpha \,, \tag{1}$$

for the charges $\mathtt{Q} = +1$ and $\mathtt{Q} = -1$, bare $e_0 \to \infty$.

Twisted masses $\mathcal{M} = \{\vec{m}_n, \vec{m}_\rho\}$ from 4D quark's bare masses.

Conformal theory

- $\#\{n^i\} = \#\{\rho^j\} = N$, $b_\beta = \sum \mathtt{Q} = 0$ (4D $b_g = 0$);
- Higgs branch: the case of (resolved if $\beta \neq 0$) *conifold* for $N = 2$;
- The central charge

$$\hat{c} \equiv \frac{c}{3} = \dim_{\mathbb{C}}\mathcal{H} = 2N - 1 \underset{N=2}{=} 3,$$

  for critical superstring;
- Coulomb branch with $\langle\sigma\rangle \neq 0$ can develop when $\beta \to 0$, at $\langle n\rangle = \langle\rho\rangle = 0$
  - Integrating out massive $\{n, \rho\}$-fields results in effective $\mathcal{N} = 2$ Liouville theory on Coulomb branch;
  - $N = 2$ corresponds to *deformed* conifold, empty for $N \geq 3$ ($\langle\sigma\bar{\sigma}\rangle \sim \delta_{N,2}$).
  - Mass-deformation of SQCD: deformation of $\mathcal{N} = 2$ Liouville background.

Coulomb action ($\mathcal{M} = 0$)

Integrating out:

- Kinetic term after $\sigma = e^{-\frac{\phi+iY}{Q}}$ with compact $Y + 2\pi Q \sim Y$ runs to

$$S_L = \frac{1}{4\pi}\int d^2x\sqrt{h}\,\left(\frac{1}{2}\,h^{\alpha\beta}(\partial_\alpha\phi\partial_\beta\phi + \partial_\alpha Y\partial_\beta Y) - \frac{Q}{2}\phi\,R^{(2)}\right)$$

  with $Q^2 = 2(N-1)$;
- Adding twisted superpotential $b\int e^{-\frac{\phi+iY}{Q}} + cc$ completes it to $\mathcal{N} = 2$ Liouville;
- Exact spectrum from mirror $\mathcal{N} = 2$ 2D black hole, or $SL(2,\mathbb{R})_k/U(1)$ coset with $k = \frac{2}{Q^2} = \frac{1}{N-1}$;
- Large N (quasiclassics in Liouville) $\leftrightarrow$ large k (quasiclassics in black hole) duality: we need (a selfdual) $k = 1$ point;
- Two deformations: superpotential ("exp" screening) and Kähler ("Wakimoto" screening: mass deformation) - expected duality ...

Coulomb action ($\mathcal{M} \neq 0$)

For the mass-deformed theory in 2D Liouville action:

- Deformed flat background

$$S_L^{\mathcal{M}} = \frac{1}{4\pi}\int d^2x\sqrt{h}\,\left(\frac{1}{2}\,h^{\alpha\beta}g(\phi)(\partial_\alpha\phi\partial_\beta\phi + \partial_\alpha Y\partial_\beta Y) + \Phi(\phi)\,R^{(2)}\right)$$

with

- Nontrivial warp factor

$$g(\phi) = \frac{1}{1-\frac{|b|^2}{|\mathcal{M}|^2}\,e^{-Q\phi}} = \frac{1}{1-e^{-Q(\phi-\phi_0)}}, \qquad \phi_0 = -\frac{1}{Q}\log\frac{|\mathcal{M}|^2}{|b|^2}$$

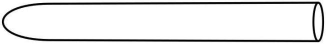
- Nonlinear dilaton

$$\Phi(\phi) = -\frac{Q}{2}\phi + \frac{1}{2}\log g(\phi) = -\frac{Q}{2}\phi - \frac{1}{2}\,\log\left[1 - e^{-Q(\phi-\phi_0)}\right]$$

- Remains intact the (super-) potential $L_{int} = b\int d^2\tilde{\theta}\,e^{-\frac{\phi+iY}{Q}}$.

String theory background

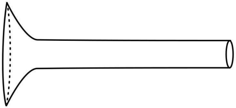
- $\mathcal{N} = 2$ Lioville theory is mirror to the 2D black-hole



with the cigar metric and dilaton

$$ds^2_{\rm cigar} = 2k(d\rho^2 + \tanh^2\rho d\theta^2), \qquad \Phi(\rho) = \Phi_0 - \log\cosh\rho$$

- The mass-deformed theory flows to the trumpet geometry



$$ds^2_{\rm trumpet} = 2k(d\rho^2 + \coth^2\rho d\vartheta^2), \qquad \Phi(\rho) = \Phi_0 - \log\sinh\rho$$

(with $e^{\frac{Q}{2}(\phi-\phi_0)} = \cosh\rho$) T-dual ... to the same 2D black hole!

Background's parameters

- String tension $\tau = \frac{1}{2\pi\alpha'} = 2\pi\xi$ by FI parameter ξ of 4D SQCD;
- In string units (fix $\alpha' = 2$) $R_{\text{cigar}} = \sqrt{2k}$, $R_{\text{trumpet}} = \sqrt{2/k}$ with $k = \frac{2}{Q^2} = \frac{1}{N-1}$:
  - Derivation and quasiclassics in $\mathcal{N} = 2$ Liouville at $Q^2 \to \infty$;
  - Quasiclassics in mirror cigar theory at $k \to \infty$;
  - Quasiclassics in T-dual trumpet theory: for the winding modes;
  - Critical $k_H = 1$ exactly for $N = 2$.
- Dilaton's constant $\Phi_0 = \log g_s$ gives

$$S_{BH} = \frac{M_{BH}}{T} = 2\pi e^{-2\Phi_0} \tag{2}$$

the entropy and/or mass of 2D black hole:
  - Target-space quasiclassics at $g_s \to 0$ or $\Phi_0 \to -\infty$;
  - We conjecture $e^{-2\Phi_0} = e^{-2\Phi_0^{(b)}} + e^{-2\Phi_0^{(\mathcal{M})}}$ or

$$\frac{M_{BH}^{\text{total}}}{Q/2} = \frac{S_{BH}^{\text{total}}}{2\pi} = |b|^{2/k} + \frac{|b|^2}{|\mathcal{M}|^2} \tag{3}$$

String states from effective action

String states

$$\Psi_{j;m_L,m_R}(\phi, Y) = e^{-\Phi} T_{j,m_L,m_R} \underset{\phi\to\infty}{\sim} e^{Q(j+\frac{1}{2})\phi + iQ(m_L Y_L - m_R Y_R)}$$

$$j = -\frac{1}{2}, -1, -\frac{3}{2}, ..., \qquad m = \pm\{j, j-1, j-2, ...\}, \qquad -\frac{k+1}{2} \le j < 0 \qquad (4)$$

give (massless!) "tachyons" $\mathcal{T}_{j,m} = e^{ip_\mu x^\mu}\, T_{j,m,-m}$, and (massive!) "gravitons" $\psi_L^\mu \psi_R^\nu\, e^{ip_\mu x^\mu}\, T_{j;m,-m}$. Effective action leads to the same Schrödinger equation

$$-\partial_\phi^2 \Psi_{j,m} + V_{\text{eff}}(\phi)\Psi_{j,m} = E_j \Psi_{j,m}$$

for their wave-functions $T_{j,m}(\phi, Y) = e^{\Phi}\, e^{iQmY}\Psi_{j,m}(\phi)$, with $E_j = -Q^2\left(j + \frac{1}{2}\right)^2$ and the potential

$$V_{\text{eff}}(\phi) = -\frac{Q^2}{4}\frac{1}{\left(e^{Q(\phi-\phi_0)} - 1\right)^2} - \frac{Q^2(m^2 - j(j+1))}{e^{Q(\phi-\phi_0)} - 1} \underset{\phi\to\phi_0}{\sim} -\frac{1/4}{(\phi-\phi_0)^2} + \dots$$

with famous Calogero singularity.

Schrödinger equation

- Has discrete spectrum, when written in trumpet background, i.e. for winding modes;
- Reduces for $\Psi(\phi) = w(\phi)\frac{\sqrt{e^{Q\phi}-1}}{e^{Qm\phi}}$ and $z = 1 - e^{Q\phi}$ to hypergeometric equation with solution

$$w\Big|_{z\to-\infty} = F(-j-m, 1+j-m, 1; z)\Big|_{z\to-\infty} \sim (-z)^j + R(-z)^{-1-j}$$

- The reflection coefficient

$$R = \frac{\Gamma(-1-2j)\Gamma(1+j-m)\Gamma(1+j+m)}{\Gamma(1+2j)\Gamma(-j-m)\Gamma(-j+m)} \tag{5}$$

  - Gives discrete spectrum $-j - |m| = -n, \qquad n = 0, 1, ...;$
  - Coincides with exact CFT result up to $1/k$-corrections;
  - For discrete spectrum $\Psi_{j,m}(\phi)\sqrt{\coth\rho}\Big|_{j=\pm m=-k/2} = \sqrt{\tanh\rho}\cosh^{1-k}\rho$

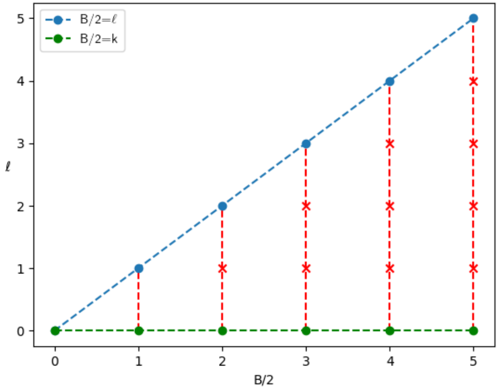
Spectrum

- String theory:
  - Parameters k and $\Phi_0 = \Phi_0(k; b, \mathcal{M})$ (also $\alpha' = \frac{1}{4\pi^2\xi}$) convert to black hole's temperature $T = \frac{1}{2\pi\sqrt{2k}}$ and entropy $S_{BH} = 2\pi e^{-2\Phi_0}$;
  - Background is stable under $1/k$-corrections (SUSY) and spectrum is independent on Φ_0 (and only "indirectly" dependent upon k);
  - Mass deformation only changes the number os states!
- SQCD: the hadron masses (from D- and F- terms)

$$m_H^2 = m_{\text{non-BPS}}^2(\xi) + |Z_{BPS}(m_n, m_\rho)|^2$$

  - $m_{\text{non-BPS}}^2 = M_{\text{string}}^2(\Delta_{j,m})$ cannot be computed in strongly-coupled SQCD, got from string theory;
  - $Z_{BPS} = i\vec{m}_n\vec{q}_n - i\vec{m}_\rho\vec{q}_\rho$ ($\vec{q}$ are $\mathfrak{u}(N) = \mathfrak{su}(N) \oplus \mathfrak{u}(1)_B$ flavor charges) vanishes for mass deformation $\mathcal{M} \sim m_n = m_\rho$;
  - High degeneracy of the spectrum with the same $m_H^2 = m_{\text{non-BPS}}^2(\xi)$ and different charges.

More baryons with mass deformation



States with different baryonic charges. Blue: $\ell = \frac{B}{2}$, $k = 0$, green: $\ell = 0$, $k = \frac{B}{2}$.
Red: new states that emerge as we reduce $\mathcal{M}$.

2D black hole entropy

- Problem with $k = 1$ it corresponds to the Hagedorn point $T = \frac{1}{2\pi\sqrt{2k}} = T_H$;
- Main contribution from effective action

$$s = \left(1 - R\frac{\partial}{\partial R}\right) \log Z \approx \left(R\frac{\partial}{\partial R} - 1\right) S_b^E$$

  for the *thermal scalar* (winding mode with $j = \pm m = -k/2$);
- Results in

$$\frac{(2\pi\alpha')^2}{V_{4D}}\, s = \frac{e^{-2\Phi_0}}{k-1} = \frac{\sqrt{2k}}{k-1} M_{BH}^{\mathrm{total}}$$

  with $e^{-2\Phi_0} = |b|^{2/k} + \frac{|b|^2}{|\mathcal{M}|^2}$.
- Divergence

$$\frac{(2\pi\alpha')^2}{V_{4D}}\, s \sim \frac{1}{k-1} \frac{|b|^2}{\mathcal{M}^2}$$

  at $k \to 1$ (Hagedorn) and $\mathcal{M} \to 0$.

Conclusions

- Critical 2D string gives hadron spectrum for conformal $U(2)$ and $N_f = 4$ SQCD at strong coupling;
- Theory possesses mass deformation "towards" $U(4)$ and $N_f = 8$ SQCD, it results in nonlinear deformation of string background;
- Mass-deformed background is T-dual to mirror Liouville's $\mathcal{N} = 2$ black hole;
- The spectrum remains intact after the deformation, but the multiplicities of the states grow up;
- This effect is justified by "extra" divergency of 2D black hole entropy.