Recent Advances in the Physics of the Inner Crust

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In collaboration:

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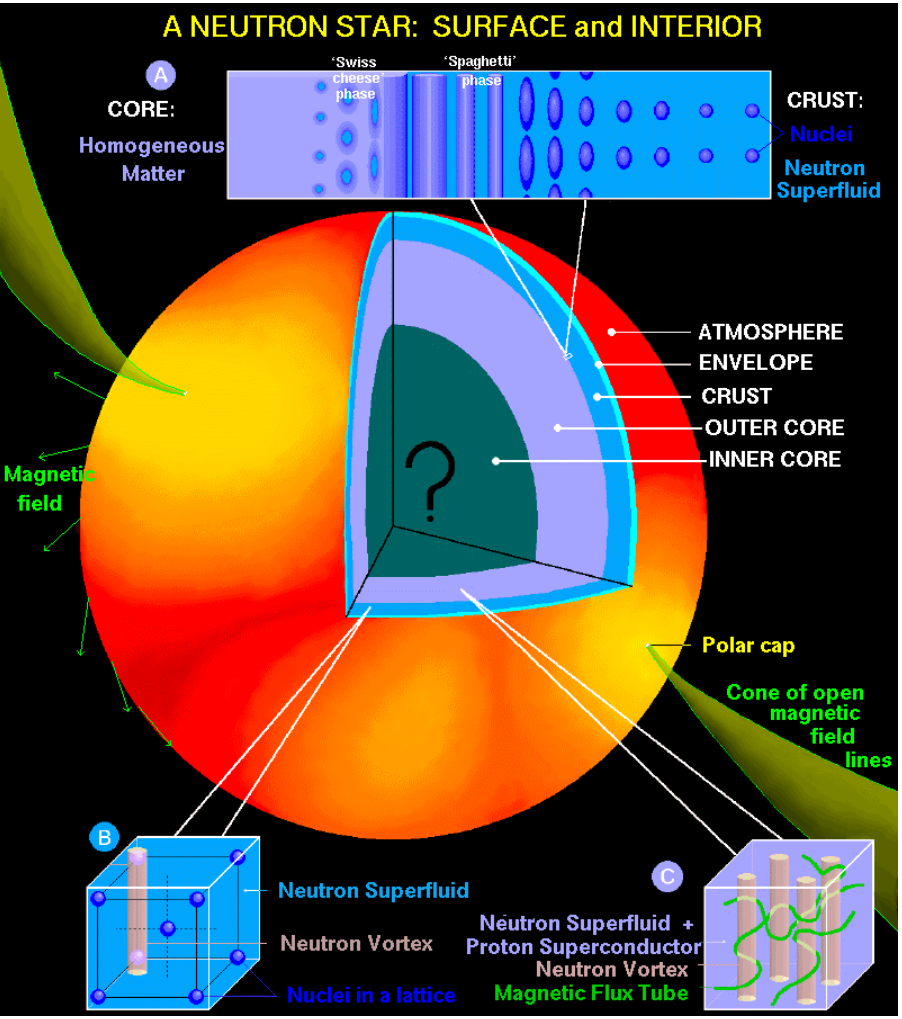
- N.N. Shchechilin, N. Chamel, & AIC (accepted) EPJA
- N. Chamel, N.N. Shchechilin, & AIC (2025), PRC
- A. Y. Potekhin, AIC, N.N. Shchechilin, & N. Chamel, Phys. Usp. (accepted)
- N.N. Shchechilin, N. Chamel, J.M. Pearson, AIC, & A.Y. Potekhin (2024), PRC
- N.A. Zemlyakov, & AIC (2022), Particles

INFINUM-2025, 12-16 May, 2025

BLTR, JINR, Dubna, Russia

Neutron star structure



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$$\rho \sim 10^{15}\ \mathrm{g/cm^3}$$

$$T \lesssim 10^{9}\ \mathrm{K}$$

$$B \sim 10^{12}\ \mathrm{G}$$

$$g \sim 10^{14}\ \mathrm{cm/s^2}$$

$$R \sim 2R_{\mathrm{g}} = 4GM/c^2$$

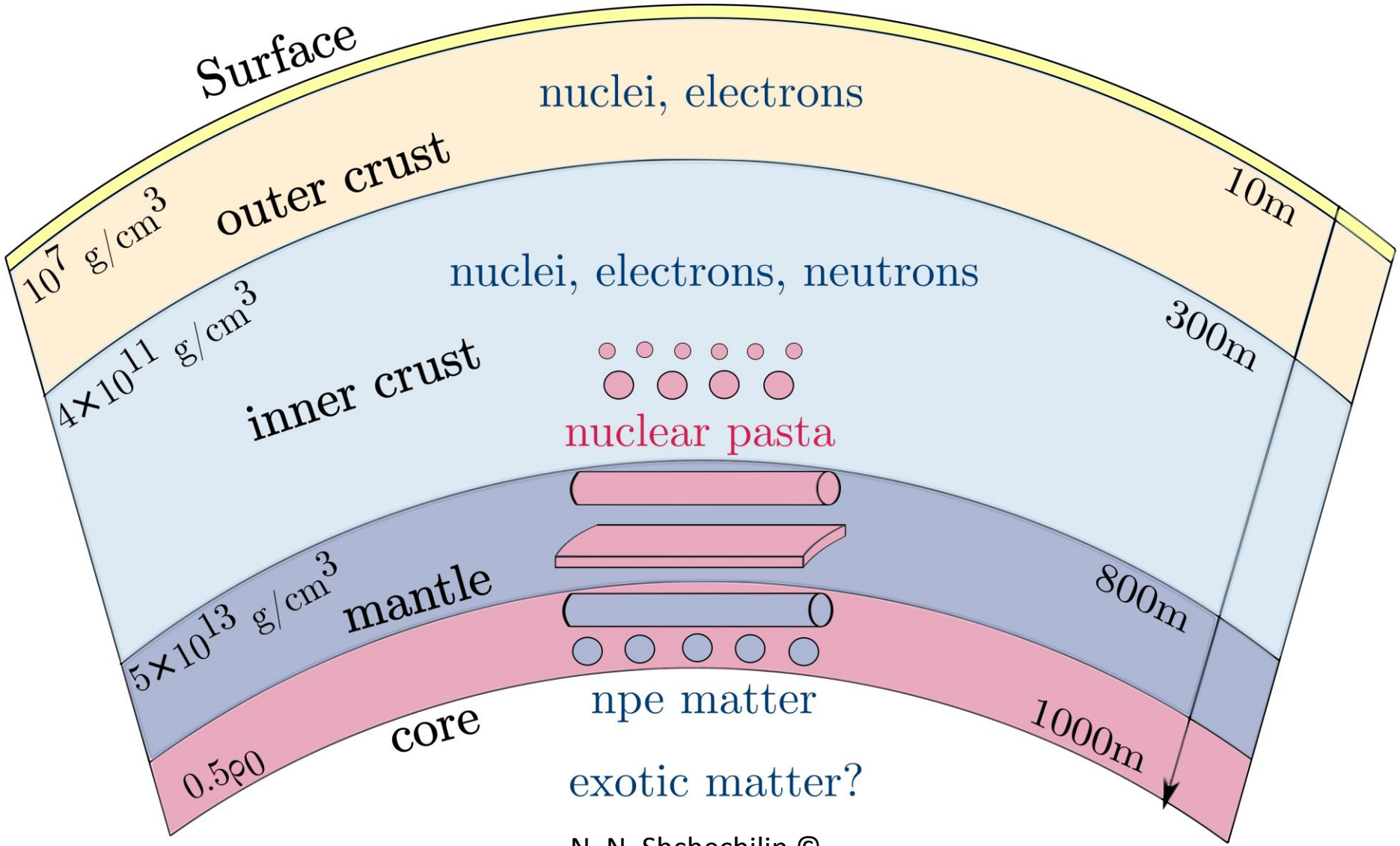
$$T_{\mathrm{cp}} \sim 10^{9}\ \mathrm{K}$$

$$T_{\mathrm{cn}} \sim 10^{8}\ \mathrm{K}$$

$$R \sim 10 - 14\ \mathrm{km},\ M \sim 1.4 M_{\odot}$$

- Neutron star are extreme objects
- They are observed
- Observations are affected by crust

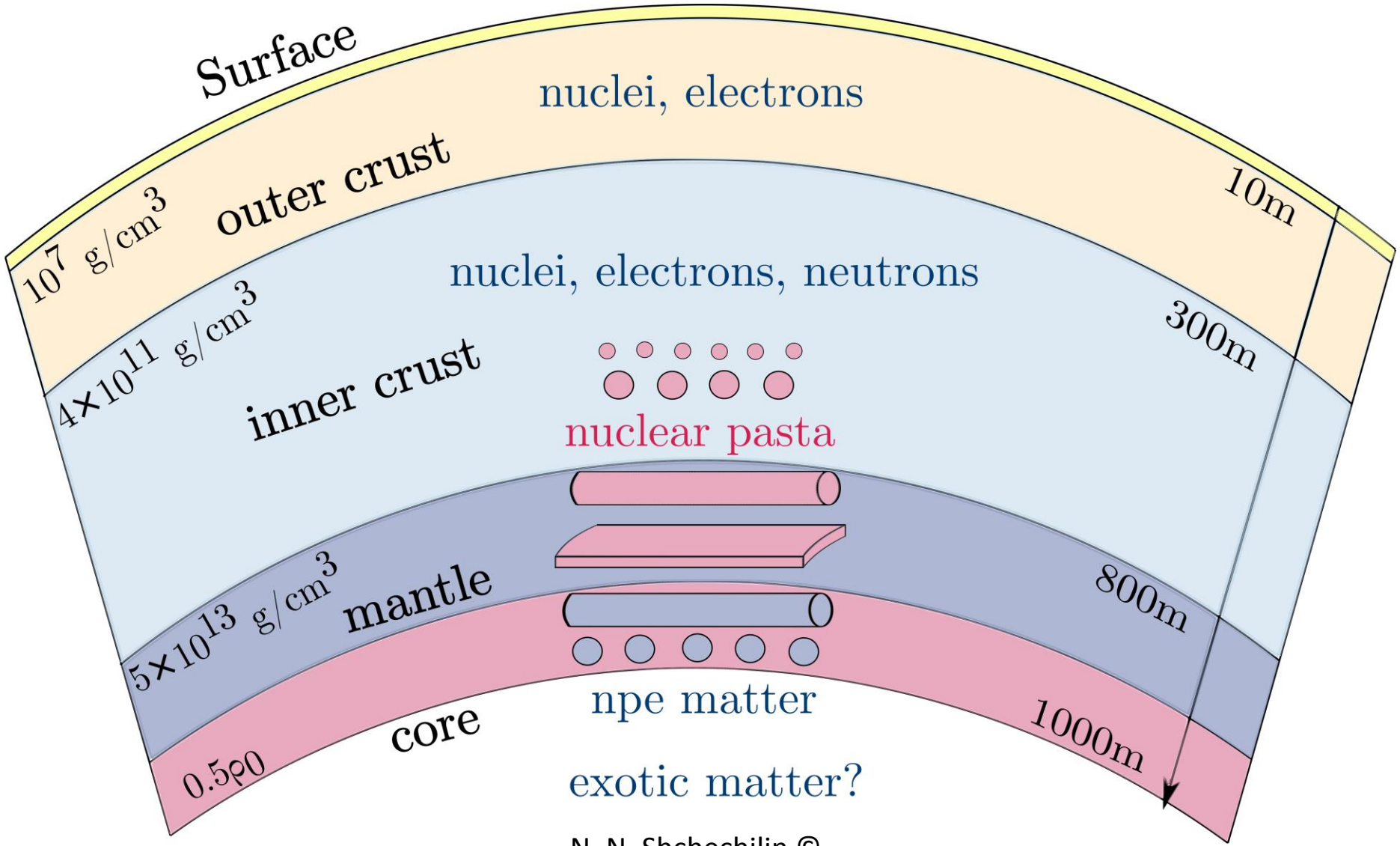
Neutron star crust



N. N. Shchechilin ©

Crust: nonunifrom nuclear matter **with neutralizing background of electrons**

Neutron star crust



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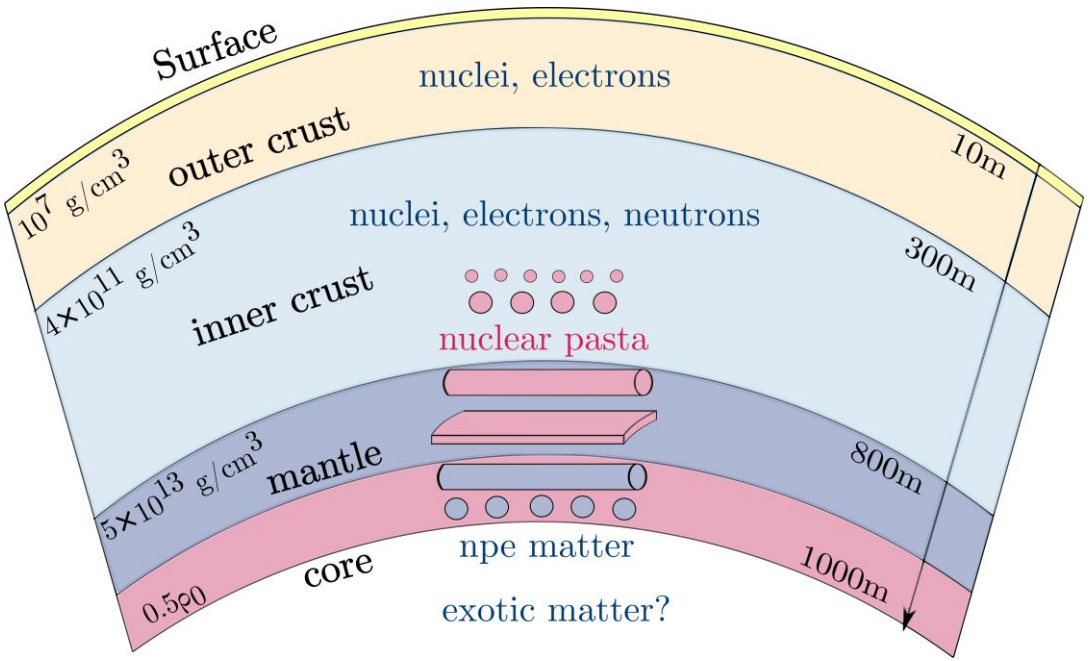
Infinite and Finite Nuclear Matter: (semi)finite clusters on infinite background

That we want to know about the crust?



N. N. Shchechilin ©

- **Composition**
  - Equilibrium
  - Nonequilibrium
- **Equation of state**
  - T=0
  - Thermal properties
  - State of matter (solid/liquid)
- **Dynamical properties One/two liquid hydro (magneto) dynamics**
- **Transport properties (kinetic coefficients)**
- **Elasticity, strength**
- ...

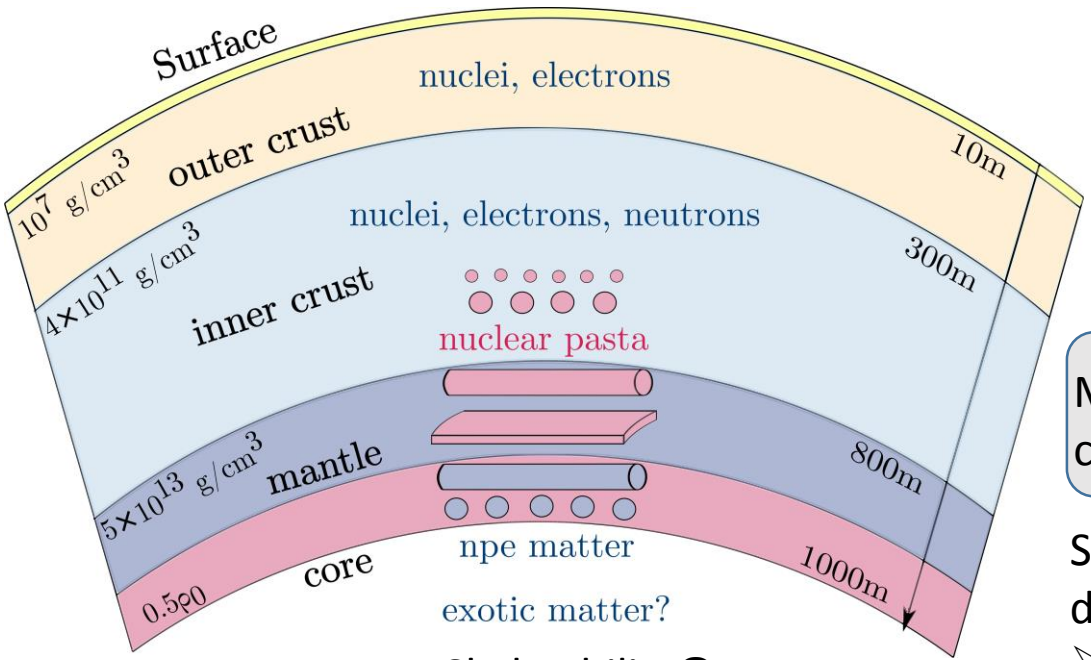
D.G. Yakovlev, HEA2017(?)
Crust as Cinderella of NS

Why???

These properties affect observations, and thus they are required for adequate interpretation of observations

Typically: the main mystery of NSs is the core. The crustal properties should be known accurately to avoid biases for the core properties

That we want to know about the crust?



N. N. Shchechilin ©

- **Composition**
  - Equilibrium
- **Equation of state**
  - T=0

Matter is neutral: proton charge is compensated by degenerate electrons

Specify for all relevant baryon number densities :

- Composition (type of clusters)
- Thermodynamics (energy density, pressure, chemical potentials,…)

Assignment: calculate these properties and provide a tractable procedure to include results into applications. *Results should be trustworthy*!

Approach:

- Calculate energy density for given baryon number density, for each considered structure
- Select optimal structure (with minimal energy)

Recent Advances in the Physics of the Inner Crust

"Everything should be made as simple as possible, but not simpler"

Attributed to Albert Einstein

According to Robinson [*Nature* **557**, 30 (2018)], it can be a compressed version of lines from a 1933 lecture by Einstein:
"*It can scarcely be denied that the supreme goal of all theory is to make the irreducible basic elements as simple and as few as possible without having to surrender the adequate representation of a single datum of experience*."

<u>How to study the crust:</u>

- Specify nucleon interaction model (energy density functional)
- Specify an approach
  - Compressible liquid drop model (CLDM)
  - Extended Thomas-Fermi (ETF)
  - Hartree-Fock(-Bogolyubov) (HFB)
- Calculate

Spherical clusters and pasta phases

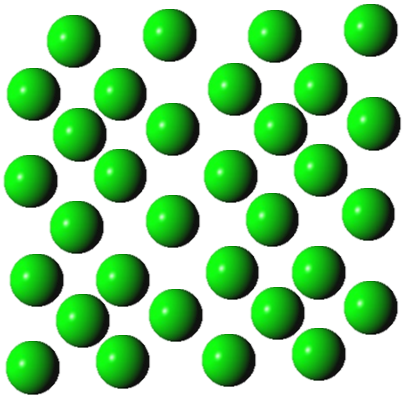
Ravenhall et al. (1984); Hashimoto et al (1984):
In the deepest layers of the crust the clusters becomes essentially nonspherical

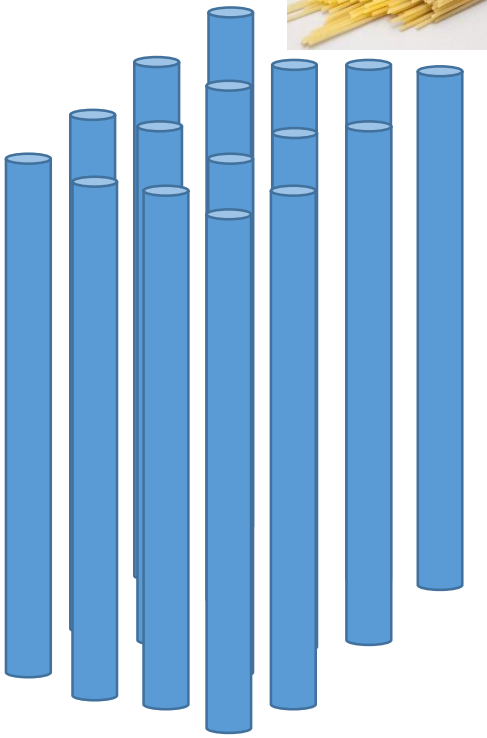
Cluster shape:

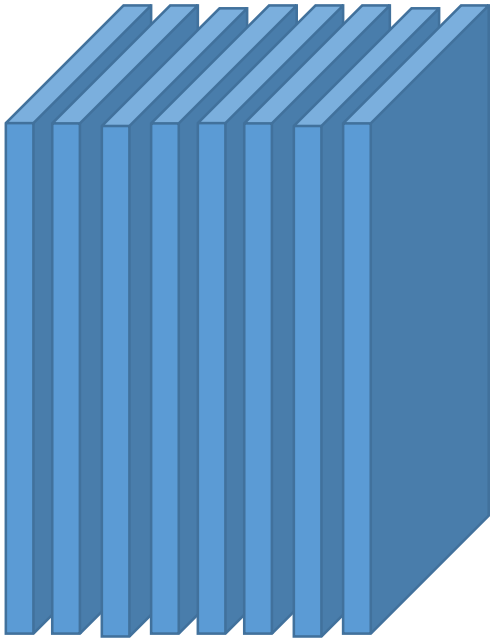
Spheres

Cylinders (spaghetti)

Plates (lasagna)







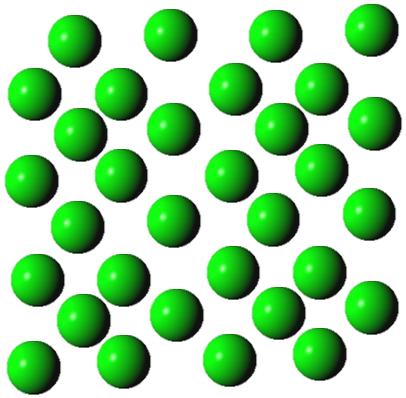
Increase of the density

Spherical clusters and pasta phases

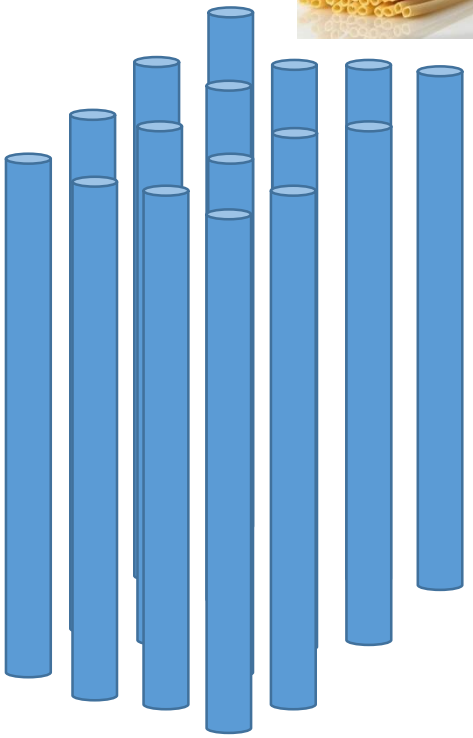
Ravenhall et al. (1984); Hashimoto et al (1984):
In the deepest layers of the crust the clusters becomes essentially nonspherical

Inverse phase (shape of the ‘holes’):

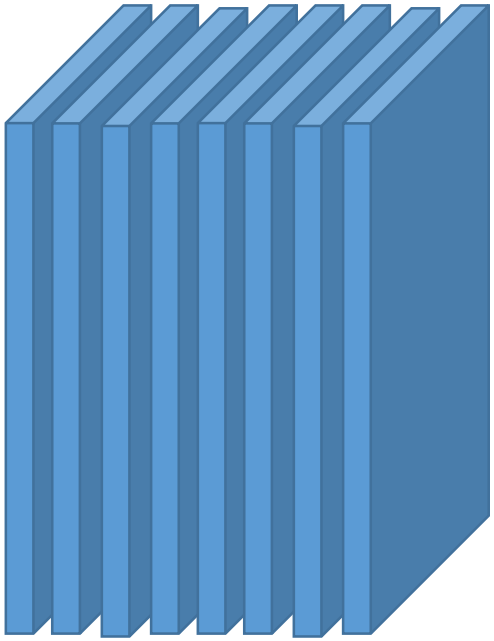
Swiss cheese



Cylinders (bucatini)



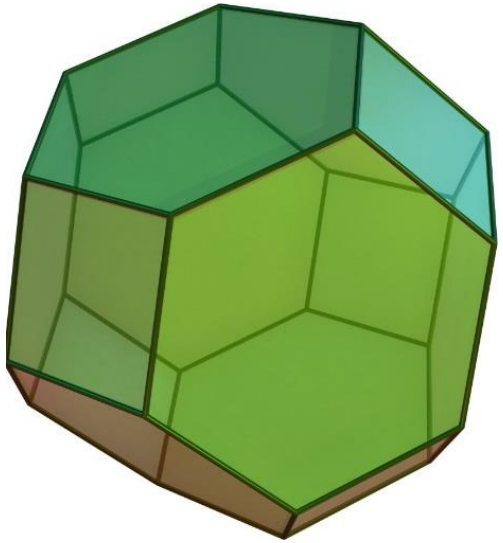
Plates (lasagna)



Increase of the density

Wigner-Seitz (WS) approximation: Spheres instead as unit cells

Clusters form a perfect regular lattice



Energy per cluster = energy of the cell

WS approximation: consider spherical cells

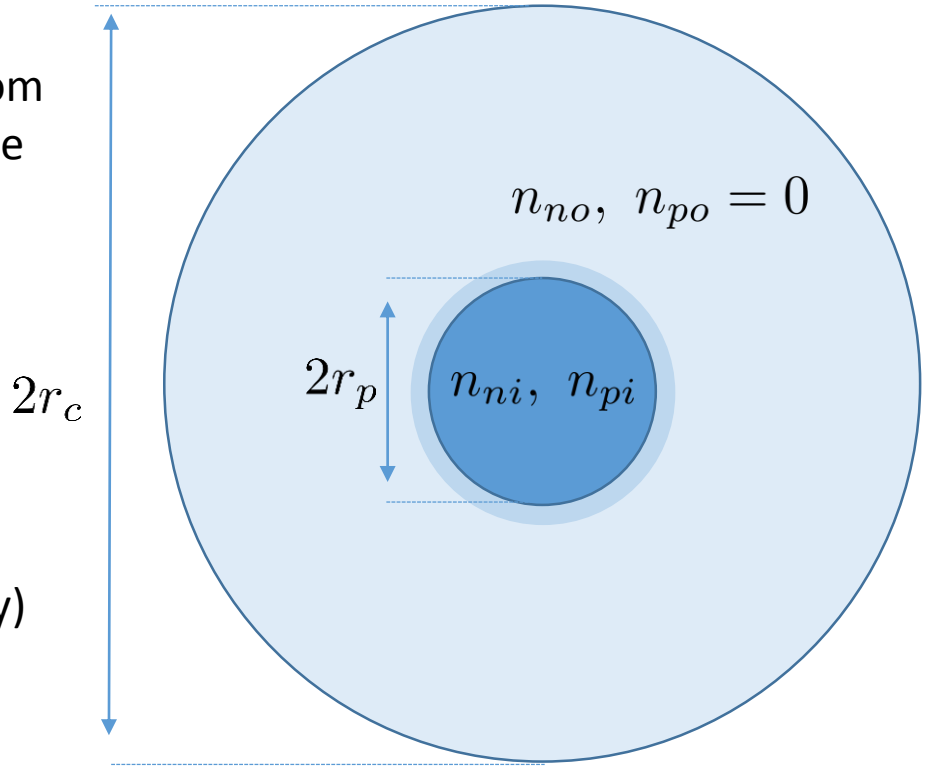
Simplify the analysis: the boundary conditions, symmetry

Compressible liquid drop model for inner crust

- CLDM is rather a class of modes, than one model
- CLDM does not assume that step-like profile for proton and neutron density is real. Rather, general feature of CLDMs is that they start from **explicit analytical expression for energy** of the cell, written as simple as possible, but (reasonable) accurate

$$E = E_{\rm in} + E_{\rm out} + E_{\rm surf} + E_C + E_e$$

- Explicit expressions for all thermodynamic quantities can be obtained analytically. It guaranties absolute (up to numerical accuracy) thermodynamic consistency of the model (if properly applied)



"Everything should be made as simple as possible, but not simpler"

Attributed to Albert Einstein

Compressible liquid drop model for inner crust

- CLDM is rather a class of modes, than one model
- General feature of these models is that they start from explicit analytical expression (for energy)

$$E = E_{\rm in} + E_{\rm out} + E_{\rm surf} + E_C + E_e$$

Nuclear energy inside cluster:

$$E_{\rm in} = \epsilon_{\rm nm}(n_{ni}, n_{pi}) V_i$$

Nuclear energy outside cluster:

$$E_{\rm out} = \epsilon_{\rm nm}(n_{no}, n_{po}) V_o$$

Surface energy

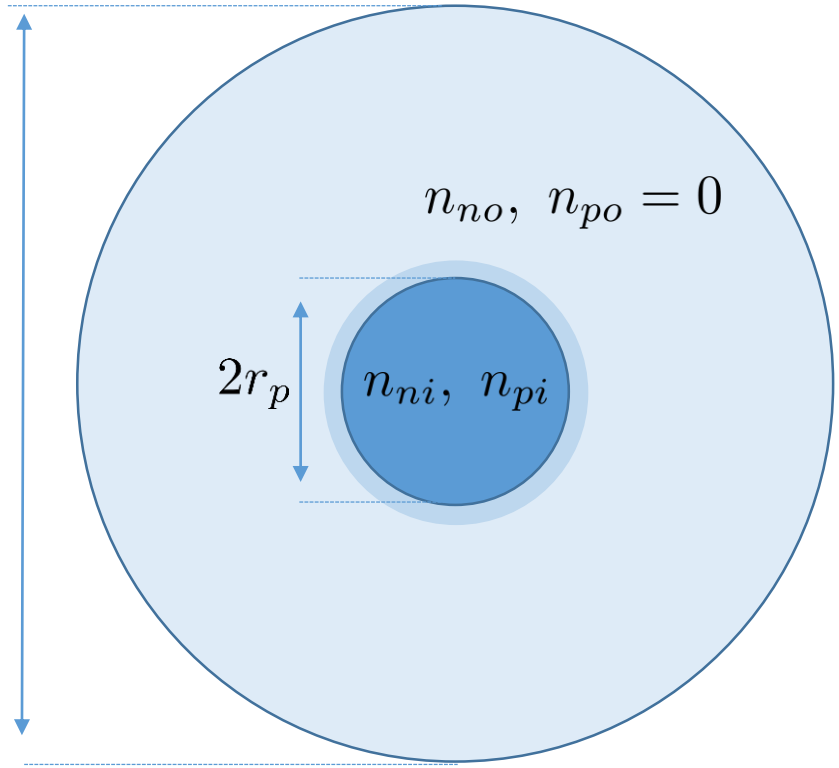
$$E_{\rm surf} = \sigma(\ldots) S + \ldots$$

Coulomb energy: two uniformly charged balls

$$E_C = \frac{3}{5}\frac{Z^2 e^2}{R_N}\left(1 - \frac{3}{2}u^{1/3} + \frac{1}{2}u\right) + \ldots$$

Energy of electrons (uniform degenerate ideal gas)

$$E_e = \epsilon_e(n_e)(V_o + V_i)$$



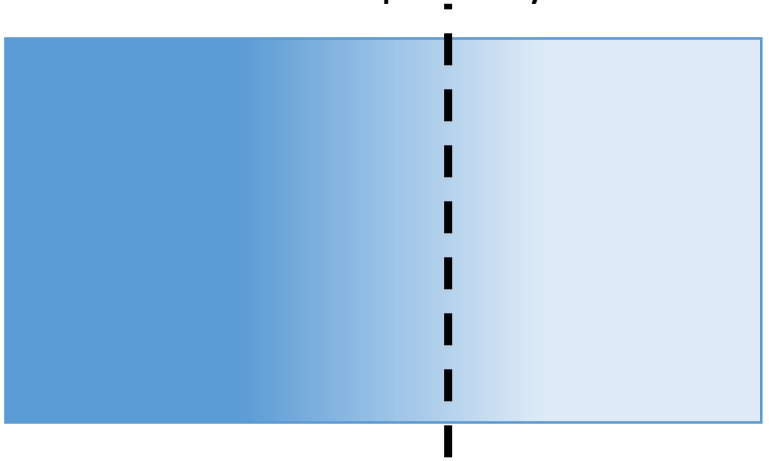
Surface energy? What is it?
Something artificial???

Surface energy is natural. It can be calculated

Centelles et al. Nucl. Phys. A, 635 (1998), 193

Realistic two-phase system

Reference two-phase system



$$E_{\text{two phase}}, N_{\text{two phase}}, Z_{\text{two phase}}$$

Surface energy is natural. It can be calculated

Centelles et al. Nucl. Phys. A, 635 (1998), 193

Realistic two-phase system

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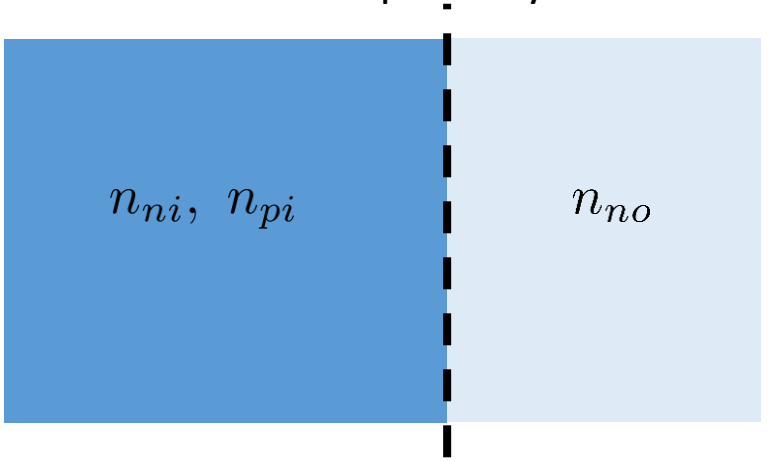


$$E_{\text{two phase}}, N_{\text{two phase}}, Z_{\text{two phase}}$$

$$\begin{aligned} E_{\text{ref}} &= \epsilon_i V_i + \epsilon_o V_o, \\ N_{\text{ref}} &= n_{ni} V_i + n_{no} V_o, \\ Z_{\text{ref}} &= n_{pi} V_i \end{aligned}$$

$$\begin{aligned} E_{\text{surf}} &= E_{\text{two phase}} - E_{\text{ref}} \\ N_{\text{surf}} &= N_{\text{two phase}} - N_{\text{ref}} \\ Z_{\text{surf}} &= Z_{\text{two phase}} - Z_{\text{ref}} \end{aligned}$$

Can be nulled by choose of the reference system

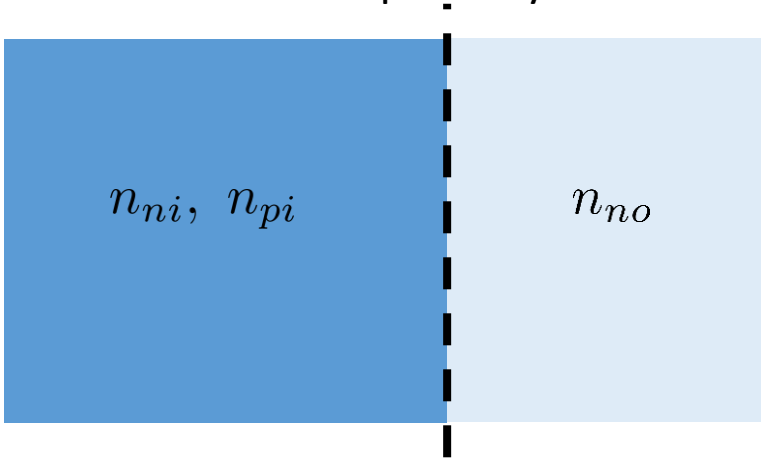
- Surface energy is a correction, required to reproduce two phase system energy
- (Neutron) adsorption is required for thermodynamically consistent description of two phase boundary
- Surface energy describe two-phase thermodynamics precisely

Surface energy is natural. It can be calculated

Centelles et al. Nucl. Phys. A, 635 (1998), 193

Realistic two-phase system

Reference two-phase system



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Can be nulled by choose of the reference system

Basic element of CLDM is a prescription to calculate the surface properties (=correction to the reference energy).
It should be:

- Tractable
- Enough accurate

Compressible liquid drop model: example

$$\epsilon = u\,\epsilon^{\rm nm}(n_{ni},\, n_{pi}) + (1-u)\,\epsilon^{\rm nm}(n_{no},\, 0) + \frac{E_s(\nu_s, r_p)}{V_c} + \frac{E_C(n_{pi}, r_p, w)}{V_c} + \epsilon_e(n_e).$$

Explicit (algebraic) expression for the energy density

6 parameters:

$$n_{ni},\, n_{pi},\, n_{no},\, \nu_s,\, V_c,\, r_p$$

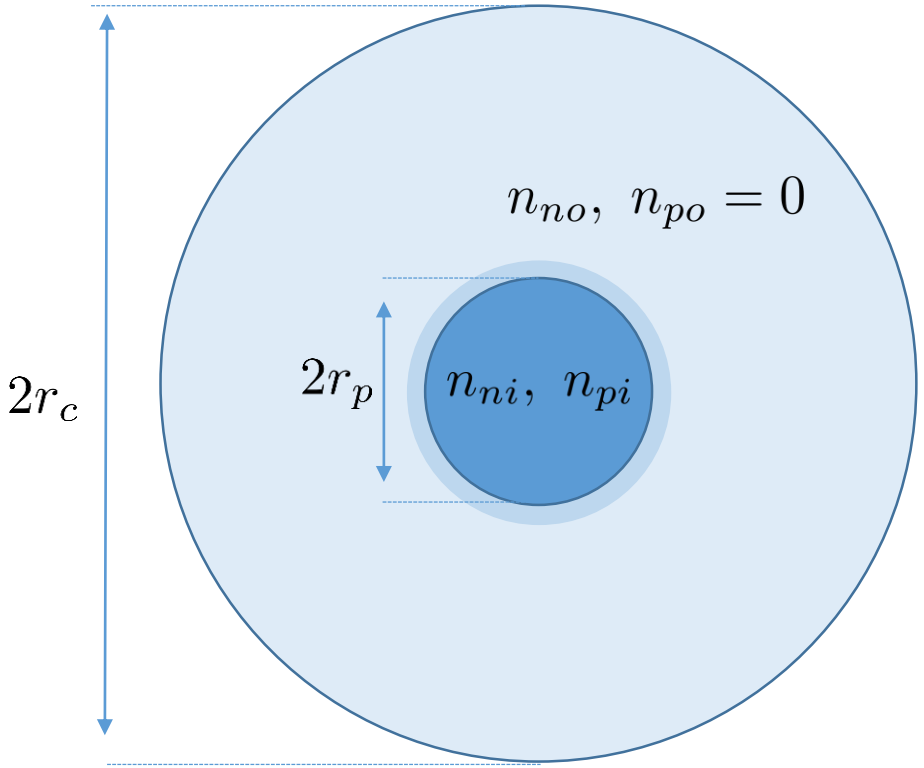
Minimization at fixed n_b

System of 5 algebraic (nonlinear) equations
With clear physical meaning:

- Chemical equilibrium inside the cell (2 eqs.)
- Beta-equilibrium (1 eq)
- Mechanical equilibrium (1 eq)
- Optimal size of the cell (1 eq)

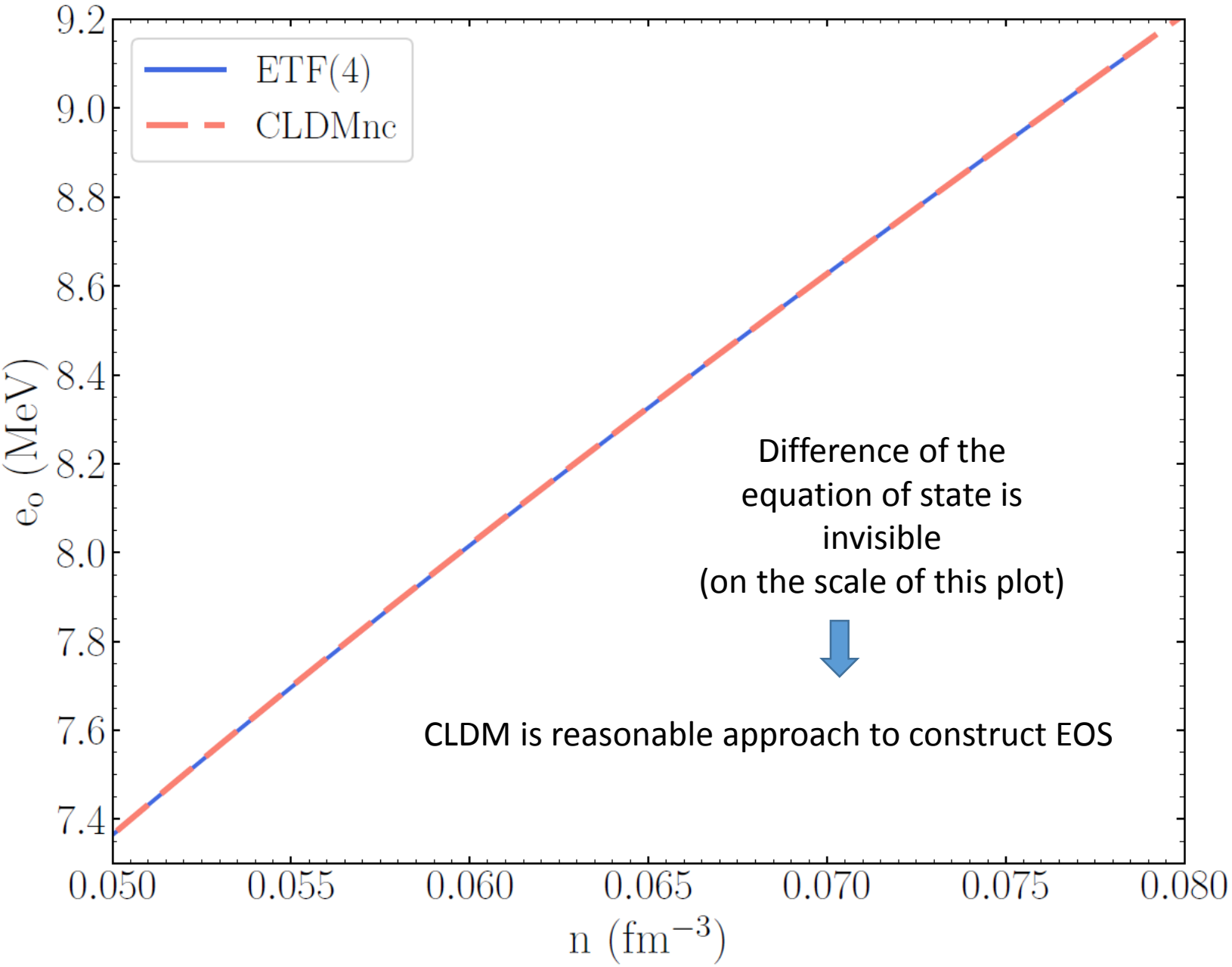
Explicit formulae for thermodynamic quantities – absolute thermodynamic consistency!

$$\mu_n,\; P,\; \ldots$$



Gusakov, AIC (2020)

Inner crust: Equation of state



Compressible liquid drop model and pasta phases: universal filling fractions for transitions

Compare two phases (A and B) with same

$$n_{ni},\ n_{pi},\ n_{no}, u = V_{\rm cluster}/V_{\rm cell}$$

$$\epsilon^A = \frac{E^A}{V^A_{\rm cell}} = u\epsilon^{\rm nm}(n_{ni}, n_{pi}) + (1-u)\epsilon^{\rm nm}(n_{no}, 0) + \frac{E^A_{\rm surf}}{V^A_{\rm cell}} + \frac{E^A_C}{V^A_{\rm cell}} + \epsilon_e(n_e = u n_{pi})$$

$$\epsilon^B = \frac{E^B}{V^B_{\rm cell}} = u\epsilon^{\rm nm}(n_{ni}, n_{pi}) + (1-u)\epsilon^{\rm nm}(n_{no}, 0) + \frac{E^B_{\rm surf}}{V^B_{\rm cell}} + \frac{E^B_C}{V^B_{\rm cell}} + \epsilon_e(n_e = u n_{pi})$$

Oyamatsu et al. (1984); Shchechilin et al. (accepted)

Compressible liquid drop model and pasta phases: universal filling fractions for transitions

Compare two phases (A and B) with same

$$n_{ni},\ n_{pi},\ n_{no}, u = V_{\rm cluster}/V_{\rm cell}$$

(adjusted to the phase A)

$$\frac{E^A_{\rm surf}}{V^A_{\rm cell}} + \frac{E^A_C}{V^A_{\rm cell}} \Leftrightarrow \frac{E^B_{\rm surf}}{V^B_{\rm cell}} + \frac{E^B_C}{V^B_{\rm cell}}$$

$$\begin{aligned} V_c &= a^3 \\ E_{\rm surf} &= \sigma_s g_s(u) a^2 \\ E_C &= (e n_e)^2 w(u) a^5 \end{aligned}$$

Size of the cell is optimal for each phase:

$$\begin{aligned} E^A_{\rm surf} &= 2E^A_C \\ E^B_{\rm surf} &= 2E^B_C \end{aligned}$$

Shape	$g_s(u)$	$w(u)$
sphere	$(6\pi^{1/2}u)^{2/3}$	$(9\pi)^{1/3}(2u^{5/3} - 3u^2 + u^{8/3})/(5\cdot 2^{1/3})$
spaghetti	$(4\pi u)^{1/2}$	$u^2(u - 1 - \ln u)/2$
lasagna	2	$\pi u^2(1-u)^2/6$

Oyamatsu et al. (1984); Shchechilin et al. (accepted)

Compressible liquid drop model and pasta phases: universal filling fractions for transitions

Shape	$g_s(u)$	$w(u)$
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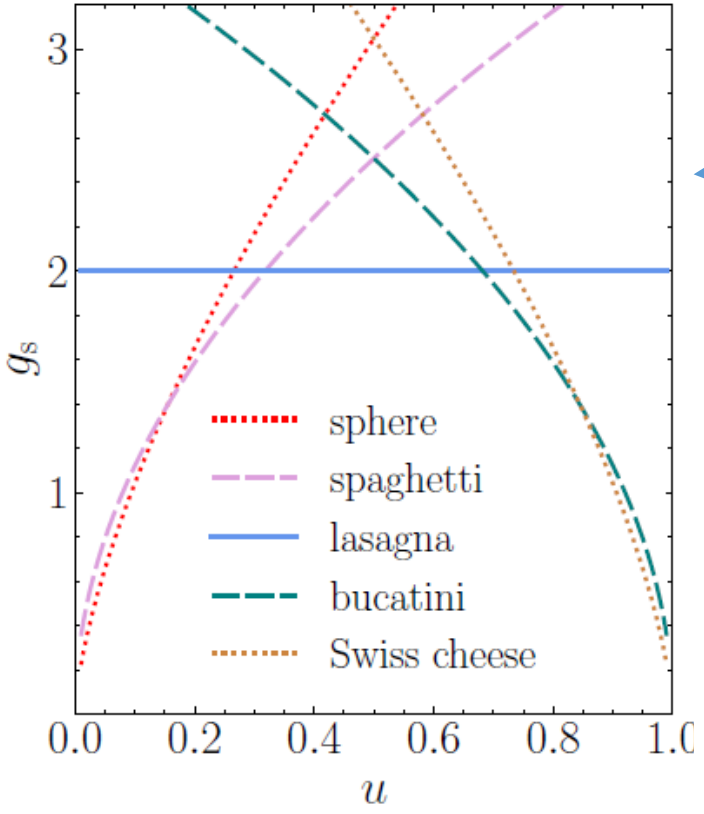
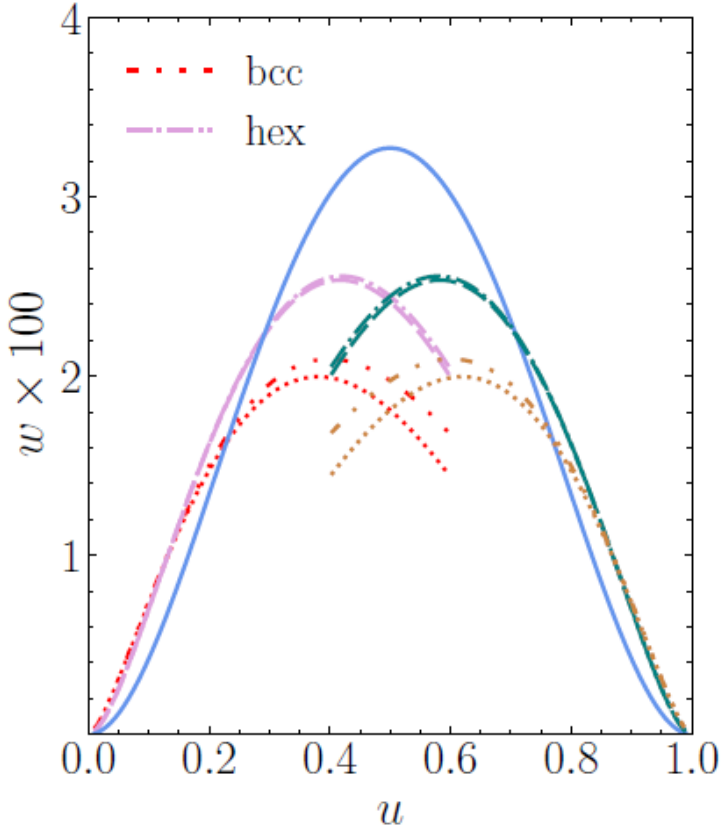


Pasta-shaped clusters can be formed without Coulomb interaction for fixed box size



Oyamatsu et al. (1984); Shchechilin et al. (accepted)

Compressible liquid drop model and pasta phases: universal filling fractions for transitions

Compare two phases (A and B) with same

$$n_{ni},\ n_{pi},\ n_{no}, u = V_{\rm cluster}/V_{\rm cell}$$

$$\frac{E^A_{\rm surf}}{V^A_{\rm cell}} + \frac{E^A_C}{V^A_{\rm cell}} \Leftrightarrow \frac{E^B_{\rm surf}}{V^B_{\rm cell}} + \frac{E^B_C}{V^B_{\rm cell}}$$

$$\begin{aligned} V_c &= a^3 \\ E_{\rm surf} &= \sigma_s g_s(u) a^2 \\ E_C &= (e n_e)^2 w(u) a^5 \end{aligned}$$

Size of the cell for each phase:

$$\begin{aligned} E^A_{\rm surf} &= 2E^A_C \\ E^B_{\rm surf} &= 2E^B_C \end{aligned}$$

$$\left(\frac{g^B_{\rm s}}{g^A_{\rm s}}\right)^{2/3} \left(\frac{w^B}{w^A}\right)^{1/3} < 1$$ ⇒ Phase A is unstable (have larger energy, than phase B)

WS approximation:

$$u_{\rm t} = [0.19, 0.35, 0.65, 0.81]$$

Phase transitions at certain filling fractions (for any nucleon interaction).

Oyamatsu et al. (1984); Shchechilin et al. (accepted)

Compressible liquid drop model and pasta phases: universal filling fractions for transitions

Compare two phases (A and B) with same

$$n_{ni},\ n_{pi},\ n_{no}, u = V_{\rm cluster}/V_{\rm cell}$$

$$\frac{E^A_{\rm surf}}{V^A_{\rm cell}} + \frac{E^A_C}{V^A_{\rm cell}} \Leftrightarrow \frac{E^B_{\rm surf}}{V^B_{\rm cell}} + \frac{E^B_C}{V^B_{\rm cell}}$$

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Size of the cell for each phase:

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$$\left(\frac{g^B_{\rm s}}{g^A_{\rm s}}\right)^{2/3} \left(\frac{w^B}{w^A}\right)^{1/3} > 1$$ ⇒ Phase A is unstable (have larger energy, than phase B)

Account for lattice structure

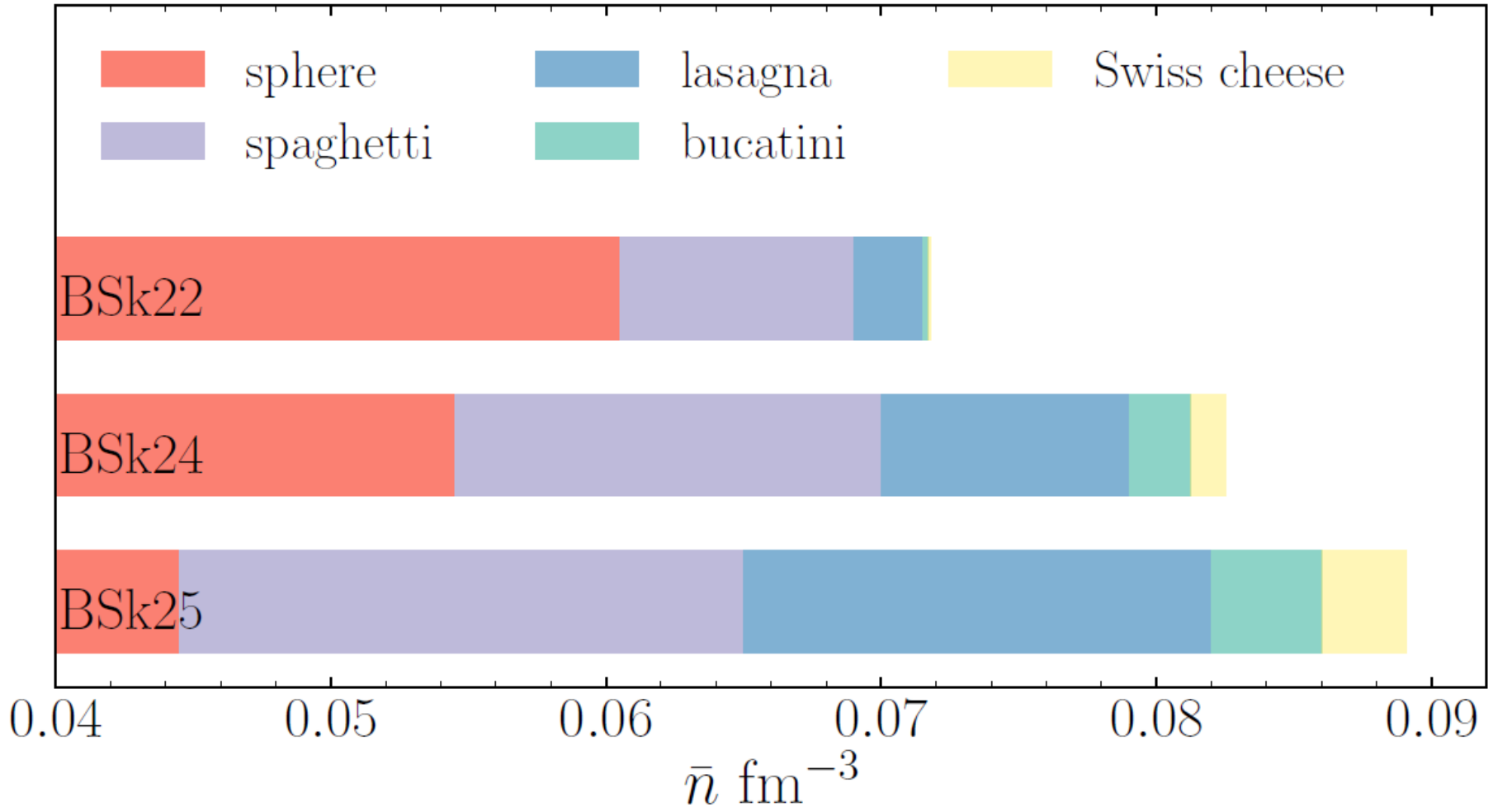
$$u_{\rm t} = [0.215, 0.355, 0.645, 0.785]$$

Phase transitions at certain filling fractions (for any nucleon interaction).

Shchechilin et al. (accepted)

Pasta within Extended Thomas-Fermi calculations

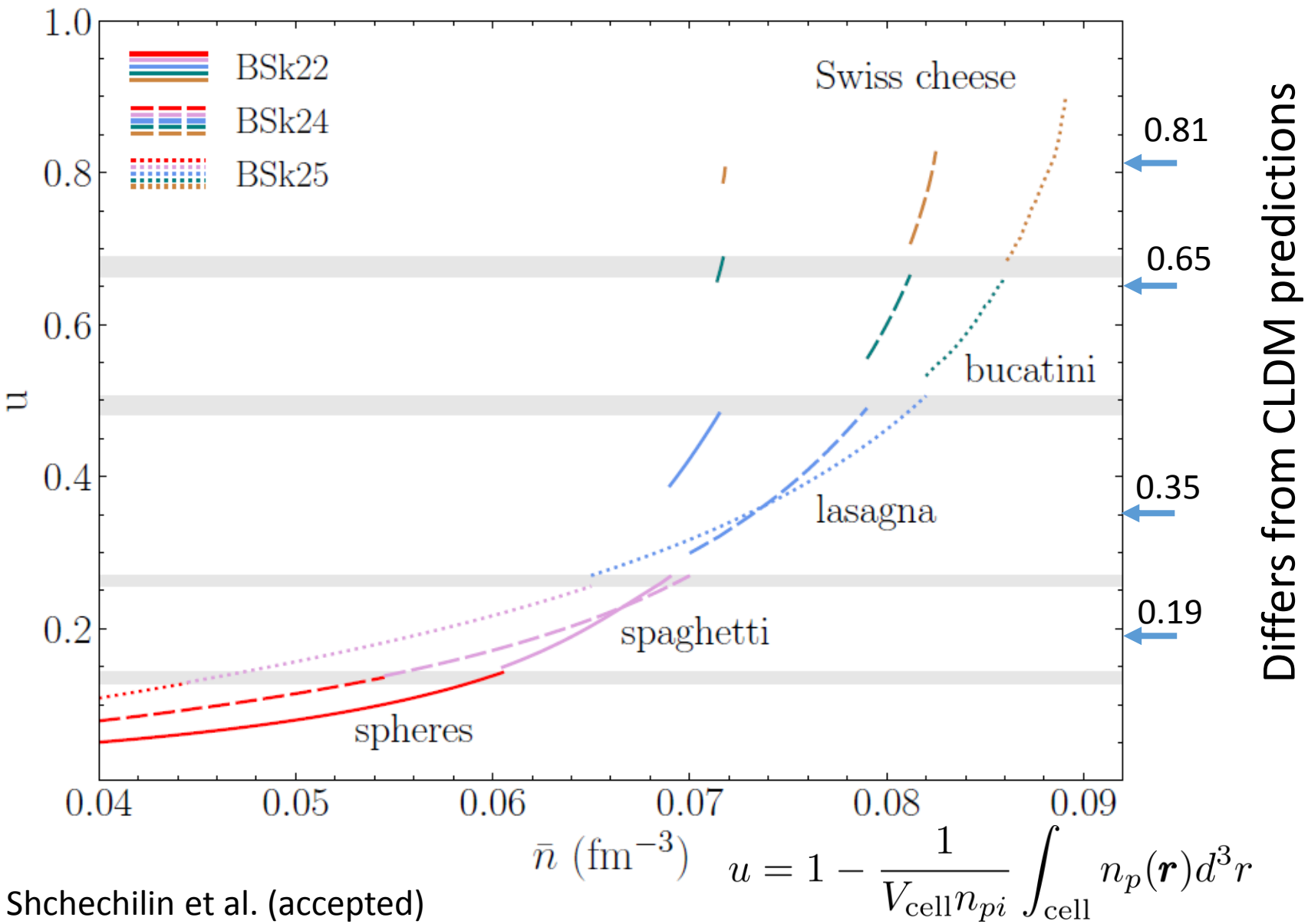
(some details will follow latter)



Amount of pasta is model-dependent

Figure from Shchechilin et al. (accepted)

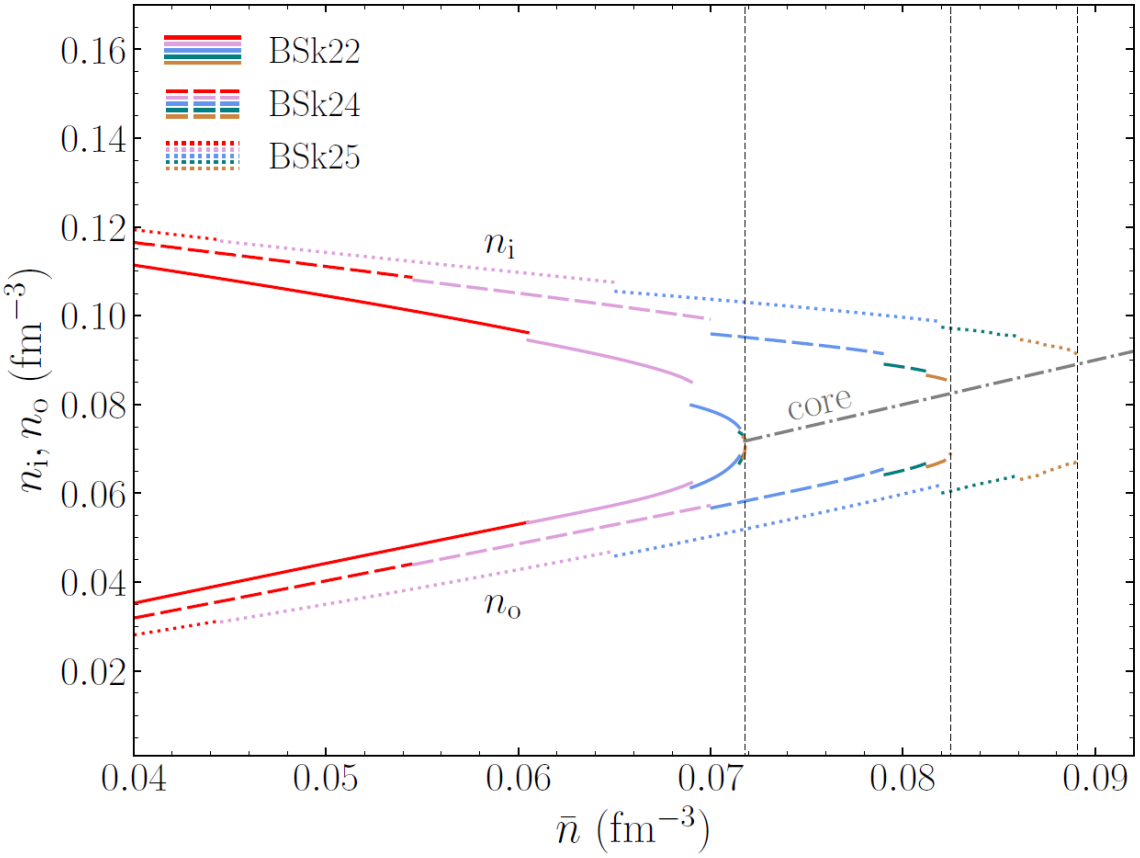
Extended Thomas-Fermi calculations
(almost) universal filling fractions for transitions



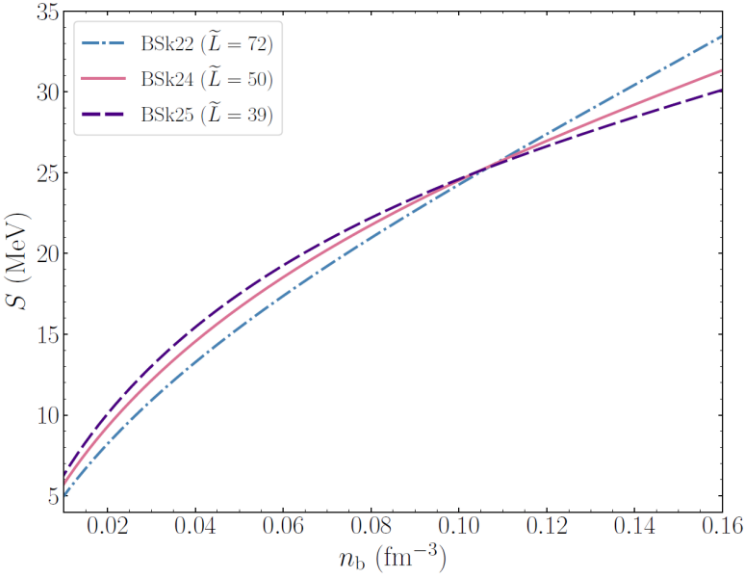
Differs from CLDM predictions

Shchechilin et al. (accepted)

Extended Thomas-Fermi calculations
(almost) universal filling fractions for transitions



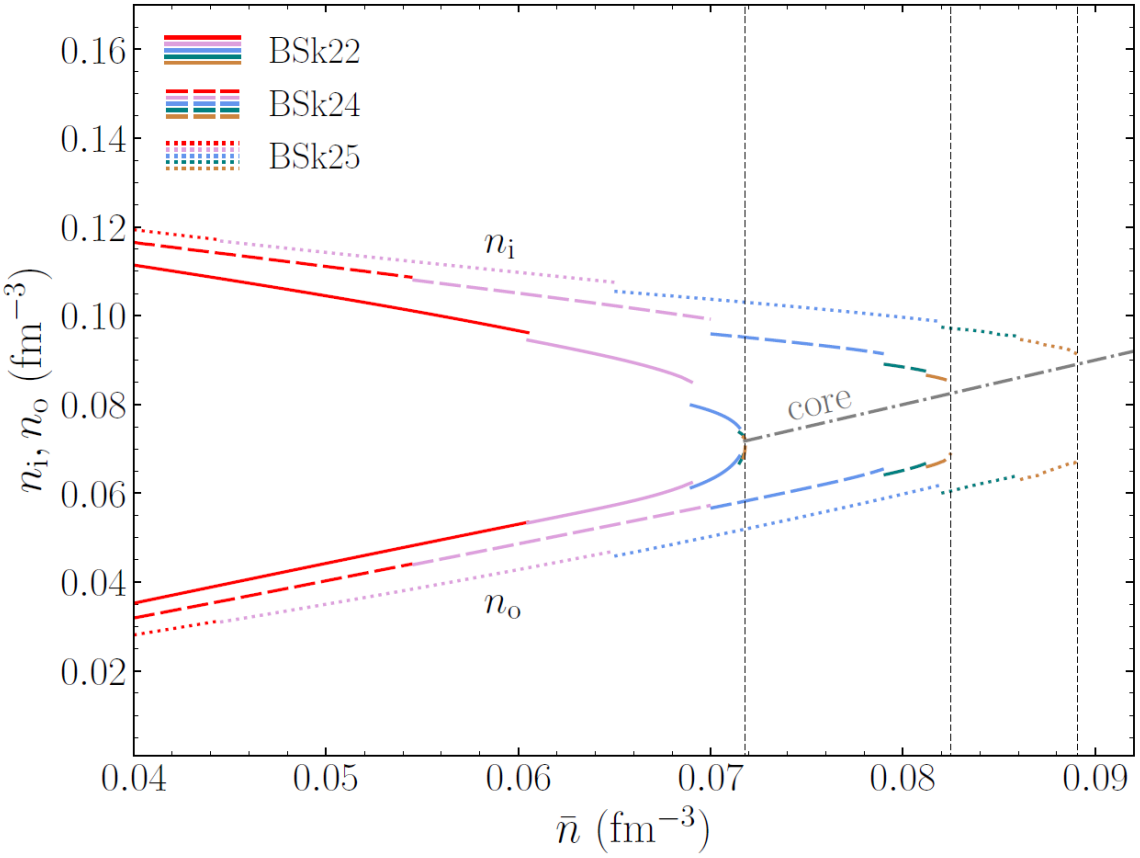
- Symmetry energy affects amount of pasta via u(n) dependence.
- The larger symmetry energy, the weaker is u(n) dependence and the larger is an amount of pasta



$$u = 1 - \frac{1}{V_{\rm cell} n_{pi}} \int_{\rm cell} n_p(\boldsymbol{r}) d^3 r$$

Shchechilin et al. (accepted)

Extended Thomas-Fermi calculations
(almost) universal filling fractions for transitions



- Symmetry energy affect amount of pasta via u(n) dependence.
- The lower symmetry energy, the weaker is u(n) dependence and the amount of pasta is larger
- Crust-core transition agrees well with the instability of the core matter
- Crust (pasta)-core phase transition predicted to be of the first order, but very weak

Zemlyakov & AIC (2022):
Spheres -> spaghetti transition is not associated with fission instability

Shchechilin et al. (accepted)

$$u = 1 - \frac{1}{V_\mathrm{cell} n_{pi}} \int_\mathrm{cell} n_p(\boldsymbol{r}) d^3 r$$

Compressible liquid drop model and pasta phases: account for curvature corrections

Compare two phases (A and B) with same

$$n_{ni},\ n_{pi},\ n_{no}, u = V_{\rm cluster}/V_{\rm cell}$$

$$\frac{E^A_{\rm surf}}{V^A_{\rm cell}} + \frac{E^A_C}{V^A_{\rm cell}} \Leftrightarrow \frac{E^B_{\rm surf}}{V^B_{\rm cell}} + \frac{E^B_C}{V^B_{\rm cell}}$$

$$\begin{aligned} V_c &= a^3 \\ E_{\rm surf} &= \sigma_s g_s(u) a^2 + \sigma_c g_c(u) a \\ E_C &= (e n_e)^2 w(u) a^5 \end{aligned}$$

Size of the cell for each phase:

$$\begin{aligned} \tilde{E}^A_{\rm surf} &= 2E^A_C \\ \tilde{E}^B_{\rm surf} &= 2E^B_C \end{aligned}$$

Shape	$g_s(u)$	$g_c(u)$	$w(u)$
sphere	$(6\pi^{1/2}u)^{2/3}$	$2(48\pi^2 u)^{1/3}$	$(9\pi)^{1/3}(2u^{5/3} - 3u^2 + u^{8/3})/(5\cdot 2^{1/3})$
spaghetti	$(4\pi u)^{1/2}$	2π	$u^2(u - 1 - \ln u)/2$
lasagna	2	0	$\pi u^2 (1-u)^2/6$

Shchechilin et al. (accepted)

Compressible liquid drop model and pasta phases: account for curvature corrections

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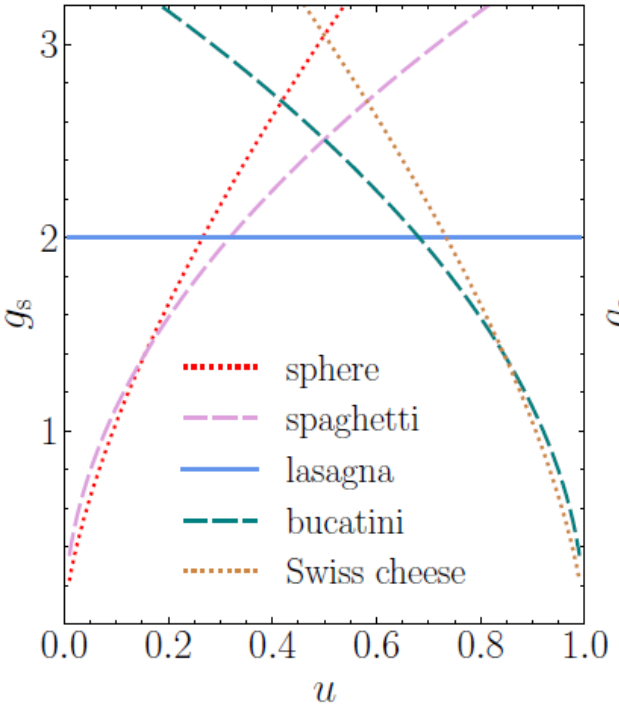
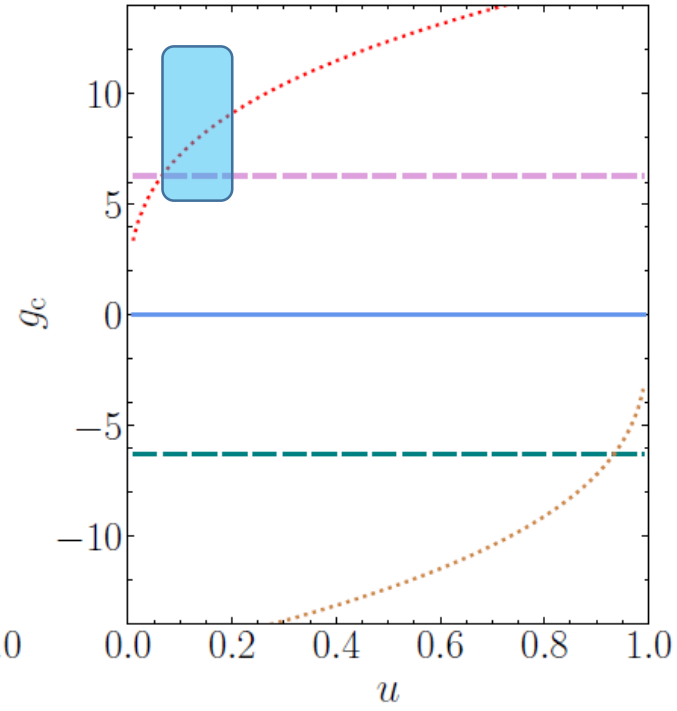
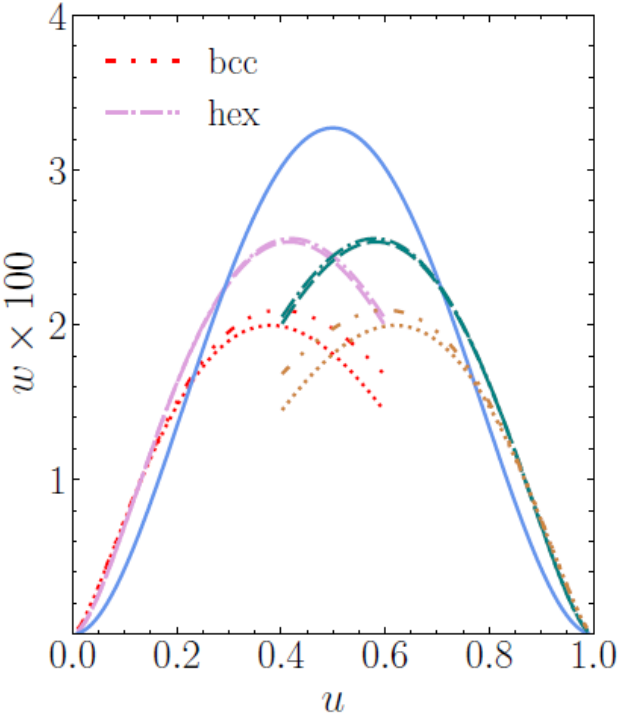
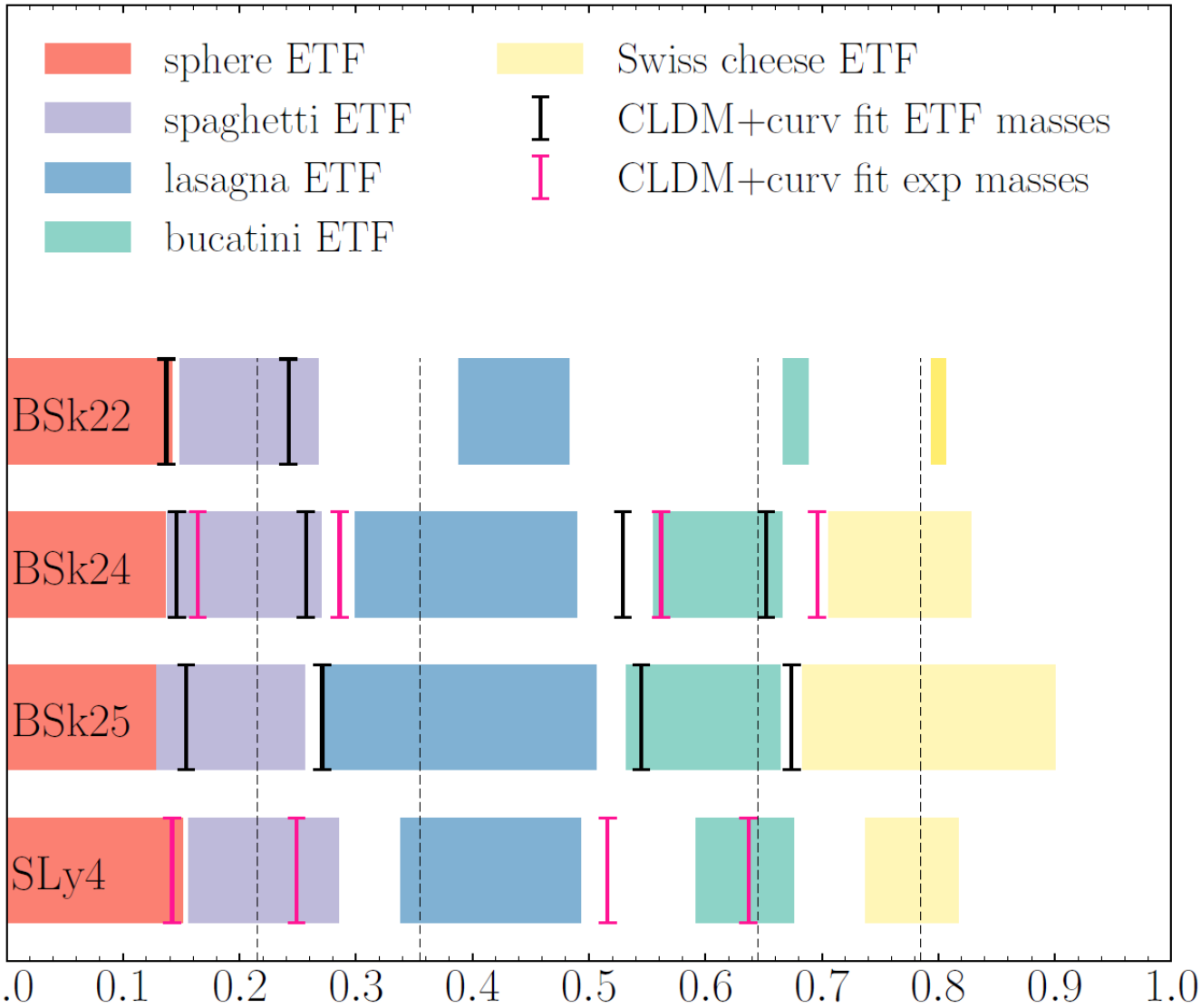






Shchechilin et al. (accepted)

Compressible liquid drop model and pasta phases: account for curvature corrections



Shchechilin et al. (accepted)

- Generally, transitions depend on σ_c/σ_s, but it almost the same (at least for considered EDFs).
- Curvature corrections should be taken into account to predict transitions within CLDM

Extended Thomas-Fermi approach

Energy density functional (Wigner-Kirkwood expansion of the Bloch density matrix)

Kirzhnits (1957), Hodges (1973), Grammaticos & Voros (1979), Brack et al. (1985)

$$E_{\mathrm{ETF}} = \int_{\mathrm{cell}} \left(\tau_4 + \epsilon_{\mathrm{nuc}} + \epsilon_C + \epsilon_e\right) dV$$

kinetic nuclear Coulomb electron

Nucleon profile optimization: (accurate) Euler–Lagrange equations

$$\frac{\partial \delta E_{\mathrm{ETF}}}{\partial \delta n_p} = \mu_p; \quad \frac{\partial \delta E_{\mathrm{ETF}}}{\partial \delta n_n} = \mu_n; \quad \mu_n - \mu_p = \mu_e$$

Real life nucleon profile optimization: minimization over parametrized profiles

$$n_q(r) = n_{Bq} + n_{\Lambda q} \frac{1}{1 + \exp\left(\frac{r - C_q}{a_q}\right)}$$

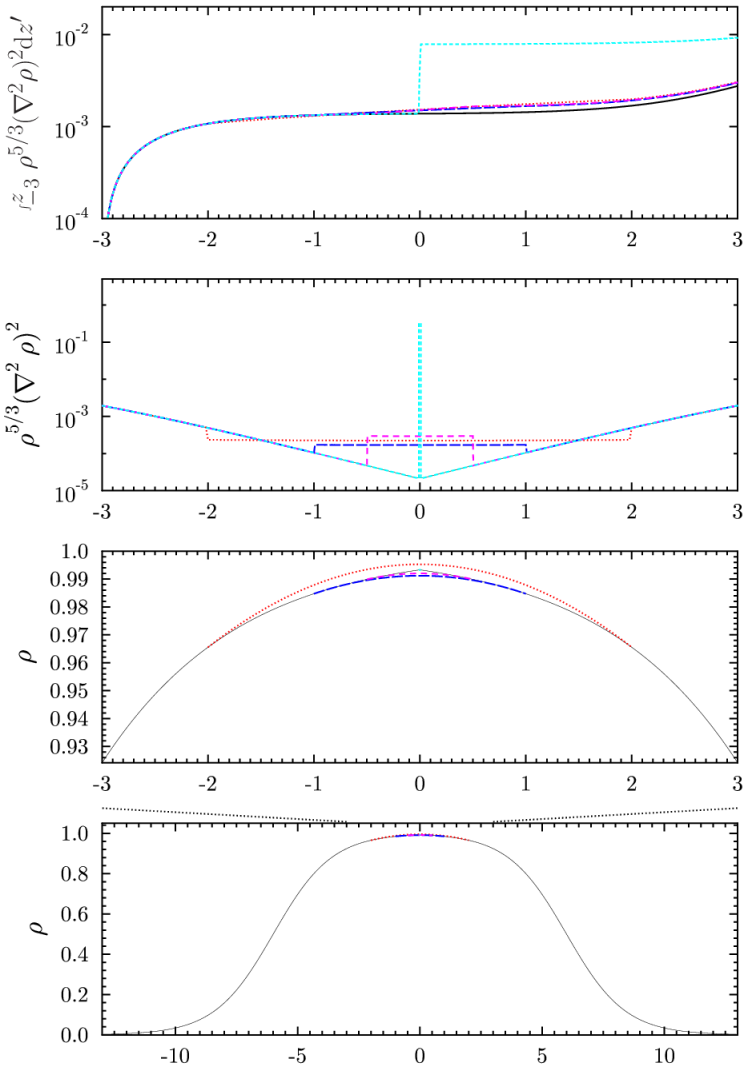
- In some sense similar to CLDM: cell energy depend on a set of parameters
- Much more computation extensive when CLDM (integrals should be taken numerically)
- **Choose of the functional form of the profiles:**

  Use of insufficiently smooth parametrizations may reduce accuracy (and affect results)

Extended Thomas-Fermi approach

- **Choose of the functional form of the profiles:**

  Use of insufficiently smooth parametrizations may reduce accuracy (and affect results)



Peak contribute to integral

Numerical integration see kink as a peak of integrand

Kink becomes visible

Profile looks smooth

$$n_q(r) = n_{Bq} + n_{\Lambda q}\frac{1}{1+\exp\left(\frac{r-C_q}{a_q}\right)}$$

Kink in parametrized profile leads to problems with gradient terms:

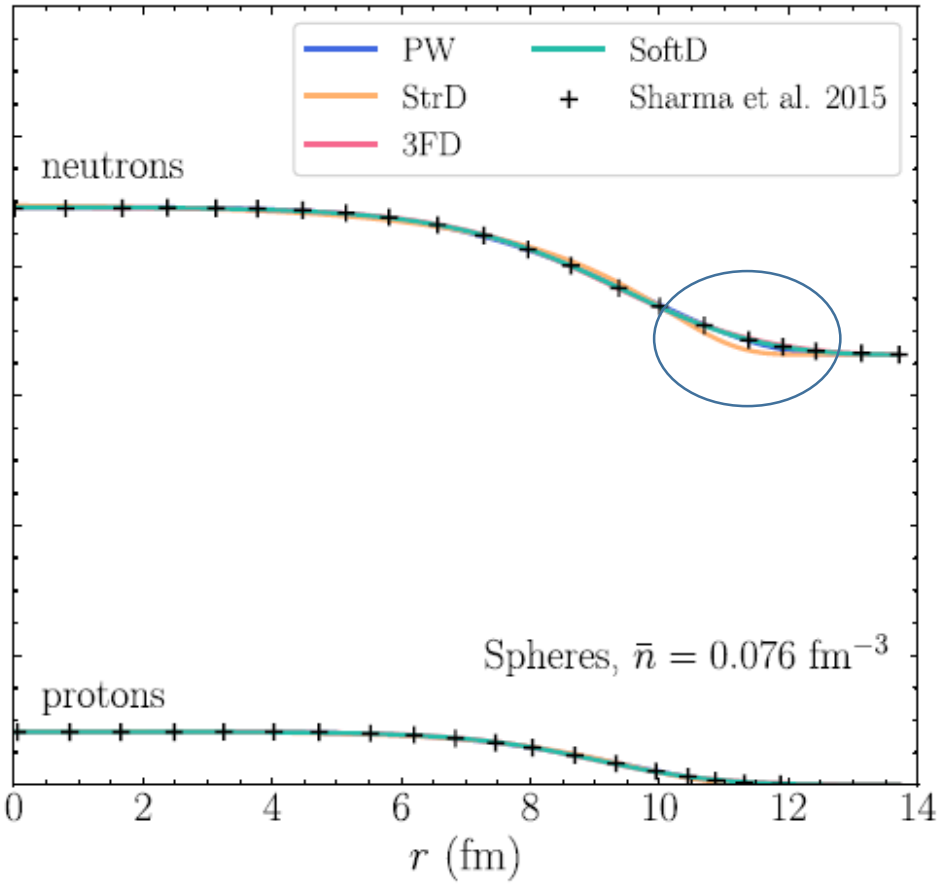
<u>Analytical analysis:</u>
The energy can even diverge!

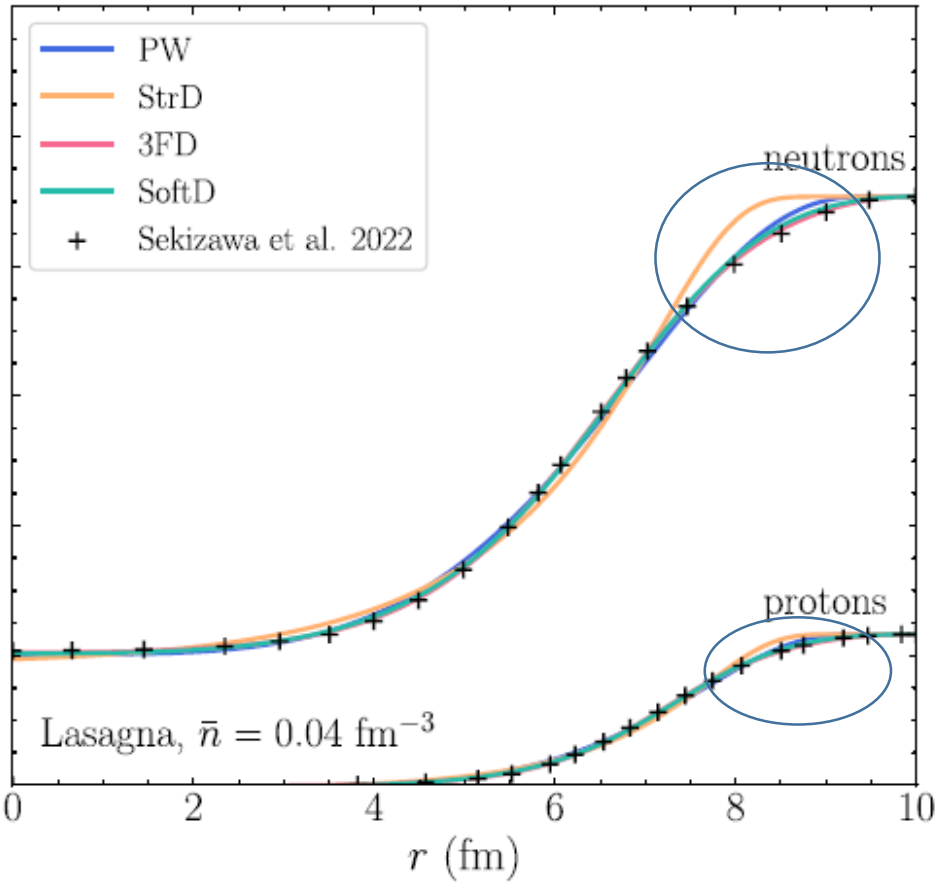
<u>Numerical analysis:</u>
Kink leads to numerical problems

Potekhin et al., Phys. Usp. (accepted)

Extended Thomas-Fermi approach

- Choose of the functional form of the profiles:
  Profiles should not be too smooth





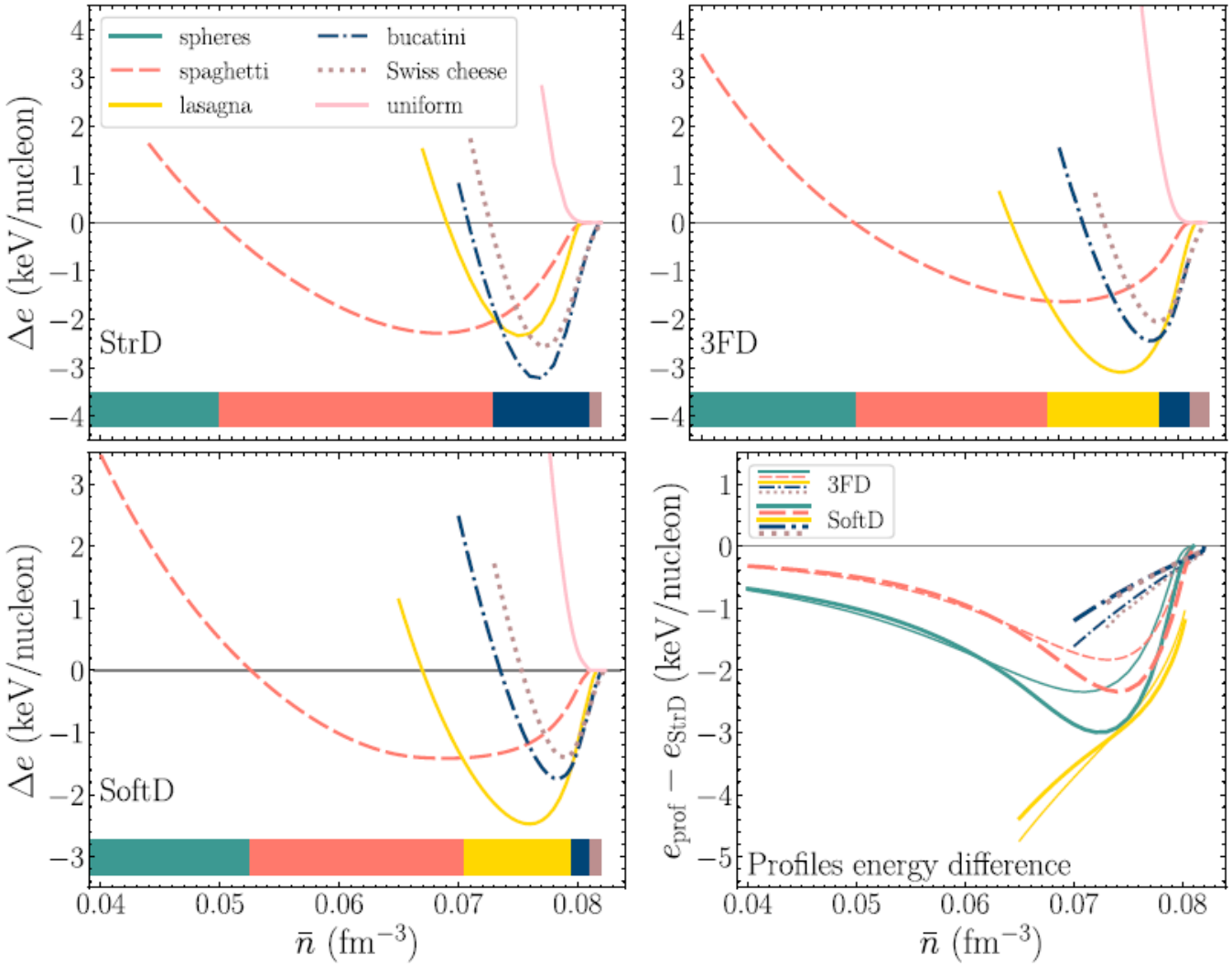
$$n_q(r) = n_{Bq} + n_{\Lambda q} \frac{1}{1 + \exp\left[\left(\frac{C_q - R}{r - R}\right)^2 - 1\right] \exp\left(\frac{r - C_q}{a_q}\right)}$$

Shchechilin et al. (2024), PRC

Extended Thomas-Fermi approach

- Choose of the functional form of the profiles:

  Profiles should not be too smooth – it is not realistic and do not allow to minimize energy properly. Even pasta sequence can be affected!



Phase transitions are controlled by

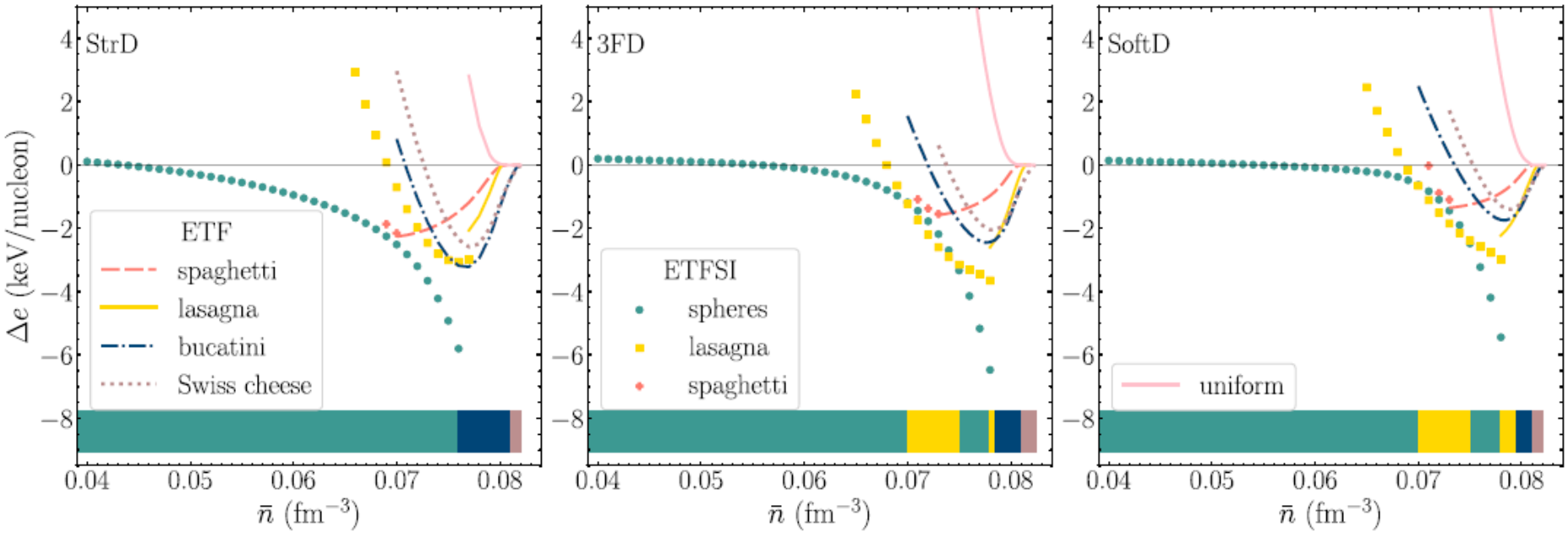
$$\sim 1 \frac{\rm keV}{\rm nucleon}$$

BSK24

Shchechilin et al. (2024), PRC

Extended Thomas-Fermi + Strutinsky intergral

BSK24



Account for shell effects (via Strutinsky integral) affect the pasta sequence

Shchechilin et al. (2024), PRC

Hartree-Fock within WS approximation

$$E_{\mathrm{HF}} = \int \mathrm{d}^3\boldsymbol{r}\, \mathcal{E}_{\mathrm{HF}}\Big[n_q(\boldsymbol{r}), \tau_q(\boldsymbol{r}), \boldsymbol{J_q}(\boldsymbol{r}), n_e(\boldsymbol{r})\Big]\,,$$

Self-consistent HF equations for the nucleons

$$\sum_{\sigma'} h_q(\boldsymbol{r})_{\sigma\sigma'} \varphi_k^{(q)}(\boldsymbol{r},\sigma') = \epsilon_k^{(q)} \varphi_k^{(q)}(\boldsymbol{r},\sigma); \quad \varphi_\alpha^{(q)}(\boldsymbol{r},\sigma) = \mathcal{R}_\alpha^{(q)}(r)\mathcal{Y}_{\ell jm}(\theta,\phi,\sigma)$$

Boundary conditions within WS approximation:

Dirichlet boundary conditions

$$\mathcal{R}_\alpha^{(q)}(R) = 0$$

Neumann boundary conditions

$$\left.\frac{d\mathcal{R}_\alpha^{(q)}}{dr}\right|_{r=R} = 0$$

Hartree-Fock within WS approximation

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Pressure:

$$P_{\rm HF} = -\mathcal{E}_{\rm HF}(R) + \frac{1}{4\pi}\sum_q\sum_{\alpha(q)} g_\alpha^{(q)}\epsilon_\alpha^{(q)}\mathcal{R}_\alpha^{(q)}(R)^2 + \tilde{\mu}_e n_e\,.$$

Boundary conditions within WS approximation:

Dirichlet boundary conditions

$$\mathcal{R}_\alpha^{(q)}(R) = 0$$

No contribution to pressure from neutrons

Neumann boundary conditions

$$\left.\frac{d\mathcal{R}_\alpha^{(q)}}{dr}\right|_{r=R} = 0$$

Affect pressure by discretization of energies

The errors incurred by the choice of approximate boundary conditions can propagate to global thermodynamic properties. Without sacrificing the Wigner-Seitz approximation, the equation of state can be more reliably calculated within the ETF approach.

Chamel et al. (2025)

Pressure within ETF approach (WS approximation)

Definition:

$$P = -\frac{dE}{dV} = -\frac{dE_{\rm cell}}{dV_{\rm cell}}$$

ETF, full minimization of nucleon profiles

Associated with gradient terms in ETF

$$P_{\rm ETF} = -\mathcal{E}_{\rm ETF}(R) + \mu_e n_e(R) + \sum_q \mu_q n_q(R) = P_{\rm hom}(n_n(R), n_p(R), n_e(R)) + \delta P^{\nabla}$$

Pressure is determined by density at the boundary!

ETF, parametrized profiles

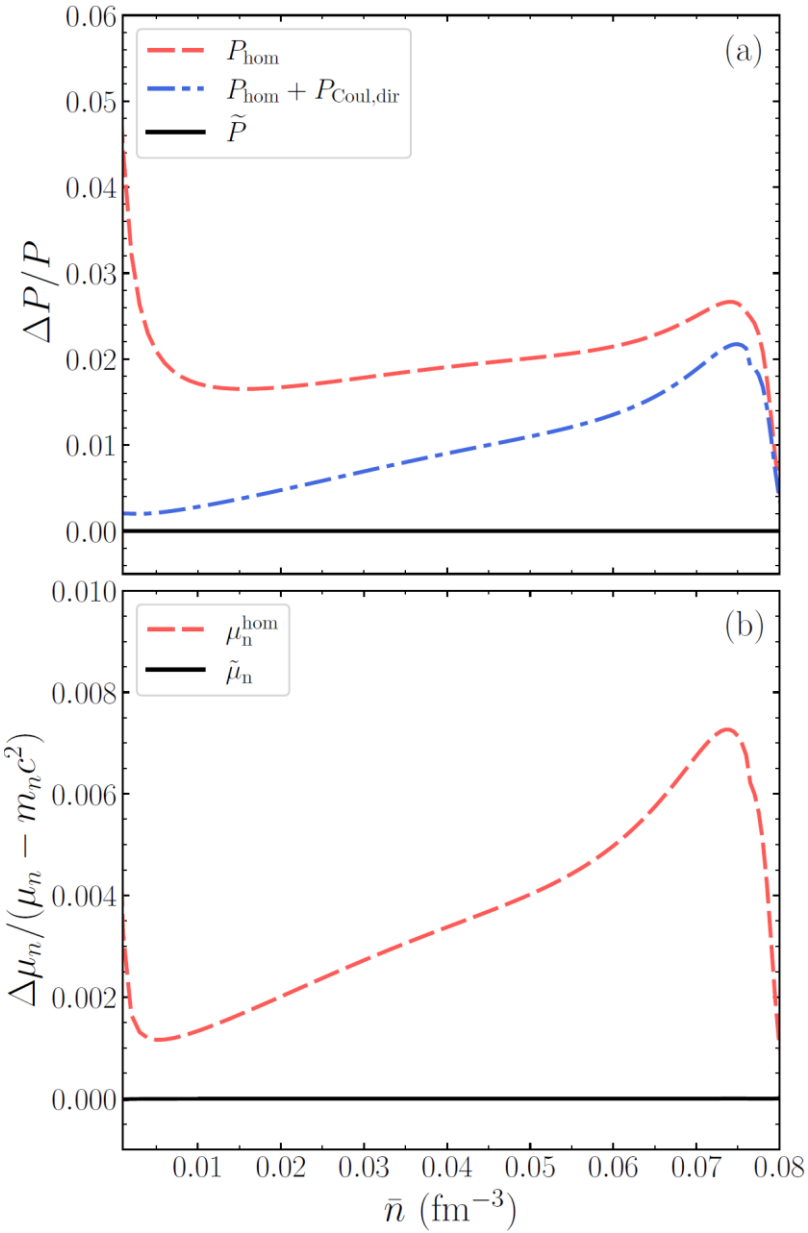
Corrections for electron polarization

$$P_{\rm ETF} = P_{\rm hom}(n_n(R), n_p(R), n_e(R)) + \delta P^{\nabla} + \delta P_{\rm Coul,dir} + \delta P_{\rm nuc}$$

Corrections, associated with non-optimal density at the boundary
[due to restricted (parametrized) profiles]

Chamel et al. (2025)

Pressure within ETF approach (and WS approximation)



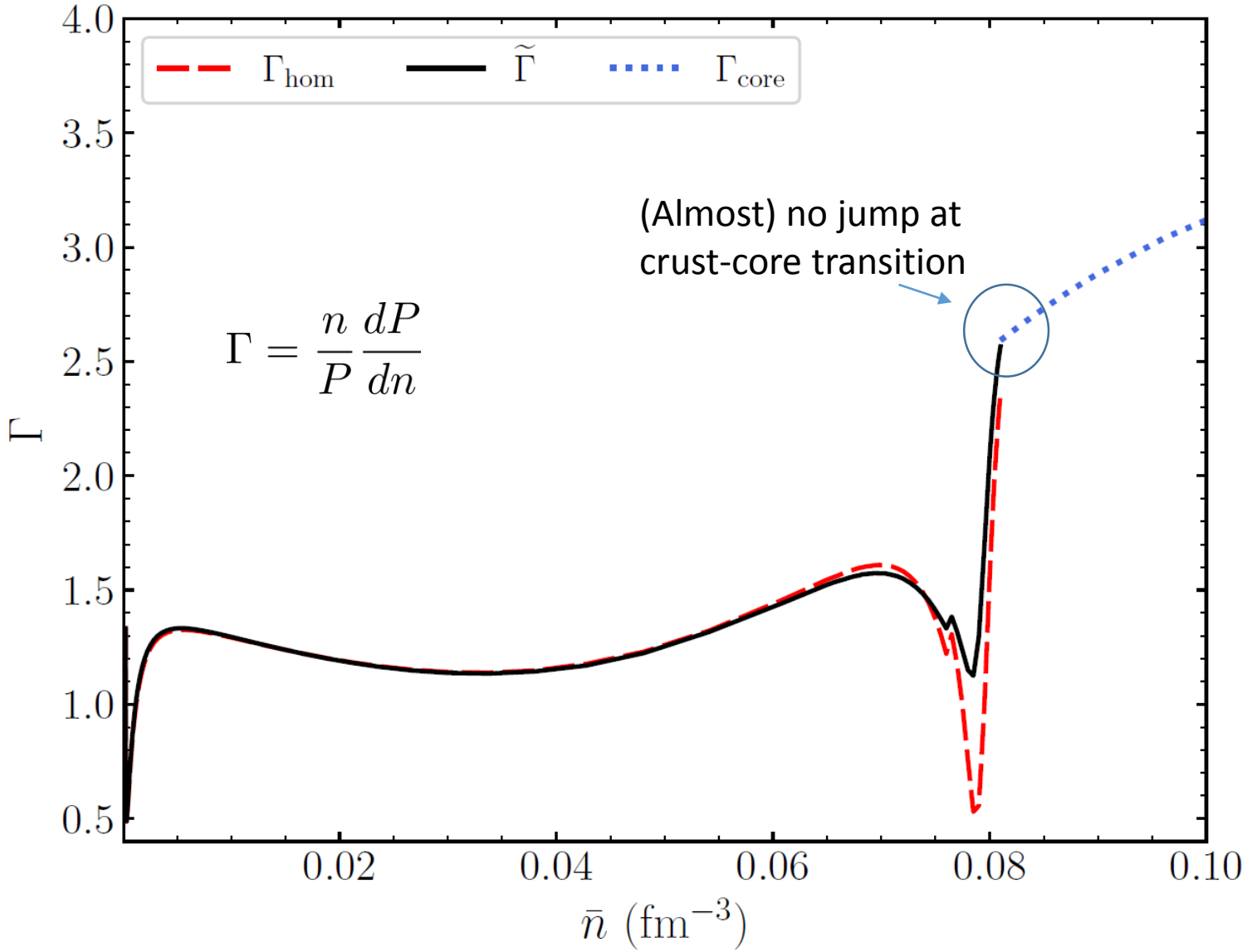
At lower densities corrections are dominated by electron polarization

$$P_{\rm Coul,dir} = -\frac{2\pi}{5} e^2 n_e^2 R^2 \left(1 - \frac{5}{3}\frac{\langle r^2\rangle}{R^2}\right)$$

At higher densities corrections are dominated by limited flexibility of parametrized nucleon profiles (number density at the cell boundary is not enough accurate)

Chamel et al. (2025)

Adiabatic index within ETF approach



Chamel et al. (2025)

Summary

- CLDM can be applied to describe $P(\rho)$
- Amount of pasta and crust-core transition is sensitive to the symmetry energy at relevant density (the larger symmetry energy, the larger is an amount of pasta)
- Pasta phase transitions are controlled by energy difference $\sim 1\mathrm{keV}/\mathrm{nucleon}$
  - At this precision, results depend on the approach (even for given density functional)
  - If shell effects are neglected, the pasta phase transition occurs at the filling fractions, which are rather model-independent (u~0.12 for spheres -> spaghetti)
  - CLDM without curvature corrections is not enough accurate to consider pasta phase transitions (but it can predict filling fractions)
  - The amount of pasta depend on the symmetry energy at the relevant densities
- Without sacrificing the Wigner-Seitz approximation, the equation of state can be more reliably calculated within the ETF approach.

D.G. Yakovlev, HEA2017(?)
Crust as Cinderella of NS

Avoiding to surrender the adequate representation of observations, controlled by the crust

Glitches, Transients (shallow heating), Magnetars,….