Sigma Models as Spin Chains Limit
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Motivation

1. We want to explore the spectrum of the Laplace-Beltrami operator $H := -\triangle_g$ on a compact manifold $(\mathcal{M}, g)$ acting on $L^2(\mathcal{M})$:

$$H\Psi(x) = -\triangle_g \Psi(x) = E_k \Psi(x), \quad k = 0, 1, \dots \tag{1}$$

   Quantization of a point particle (1D sigma model):

$$\mathcal{S}[x] = \frac{1}{2} \int\limits_{\mathbb{R}} \mathrm{d}t \left(g_{ij} \dot{x}^i \dot{x}^j \right) \tag{2}$$

   with phase space $\mathrm{T}^*\mathcal{M}$.

2. Main difficulty: Solving PDEs is hard.
3. Reasonable simplification: Find first few E_k. Can we find a suitable finite-dimensional truncation

$$H^{(p)} \Psi^{(p)} = E_k \Psi^{(p)}, \quad k = 0, 1, \dots, p, \tag{3}$$

   where $H^{(p)}$ is a finite matrix?

Strategy

- Idea: embed everything into a product of "smallest quantizable objects".
- "Smallest quantum object": an irreducible representation ρ_λ of a symmetry group G.
- Classical analog of ρ_λ?

Answer: (co)adjoint orbit $\mathscr{O}_\lambda \subset \mathfrak{g}^*$ [Kirillov'61]

Key property: $\mathscr{O}_\lambda$ is naturally symplectic ("phase space quanta").

Geometric idea

[Bykov'12], [Bykov,Kuzovchikov'24]

- Suppose $\mathrm{T}^*\mathcal{M}$ is almost symplectomorphic to $\mathscr{O}_\lambda \times \mathscr{O}_{\lambda'} \times \cdots$ (maybe in some limit in $\lambda, \lambda', \ldots$). Then, after geometric quantization, the harmonics of the Laplace operator factorize as $\rho_\lambda \otimes \rho_{\lambda'} \otimes \cdots$ (in this limit):

$$\mathrm{T}^*\mathcal{M} \simeq \lim_{\lambda,\lambda',\ldots} \mathscr{O}_\lambda \times \mathscr{O}_{\lambda'} \times \cdots \quad \Longleftrightarrow \quad L^2(\mathcal{M}) \simeq \lim_{\lambda,\lambda',\ldots} \rho_\lambda \otimes \rho_{\lambda'} \otimes \cdots \tag{4}$$

- Finite λ's provide a natural "spin chain" truncation.
- $H^{(p)}$ is typically an all-to-all spin chain Hamiltonian.
- How to find such "almost symplectomorphism"? Recipe: find a Lagrangian embedding

$$\mathcal{M} \hookrightarrow \lim_{\lambda,\lambda',\ldots} \mathscr{O}_\lambda \times \mathscr{O}_{\lambda'} \times \cdots \tag{5}$$

  with a sufficiently large/dense Weinstein tubular neighborhood which is almost $\mathrm{T}^*\mathcal{M}$ (maybe in a limit in λ's).

Consider a compact semisimple classical group G witn an irrep ρ_λ. Geometric quantization of its orbits is described by *the Borel-Weil-Bott theorem*: there exists a correspondence

$$(\mathscr{O}_\lambda, \omega) \quad \leftrightsquigarrow \quad \mathcal{L}_\lambda \longrightarrow \mathscr{O}_\lambda, \tag{6}$$

where

$$c_1(\mathcal{L}_\lambda) = [\omega] \in H^2(\mathscr{O}_\lambda, \mathbb{Z}) \quad \text{and} \quad \Gamma^{\mathrm{hol}}(\mathcal{L}_\lambda) := H^0(\mathscr{O}_\lambda, \mathcal{L}_\lambda) \simeq \rho_\lambda. \tag{7}$$

Example: $G = \mathrm{SU}(2)$, $\lambda \in \mathbb{Z}_+$ (spin), $\mathscr{O}_\lambda \simeq \mathrm{SU}(2)/\mathrm{U}(1) \simeq \mathbb{S}^2$, $\omega = \lambda \frac{\mathrm{d}z \wedge \mathrm{d}\bar{z}}{(1+|z|^2)^2}$, $\mathcal{L}_\lambda \simeq \mathcal{O}(\lambda)$,

$$\Gamma^{\mathrm{hol}}(\mathcal{O}(\lambda)) \simeq \mathrm{Sym}^\lambda(\mathbb{C}^2) \simeq \text{Fock space of two oscillators with } a^\dagger \circ a = \lambda. \tag{8}$$

Simplest example

1. Particle mechanics on the sphere $\left(\mathrm{T}^*\mathbb{CP}^1\right)$ is almost symplectomorphic to a $\mathbb{CP}^1 \times \mathbb{CP}^1$ spin chain with $\omega = \lambda\left(\omega_1^{FS} + \omega_2^{FS}\right)$ in the large spin limit $\lambda \to \infty$. Intuitively, $\mathbb{CP}^1 \times \mathbb{CP}^1$ is a one-point fiber compactification of $\mathrm{T}^*\mathbb{CP}^1$.
2. Corresponding Lagrangian embedding: $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$

$$\bar{z}_1 \circ z_2 = 0 \quad \leftarrow \quad \left(z_1, z_2\right) \tag{9}$$

where $K_{12} := \bar{z}_1 \circ z_2$ is a "momentum" in $\mathrm{T}^*\mathbb{CP}^1$.

Quantization of $\mathbb{CP}^1 \times \mathbb{CP}^1$

- Quantization yields the Hilbert space:

$$\mathscr{H} \simeq \mathrm{Sym}_\lambda(\mathbb{C}^2) \otimes \mathrm{Sym}_\lambda(\mathbb{C}^2) \simeq \bigoplus_{k=0}^{\lambda} \mathrm{Sym}_{2k}(\mathbb{C}^2). \tag{10}$$

Oscillators $z_i^\alpha \mapsto \lambda^{-\frac{1}{2}} a_i^\alpha$ satisfy:

$$\left[a_i^\alpha, (a_j^\dagger)^\beta\right] = \delta^{\alpha\beta}\delta_{ij}, \quad a_i^\dagger \circ a_i = \lambda, \quad i,j = 1,2, \quad \alpha,\beta = 1,2. \tag{11}$$

States have the form:

$$\Psi_{\alpha_1\ldots\alpha_\lambda|\beta_1\ldots\beta_\lambda} (a_1^\dagger)^{\alpha_1} \cdots (a_1^\dagger)^{\alpha_\lambda} (a_2^\dagger)^{\beta_1} \cdots (a_2^\dagger)^{\beta_\lambda} |0\rangle. \tag{12}$$

- Hamiltonian is quadratic in momenta:

$$H = (a_1^\dagger \circ a_2)(a_2^\dagger \circ a_1) \quad \Longrightarrow \quad E_k = k(k+1), \quad k = 0, 1, \ldots, \lambda. \tag{13}$$

- Observation: limit $\lambda \to \infty$ recovers the spherical harmonics spectrum.

Spin chain limit

- Classical spin chain action:

$$\mathcal{S}[z_1, z_2] = \int_{\mathbb{R}} \mathrm{d}t \Big(i\bar{z}_1 \circ \dot{z}_1 + i\bar{z}_2 \circ \dot{z}_2 - (\bar{z}_1 \circ z_2)(\bar{z}_2 \circ z_1) \Big), \quad \bar{z}_i \circ z_i = \lambda. \tag{14}$$

- <u>Statement:</u> As $\lambda \to \infty$, this action becomes equivalent to the sphere sigma model:

$$\mathcal{S}[u] = \int_{\mathbb{R}} \mathrm{d}t \left(\frac{\dot{u}\dot{\bar{u}}}{\left(1 + |u|^2\right)^2} \right). \tag{15}$$

- Proof idea: "Integrate out" the momenta $K_{12} := \bar{z}_1 \circ z_2$ and $K_{21} := \bar{z}_2 \circ z_1$ using polar decomposition. Define $\mathcal{Z} = (z_1 \;\; z_2)$ as

$$\mathcal{Z} = UH, \quad H^2 := K = \mathcal{Z}^\dagger \mathcal{Z} = \begin{pmatrix} \lambda & K_{12} \\ K_{21} & \lambda \end{pmatrix} \tag{16}$$

where U is unitary. The momenta K_{12} and K_{21} are unrestricted when $\lambda \to \infty$ and can be integrated out. The matrix U can be parametrized by the sphere coordinate u.

Further results

1. Method applies to flag manifolds [Bykov, Kuzovchikov '24]:

$$\mathscr{F}_{n_1,\ldots,n_m} \simeq \frac{\mathrm{U}(n)}{\mathrm{U}(n_1)\times\cdots\times\mathrm{U}(n_m)} \hookrightarrow \mathrm{Gr}(n_1,n)\times\cdots\times\mathrm{Gr}(n_m,n). \tag{17}$$

   Spectrum for various metrics can be computed.
2. Magnetic monopole (Bochner) Laplacian describable via "twisting" of symplectic forms.
3. Generalization to $\mathcal{N}=2$ and $\mathcal{N}=4$ SUSY cases (Dolbeault/de Rham Laplacians, Dirac operator) [Bykov, Krivorol, Kuzovchikov '25].
4. First steps for SO and Sp cases [Bykov, Kuzovchikov '24].
5. Current project: Particle on Lobachevsky plane as two-site $\mathrm{SL}(2,\mathbb{R})$ spin chain.

Discussion

Results:

1. We present finite-dimensional spin chain truncations for the sigma model on the sphere, providing exact solutions for a finite number of harmonics.
2. The spin chain – sigma model connection is established via “polar decomposition variables”.
3. The framework generalizes to flag manifolds with SUSY and monopole field.

Open questions:

1. Extension to other groups: SO, Sp, exceptional, non-compact, infinite-dimensional cases
2. Can we construct “long spin chains” using this method? Are they integrable?