Fluctuation-induced first-order superfluid transition in unitary $\mathrm{SU}(N)$ Fermi gases

George Kalagov
BLTP $\in$ JINR

INFINUM-2025

15 May 2025

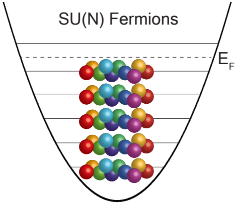
Physics: Quantum gases

Why cold atoms are interesting?

- ✓ Ultracold gases are "simple" systems
- ✓ Both fermions (^{6}Li, ^{87}Sr, $\ldots$) and bosons (^{87}Rb, ^{23}Na, $\ldots$) can be studied
- ✓ Interaction can be tuned to arbitrary values (via resonance scattering)
- ✓ Various internal symmetries ($\mathrm{SU}(N), \mathrm{SO}(N), \mathrm{Sp}(N) \ldots$) can be realized

$$N = 2I + 1$$

- ^{171}Yb $\quad I = 1/2 \quad \mathrm{SU}(2)$
- ^{173}Yb $\quad I = 5/2 \quad \mathrm{SU}(6)$
- ^{87}Sr $\quad I = 9/2 \quad \mathrm{SU}(10)$
- ^{171}Yb $+^{173}$ Yb $\quad \mathrm{SU}(2) \times \mathrm{SU}(6)$



pic. by Sonderhouse

Scales in cold atoms

Lenght scales (in units of Bohr radius)

- Van der Waals length $\ell_{VdW} \sim 10 \div 100$
- s-wave scattering length $a \sim 10 \div 200$
- interparticle distance $\ell \sim 800 \div 3000$
- de Broglie wavelength $\ell_T \sim (1 \div 4) \times 10^4$
- size of the system $L \sim 10^5$

$$\ell_{VdW} \lesssim a \lesssim \ell \ll \ell_T \ll L$$

Scales in cold atoms

Lenght scales (in units of Bohr radius)

- Van der Waals length $\ell_{VdW} \sim 10 \div 100$
- s-wave scattering length $a \sim 10 \div 200$
- interparticle distance $\ell \sim 800 \div 3000$
- de Broglie wavelength $\ell_T \sim (1 \div 4) \times 10^4$
- size of the system $L \sim 10^5$

$$\ell_{VdW} \lesssim a \lesssim \ell \ll \ell_T \ll L$$

Energy scales (e.g. ^{6}Li)

- $T_c \sim 0.16\, T_F \sim 0.1 \mu K$
- $T_c \sim 10^{-11}$eV

  where $T_F \sim n^{2/3}$
  and $n \sim 10^{13}$cm^{-3}

Scales in cold atoms

Lenght scales (in units of Bohr radius)

- Van der Waals length $\ell_{VdW} \sim 10 \div 100$
- s-wave scattering length $a \sim 10 \div 200$
- interparticle distance $\ell \sim 800 \div 3000$
- de Broglie wavelength $\ell_T \sim (1 \div 4) \times 10^4$
- size of the system $L \sim 10^5$

$$\ell_{VdW} \lesssim a \lesssim \ell \ll \ell_T \ll L$$

Energy scales (e.g. ^{6}Li)

- $T_c \sim 0.16\ T_F \sim 0.1 \mu K$
- $T_c \sim 10^{-11}$eV

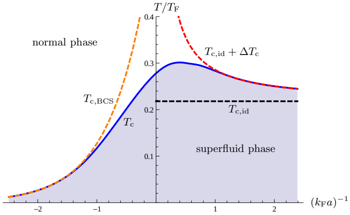
  where $T_F \sim n^{2/3}$
  and $n \sim 10^{13}$cm^{-3}

HTS vs Gases

- $T_c^{\text{HTS}} \sim 0.01\ T_F^{\text{HTS}} \sim 100K$

cold atoms are "hot" matter

Key observables



The phase diagram of interacting $\mathrm{SU}(2)$ Fermi gas [pic. by Boettcher]

Method	Δ/T_F
Schirotzek (Exp)	0.44(3)
Carlson (QMC)	0.50(5)
Floerchinger (FRG)	0.46
Haussmann (LW)	0.46

The superfluid gap Δ at $T = 0$ in $\mathrm{SU}(2)$ systems in the unitary regime

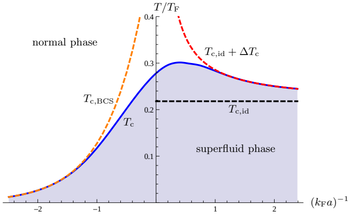
Method	T_c/T_F
Nascimbene (Exp)	0.157(15)
Horikoshi (Exp)	0.17(1)
Ku (Exp)	0.167(13)
Goulko (QMC)	0.171(5)
Floerchinger (FRG)	0.248
Haussmann (LW)	0.160

The critical temperature of superfluid (*2nd order*) phase transition in $\mathrm{SU}(2)$ systems in the unitary regime:

$$k_F a \to -\infty.$$

✓ Critical exponents $\in$ XY-model universality class.

Key observables



The phase diagram of interacting $SU(2)$ Fermi gas [pic. by Boettcher]

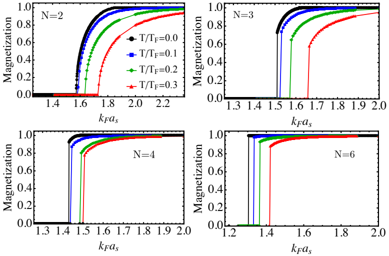
Method	Δ/T_F
Schirotzek (Exp)	0.44(3)
Carlson (QMC)	0.50(5)
Floerchinger (FRG)	0.46
Haussmann (LW)	0.46

The superfluid gap Δ at $T = 0$ in $SU(2)$ systems in the unitary regime

Method	T_c/T_F
Nascimbene (Exp)	0.157(15)
Horikoshi (Exp)	0.17(1)
Ku (Exp)	0.167(13)
Goulko (QMC)	0.171(5)
Floerchinger (FRG)	0.248
Haussmann (LW)	0.160

The critical temperature of superfluid (*2nd order*) phase transition in $SU(2)$ systems in the unitary regime: $k_F a \to -\infty$.

✓ Critical exponents $\in$ XY-model universality class.



Magnetization in $SU(N)$ system within the HF-method [Cazalilla,New J. Phys'23]

- MF and HF predict 2nd order phase transition for $SU(2)$, and the 1st o.p.t. for $SU(N > 2)$, due to the cubic term in the Landau free energy

$$\mathcal{F} = aM^2 + bM^3 + cM^4$$

- Qualitative agreement with QMC

Microscopic action

- ✓ Partially bosonized $\mathrm{U}(N)$-symmetric fermionic action $(k_B = \hbar = 2m = 1, \beta = 1/T)$

$$S = \int_0^\beta d\tau \int_{\mathbf{x}} \left[\bar{\psi}_a \left(\partial_\tau - \nabla^2 - \mu \right) \psi_a - \frac{1}{2\lambda_\Lambda} \mathrm{tr}(\varphi^\dagger \varphi) + \frac{1}{2} (\bar{\psi}_a \varphi_{ab} \bar{\psi}_b + \psi_a \varphi^\dagger_{ab} \psi_b) \right]$$

$1/\lambda_\Lambda = 1/8\pi a - C\Lambda$

- ✓ Phase transitions — nonzero expectation value $\langle \psi_a \psi_b \rangle \sim \langle \varphi_{ab} \rangle$

$$\langle \varphi \rangle = \Delta\, \mathrm{J} \equiv \Delta \begin{pmatrix} 0 & \mathrm{I}_{N/2} \\ -\mathrm{I}_{N/2} & 0 \end{pmatrix}$$

- ✓ Symmetry breaking $\mathrm{U}(N) \to \mathrm{USp}(N) = \{ h \in \mathrm{SU}(N) \mid h\, \mathrm{J}\, h^T = \mathrm{J} \}$

- ✓ Mean-field predictions:
  - continuous phase transition $\forall N$
  - The critical temperature $T_c/\mu \approx 0.66$ and the zero-temperature gap $\Delta/\mu = 1.16$ in the unitary regime $(1/a \to 0^-)$ $\forall N$

Functional RG in a nutshell

Computation of free energy

- ✓ The partition functional is given by the classical action

$$Z[J] = \int \mathcal{D}\phi \exp\{-S[\phi] + J\phi\},$$

- ✓ The Legendre transform yields the quantum effective action (depends on φ: magnetization, superfluid gap, and etc.)

$$\Gamma[\varphi] = J_\varphi \varphi - \ln Z[J_\varphi], \quad \varphi = \frac{\delta}{\delta J} \ln Z[J]\bigg|_{J=J_\varphi}.$$

- ✓ The grand potential, pressure and entropy density

$$\Omega = T \min_\varphi \Gamma[\varphi], \quad P = -\frac{\Omega}{\mathcal{V}}, \quad s = \left(\frac{\partial P}{\partial T}\right)_\mu.$$

Effective average action

✓ The modified partition functional

$$Z_k[J] = \int \mathcal{D}\phi \exp\left\{-S[\phi] - \Delta S_k[\phi] + J\phi\right\},$$

with the quadratic additive

$$\Delta S_k[\phi] = \frac{1}{2}\int_p \phi(-p)\, R_k(p)\, \phi(p).$$

Effective average action

✓ The modified partition functional

$$Z_k[J] = \int \mathcal{D}\phi \exp\left\{-S[\phi] - \Delta S_k[\phi] + J\phi\right\},$$

with the quadratic additive

$$\Delta S_k[\phi] = \frac{1}{2}\int_p \phi(-p)\, R_k(p)\, \phi(p).$$

$R_k(p) \to \infty$	for $k \to \infty$
$R_k(p) \to 0$	for $k \to 0$
$R_k(p) \sim k^2$	for $p \ll k$
$R_k(p) \sim 0$	for $p \gg k$

Effective average action

✓ The modified partition functional

$$Z_k[J] = \int \mathcal{D}\phi \exp\{-S[\phi] - \Delta S_k[\phi] + J\phi\},$$

with the quadratic additive

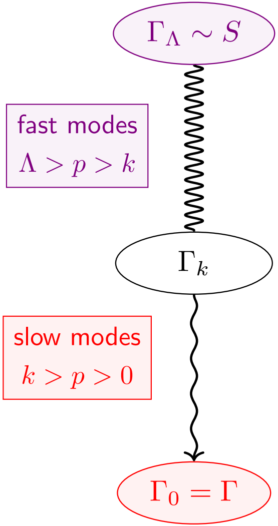
$$\Delta S_k[\phi] = \frac{1}{2}\int_p \phi(-p)\, R_k(p)\, \phi(p).$$

$R_k(p) \to \infty$	for $k \to \infty$
$R_k(p) \to 0$	for $k \to 0$
$R_k(p) \sim k^2$	for $p \ll k$
$R_k(p) \sim 0$	for $p \gg k$

✓ The effective average action functional

$$\Gamma_k[\varphi] = J_{k,\varphi}\varphi - \ln Z_k[J_{k,\varphi}] - \Delta S_k[\varphi], \quad \varphi = \frac{\delta}{\delta J}\ln Z_k[J]\Big|_{J=J_{k,\varphi}}.$$

From functional integral to functional DE



Flow equation (Wetterich, Moris)

✓ $\Gamma_{k=\Lambda} \sim S$ – classical action – input

$$\partial_t \Gamma_k[\varphi] = \frac{1}{2}\,\mathrm{Tr}\left\{(\Gamma_k^{(2)}[\varphi] + R_k)^{-1}\partial_t R_k\right\},\ t = \ln(k/\Lambda)$$

✓ $\Gamma_{k=0} = \Gamma$ – quantum effective action – output

Some features:

- The flow equation is exact
- Non-perturbative approximation is possible
- Equilibrium and far-from-equilibrium systems

Treatment of large spin fermions

Non-perturbative approximation

✓ The truncated effective average action Γ_k in the leading order of the derivative expansion

$$\Gamma_k = \int_0^\beta d\tau \int_{\mathbf{x}} \Big[\bar{\psi}_a \left(\partial_0 - \nabla^2 - \mu\right) \psi_a + Y_k \,\mathrm{tr}(\phi^\dagger \partial_0 \phi) + Z_k \,\mathrm{tr}(\nabla\phi^\dagger \nabla\phi) + V_k(\phi^\dagger\phi)$$

$$+ \frac{g_k}{2}(\bar{\psi}_a \phi_{ab} \bar{\psi}_b + \psi_a \phi^\dagger_{ab} \psi_b) + \mathcal{O}(\partial_0^2, \nabla^4, \dots) \Big]$$

✓ The initial conditions at the UV scale

$$Z_\Lambda = 0, \quad Y_\Lambda = 0, \quad g_\Lambda = 1, \quad V_\Lambda(\phi^\dagger\phi) \sim -\frac{1}{2\lambda_\Lambda}\,\mathrm{tr}(\phi^\dagger\phi)$$

✓ We use θ-regulators. This allows for an analytical treatment of the loop integrals

$$R_k^{(\mathcal{B})} \sim (k^2 - \mathbf{q}^2)\theta(k^2 - \mathbf{q}^2), \quad R_k^{(\mathcal{F})} \sim \mathrm{sgn}(\mathbf{q}^2 - \mu)(k^2 - |\mathbf{q}^2 - \mu|)\theta(k^2 - |\mathbf{q}^2 - \mu|),$$

FRG flow equations

General form of the scale-dependent potential $V_k(\phi^\dagger\phi) \equiv \Gamma_k/\mathcal{V}\beta = V_k(\rho_1, \rho_2, \dots)$, where

$$\rho_1 \equiv \mathrm{tr}(\phi^\dagger\phi), \quad \rho_a \equiv \mathrm{tr}\left(\phi^\dagger\phi - I_N\frac{\rho_1}{N}\right)^a, \quad a = 2, 3, \dots$$

The projection of the FRG equation onto the background $\Delta\,\mathrm{J}$ yields

$$\partial_t U_k = \partial_t\Gamma_k/\mathcal{V}\beta, \quad U_k(\rho_1) \equiv V_k(\rho_1, 0, 0, \dots), \quad P = -\min_{\Delta} U_{k\to 0}$$

FRG flow equations

General form of the scale-dependent potential $V_k(\phi^\dagger\phi) \equiv \Gamma_k/\mathcal{V}\beta = V_k(\rho_1, \rho_2, \dots)$, where

$$\rho_1 \equiv \mathrm{tr}(\phi^\dagger\phi), \quad \rho_a \equiv \mathrm{tr}\Big(\phi^\dagger\phi - I_N\frac{\rho_1}{N}\Big)^a, \quad a = 2, 3, \dots$$

The projection of the FRG equation onto the background Δ J yields

$$\partial_t U_k = \partial_t\Gamma_k/\mathcal{V}\beta, \quad U_k(\rho_1) \equiv V_k(\rho_1, 0, 0, \dots), \quad P = -\min_{\Delta} U_{k\to 0}$$

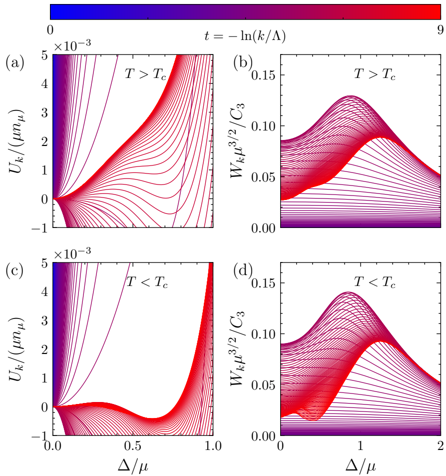
$$\partial_t U_k = k^{d+2}\left(\frac{a_\alpha + a_\sigma}{2\sqrt{a_\alpha a_\sigma}}(1 + 2n_B(\sqrt{a_\alpha a_\sigma})) + (N-2)(N+1)\frac{a_\pi + a_\zeta}{\sqrt{a_\pi a_\zeta}}(1 + 2n_B(\sqrt{a_\pi a_\zeta}))\right)$$
$$-Nk^2\frac{1}{a_\psi}(1 - 2n_F(a_\psi)) \times \Big((\mu + k^2)^{\frac{d}{2}} - \theta(\mu - k^2)(\mu - k^2)^{\frac{d}{2}}\Big),$$

here

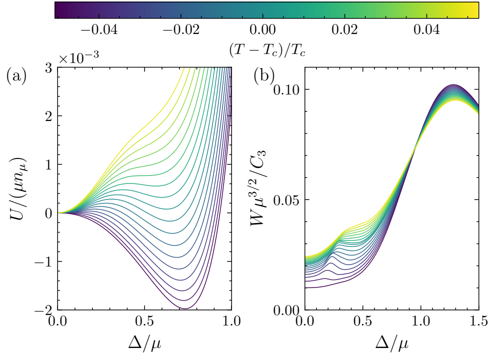
$$a_\alpha = a_\pi = Z_k k^2 + U_k', \quad a_\sigma = Z_k k^2 + U_k' + 2\rho_1 U_k'', \quad a_\zeta = Z_k k^2 + U_k' + \frac{4\rho_1}{N}W_k,$$

$$W_k = W_k(\rho_1) \equiv \frac{\partial V_k(\rho_1, \rho_2, \rho_3, \dots)}{\partial\rho_2}\bigg|_{\rho_2=0, \rho_3=0, \dots}, \qquad a_\psi^2 = k^4 + g_k^2\rho_1/N$$

Discontinuous phase transition $(N \geq 4)$



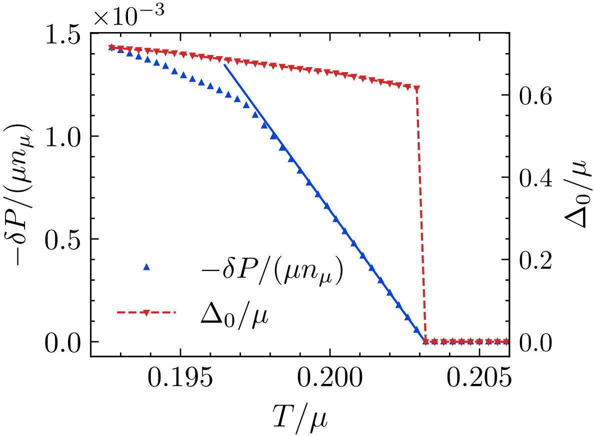
Scale dependence of the running potential U_k and the coefficient of the invariant expansion W_k.
$[n_\mu = C_3 N \mu^{3/2}, C_3 = 1/6\pi^2]$



Temperature dependence of the full effective potential $U \equiv U_{k\to 0}$ and $W \equiv W_{k\to 0}$.

- ✓ 1st-order transition unlike MF predictions
- ✓ Fermionic part of FRG eq. yields MF results

Pressure and gap $(N \geq 4)$



Typical temperature dependence of the pressure change and the superfluid gap in the vicinity of the first-order phase transition.

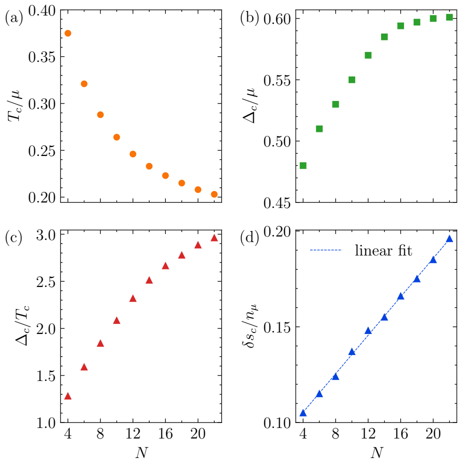
✓ The pressure change across the transition is

$$\delta P \equiv P_N - P_S$$
$$= \big(-U(0)\big) - \big(-U(\Delta_0)\big)$$

✓ The entropy density jump is

$$\delta s \equiv s(T_c + 0) - s(T_c - 0)$$
$$= \left(\frac{\partial}{\partial T}\delta P\right)_\mu$$

Thermodynamic parameters of the first-order transition



(a) critical temperature T_c/μ; (b) energy gap Δ_c/μ at T_c; (c) ratio of the gap to the critical temperature Δ_c/T_c; (d) the entropy density jump $\delta s_c/n_\mu$.

Critical temperature T_c/μ and zero-temperature gap $\Delta^{(0)}/\mu$ for the $\mathrm{SU}(2)$ system, which undergoes a continuous phase transition.

Method	T_c/μ	$\Delta^{(0)}/\mu$
Experiment (Ku'12)	0.40	
FRG (Boettcher'14)	0.38(2)	1.04(15)
LW (Haussmann'07)	0.40	1.27
QMC (Burovski'06)	0.31(2)	
Mean field	0.66	1.16

And in conclusion...

- **+** We investigated the superfluid phase transition in an $\mathrm{SU}(N)$-symmetric Fermi gas with N distinct spin states using the functional renormalization group. Our results reveal a fluctuation-induced first-order phase transition for $N \geq 4$, which is absent at the mean-field level.
- **+** We provided quantitative predictions for the critical temperature, and for the jumps in the superfluid gap and entropy density as functions of N. With increasing N, the critical temperature decreases, while the discontinuities become more pronounced, indicating a stronger first-order transition.

- **–** We did not address the determination of the equation of state $n = n(\mu, T, a_s)$.
- **–** We also did not explore possible refinements of our results that could arise from extending the truncation of the effective action or employing a frequency-dependent regulator $R_k(\omega_n, \mathbf{p})$ (typically affecting values by a few percent).
- **–** ...

And in conclusion...

- ✚ We investigated the superfluid phase transition in an $\mathrm{SU}(N)$-symmetric Fermi gas with N distinct spin states using the functional renormalization group. Our results reveal a fluctuation-induced first-order phase transition for $N \geq 4$, which is absent at the mean-field level.
- ✚ We provided quantitative predictions for the critical temperature, and for the jumps in the superfluid gap and entropy density as functions of N. With increasing N, the critical temperature decreases, while the discontinuities become more pronounced, indicating a stronger first-order transition.

- − We did not address the determination of the equation of state $n = n(\mu, T, a_s)$.
- − We also did not explore possible refinements of our results that could arise from extending the truncation of the effective action or employing a frequency-dependent regulator $R_k(\omega_n, \mathbf{p})$ (typically affecting values by a few percent).
- − ...

The end!