Spectral functions of the $O(N)$ model from the functional renormalization group approach

N. M. Lebedev

(in collaboration with G. A. Kalagov and J. M. Pawlowski)

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Outline

1. Functional renormalization group (FRG) machinery
2. Real-time quantities from the FRG flow: known problems
3. Our approach
4. The outcomes achieved

1PI-functional

✓ The generating functional of connected Green's functions

$$W[J] = \ln \int \mathcal{D}\phi \exp\left\{-S[\phi] + J\phi\right\}.$$

✓ The Legendre transformation – 1PI Green's functions

$$\Gamma[\varphi] = J_\varphi \varphi - W[J_\varphi],$$

where J_φ meets the equation

$$\left.\frac{\delta W[J]}{\delta J}\right|_{J=J_\varphi} = \varphi.$$

Mode decoupling

✓ The generating functional of connected Green's functions

$$W_k[J] = \ln \int \mathcal{D}\phi \exp\left\{-S[\phi] - \Delta S_k[\phi] + J\phi\right\},$$

with the quadratic additive

$$\Delta S_k[\phi] = \frac{1}{2}\phi\, R_k\, \phi.$$

✓ The Legendre transformation – 1PI Gren's functions

$$\Gamma_k[\varphi] = J_{k,\varphi}\varphi - W[J_{k,\varphi}] - \Delta S_k[\varphi],$$

where $J_{k,\varphi}$ meets the equation

$$\left.\frac{\delta W_k[J]}{\delta J}\right|_{J=J_{k,\varphi}} = \varphi.$$

The cut-off kernel R_k

Properties of R_k:

- ✓ $R_k(\mathbf{p}) \to \infty$ as $k \to \Lambda$ (or ∞): fluctuations are frozen, thus $\Gamma_{k\to\Lambda}[\varphi] \to S[\varphi]$ – the mean-field free energy.

- ✓ $R_k(\mathbf{p}) \to 0$ as $k \to 0$: all fluctuations are integrated out, thus $\Gamma_{k\to 0}[\varphi] \to \Gamma[\varphi]$ – the full free energy.

Widely used kernels:

- the exponential shape

$$R_k(\mathbf{p}) = \frac{p^2}{e^{p^2/k^2} - 1}$$

- the theta-regulator

$$R_k(\mathbf{p}) = (k^2 - p^2)\Theta(k^2 - p^2)$$

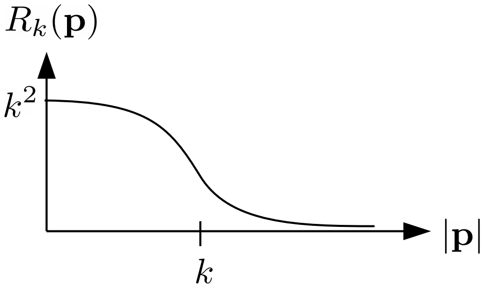


Figure 1: A typical shape of the cut-off function[1].

[1] **Dupuis; 2021**.

The Wetterich equation

✓ Flow in functional space (Wetterich, 1990's)

$$\partial_k \Gamma_k[\varphi] = \frac{1}{2}\,\mathrm{Tr}\left\{ (\Gamma_k^{(2)}[\varphi] + R_k)^{-1} \partial_k R_k \right\},$$

where $\Gamma_k^{(2)}[\varphi]$ is given by the second functional derivative of $\Gamma_k[\varphi]$.

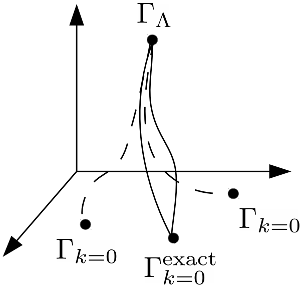


Figure 2: Schematic flows for two different cut-off shapes R_k.

Widely used truncations:

- derivative expansion

  $$\Gamma_k[\varphi] = Z_k(\varphi)(\partial\varphi)^2 + U_k(\varphi) + \text{higher order derivatives}$$

- vertex expansion

  $$\Gamma_k[\varphi] = \sum_n \frac{1}{n!} \int_x \Gamma_k^{(n)}(x_1, \ldots, x_n)\varphi(x_1)\ldots\varphi(x_n)$$

Euclidean flow equations

The two point function in $O(4)$-model

$$\Gamma_k^{(2)}[\varphi] = P_\perp \times \Gamma_{\pi,k}^{(2)} + P_\parallel \times \Gamma_{\sigma,k}^{(2)}.$$

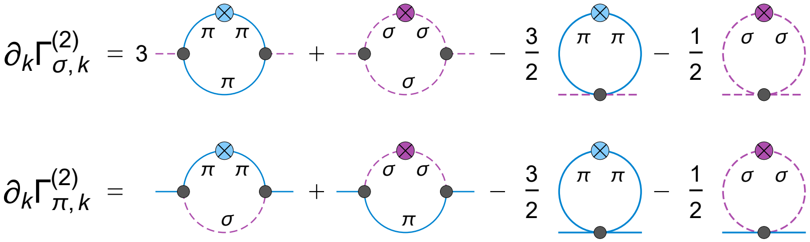


Figure 3: The flow equations for the sigma and pion two-point functions (pic. from Jung et al. 2021)

Analytical continuation

✓ Källén-Lehman spectral representation

$$G^E(p_E) = \int_0^\infty d\lambda \, \frac{\lambda}{p_E^2 + \lambda^2} \rho(\lambda), \qquad p_E^2 = p_0^2 + \mathbf{p}^2.$$

✓ From Euclidean to Minkowski space

$$p_0 = -i(\omega + i\epsilon), \quad \epsilon \to +0, \qquad p^2 = \omega^2 - \mathbf{p}^2.$$

✓ Retarded Green's function

$$G^R(\omega, \mathbf{p}) = -G^E(-i(\omega + i\epsilon), \mathbf{p}), \quad \epsilon \to +0,$$

$$\Gamma^{(2),R}(\omega, \mathbf{p}) = -\Gamma^{(2),E}(-i(\omega + i\epsilon), \mathbf{p}), \quad \epsilon \to +0, .$$

✓ Spectral function

$$\rho(\omega, \mathbf{p}) = -\frac{2}{\pi} \operatorname{Im} G^R(\omega, \mathbf{p}).$$

Regularization by ϵ

It is possible to avoid problems concerned with singularities lying on the real-time momenta axis by keeping a small but finite imaginary part of the momentum

$$\rho(\omega, \mathbf{p}) = \frac{2}{\pi} \frac{\operatorname{Im} \Gamma^{(2),E}(-i(\omega + i\epsilon), \mathbf{p})}{\left[\operatorname{Re}(\Gamma^{(2),E}(-i(\omega + i\epsilon), \mathbf{p})\right]^2 + \left[\operatorname{Im} \Gamma^{(2),E}(-i(\omega + i\epsilon), \mathbf{p})\right]^2}$$

Dependence of the meson spectral functions on the regularization parameter ϵ

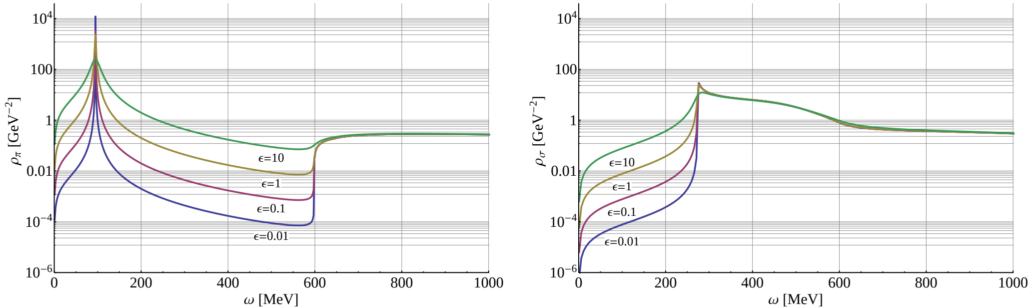


Figure 4: The pion (left) and sigma (right) spectral function, $\rho_\pi(\omega, \vec{p})$ and $\rho_\sigma(\omega, \vec{p})$, calculated in effective quark-meson model withing LPA approach are shown versus external energy ω for different values of the parameter ϵ: 0.01 MeV (blue), 0.1 MeV (magenta), 1 MeV (ochre) and 10 MeV (green) [2].

[2] Tripolt et al. Phys. Rev. D 90, 074031 (2014)

Known problems

- ✓ Keeping finite regulator ϵ causes blurring of thresholds
- ✓ Litim regulators of the form $R_k(\mathbf{p}) = (k^2 - p^2)\Theta(k^2 - p^2)$ violate Lorentz symmetry
- ✓ Regulators with rapid decay in the UV region unavoidably produce unphysical poles and cuts in the momentum complex plane
- ✓ Gradient expansion does not allow to track flow of the pole mass which leads to inconsistent thresholds position

Callan–Symanzik regulator and our ansatz

- ✓ Momentum independent mass-like regulator function

$$R_k(p) = Z\,k^2$$

- ✓ Ansatz for the running effective actions

$$\Gamma_k(\varphi) = \int \mathrm{d}^4p\,\Big[\sigma(p)\,\Sigma_k^\sigma(p)\,\sigma(-p) + \vec{\pi}(p)\,\Sigma_k^\pi(p)\,\vec{\pi}(-p) + U_k(\varphi) - c\,\sigma\Big], \quad \varphi = (\sigma, \vec{\pi}).$$

- ✓ The potential is expanded over a constant background [3]

$$U_k(\varphi) = m_k^2(\rho - \rho_0) + \frac{\lambda_k}{2}(\rho - \rho_0)^2, \quad \rho \equiv \varphi^2/2.$$

- ✓ We keep full momentum dependence of two-point functions $\Gamma_k^{(2)}(p)$ neglecting nevertheless momenta dependencies of the higher vertex functions $\Gamma_k^{(n)}$ $n > 2$.

[3] J. M. Pawlowski et al Phys. Rev. D 90, 076002 (2014)

Euclidean flow equations

✓ Analytical expression for the Euclidean flow

$$\partial_k \Gamma^{(2),E}_{\sigma,k}(p_E) = 3\,(\Gamma^{(3)}_{\pi\pi\sigma,k})^2\, F_{\pi\pi\pi,k}(p_E) + (\Gamma^{(3)}_{\sigma\sigma\sigma,k})^2\, F_{\sigma\sigma\sigma,k}(p_E) \underbrace{\boxed{-\frac{3}{2}(\Gamma^{(4)}_{\sigma\sigma\pi\pi,k})^2\, T^{(2)}_{\pi,k} - \frac{1}{2}(\Gamma^{(4)}_{\sigma\sigma\sigma\sigma,k})^2\, T^{(2)}_{\sigma,k}}}_{\text{tadpole contributions}},$$

$$\partial_k \Gamma^{(2),E}_{\pi,k}(p_E) = (\Gamma^{(3)}_{\pi\pi\sigma,k})^2\, \left[F_{\pi\pi\sigma,k}(p_E) + F_{\sigma\sigma\pi,k}(p_E)\right] \underbrace{\boxed{-\frac{3}{2}(\Gamma^{(4)}_{\pi\pi\pi\pi,k})^2\, T^{(2)}_{\pi,k} - \frac{1}{2}(\Gamma^{(4)}_{\sigma\sigma\pi\pi,k})^2\, T^{(2)}_{\sigma,k}}}_{\text{tadpole contributions}}.$$

✓ The momentum independent vertex functions

$$\Gamma^{(3)}_{\pi\pi\sigma,k} = (2\rho)^{1/2}\, U''_k, \quad \Gamma^{(3)}_{\sigma\sigma\sigma,k} = 3\,(2\rho)^{1/2}\, U''_k + (2\rho)^{3/2}\, U'''_k, \quad \ldots.$$

Momentum structure of the flow

✓ The momentum dependence of the two-point function

$$F_{\alpha\beta\gamma,k}(p_E) \equiv \int_{\{\lambda\}} \{\lambda d\lambda\} \left(\int_{q_E} \frac{1}{(q_E^2+\lambda_1^2)} \frac{1}{(q_E^2+\lambda_2^2)} \frac{1}{((q_E-p_E)^2+\lambda_3^2)} \right) \times \rho_{\alpha,k}(\lambda_1)\, \rho_{\beta,k}(\lambda_2)\, \rho_{\gamma,k}(\lambda_3).$$

✓ The internal integral over q_E is analytically evaluated

$$\left(\dots\right) = \frac{1}{(4\pi)^2(\lambda_1^2-\lambda_2^2)} \int_0^1 dx \ln \frac{x(1-x)\, p_E^2 + (1-x)\,\lambda_3^2 + x\lambda_1^2}{x(1-x)\, p_E^2 + (1-x)\,\lambda_3^2 + x\lambda_2^2}.$$

✓ The continuation procedure yields a non-trivial imaginary part

$$\mathrm{Im}\left(\dots\right) = -\frac{\pi\,\mathrm{sign}(\omega)}{(4\pi)^2}\frac{1}{\lambda_1^2-\lambda_2^2}\left[\sqrt{b^2-4c}\,\Theta(b^2-4c)\,\Theta(b)\,\Theta(2-b) - \{\lambda_1 \to \lambda_2\}\right],$$

where $b \equiv (\lambda_3^2 - \lambda_1^2 + p^2)/p^2$, $c \equiv \lambda_3^2/p^2$.

UV renormalization

✓ Tadpole contributions yield mass-like divergences

$$T^{(2)}_{\alpha\beta,k}(p_E) \equiv \int_{\{\lambda\}} \{\lambda d\lambda\} \left(\int_{q_E} \frac{1}{(q_E^2+\lambda_1^2)} \frac{1}{(q_E^2+\lambda_2^2)} \right) \times \rho_{\alpha,k}(\lambda_1)\, \rho_{\beta,k}(\lambda_2).$$

✓ Renormalization condition

$$\partial_t \Gamma^{(2),E}_{k,R}(p_E) = \partial_t \Gamma^{(2),E}_{k}(p_E) - \partial_t \Gamma^{(2),E}_{k}(p_E=\mu).$$

✓ After performing Wick rotation the condition takes the form

$$\partial_t \Gamma^{(2)}_{k,R}(p) = \partial_t \Gamma^{(2)}_{k}(p) + \partial_t \Gamma^{(2),E}_{k}(p_E=\mu).$$

✓ Resulting renormalized flow

$$\partial_k \Gamma^{(2)}_{\sigma,k}(p) = 3\,(\Gamma^{(3)}_{\pi\pi\sigma,k})^2 \Big[F_{\pi\pi\pi,k}(p) - F_{\pi\pi\pi,k}(p_E=\mu) \Big] + (\Gamma^{(3)}_{\sigma\sigma\sigma,k})^2 \Big[F_{\sigma\sigma\sigma,k}(p) - F_{\sigma\sigma\sigma,k}(p_E=\mu) \Big],$$

$$\partial_k \Gamma^{(2)}_{\pi,k}(p) = (\Gamma^{(3)}_{\pi\pi\sigma,k})^2 \Big[F_{\pi\pi\sigma,k}(p) - F_{\pi\pi\sigma,k}(p_E=\mu) + F_{\sigma\sigma\pi,k}(p) - F_{\sigma\sigma\pi,k}(p_E=\mu) \Big].$$

Convergence of spectral integrals

✓ Ansatz for the spectral function

$$\rho_{\alpha,k}(p) = \frac{1}{Z_{\alpha,k}}\delta(p^2 - m^2_{\alpha,k}) + f_{\alpha,k}(p)$$

✓ One has to account for the δ-function analytically

$$F_{\alpha\beta\gamma,k}(p) \equiv \int_{\{\lambda\}} \{\lambda d\lambda\} \Big(\dots \Big) \times \rho_{\alpha,k}(\lambda_1)\rho_{\beta,k}(\lambda_2)\rho_{\gamma,k}(\lambda_3).$$

Which can lead to a singular integrands

$$\Big(\dots \Big) = \frac{1}{(4\pi)^2(\lambda_1^2 - \lambda_2^2)} \int_0^1 dx \ln \frac{-x(1-x)\,p^2 + (1-x)\,\lambda_3^2 + x\lambda_1^2}{-x(1-x)\,p^2 + (1-x)\,\lambda_3^2 + x\lambda_2^2}.$$

✓ Thought intergrable due to presence of spectral integrals

$$\mathrm{Re}\Big(\dots \Big)\Big|_{\lambda_1=\lambda_2} = \frac{1}{(4\pi)^2} \int_0^1 dx\, \frac{x}{-p^2x(1-x) + x\lambda_1^2 + (1-x)\lambda_3^2} = \frac{1}{p^2(4\pi)^2} \int_0^1 dx\, \frac{x}{(x-x_1)(x-x_2)},$$

$$\mathrm{Im}\Big(\dots \Big)\Big|_{\lambda_1=\lambda_2} = \frac{\pi\,\mathrm{sign}(\omega)}{p^2(4\pi)^2} \frac{b}{\sqrt{b^2-4c}}\Theta(b^2-4c)\Theta(b)\Theta(2-b).$$

Wave mass and function renormalization flow

- ✓ Ansatz at the small enough vicinity of pole

$$\rho_{\alpha,k}(p) = \frac{1}{Z_{\alpha,k}}\delta(p^2 - m_{\alpha,k}^2) \quad \Rightarrow \quad \partial_t \rho_k(p) = -\frac{\partial_t Z_k}{Z_k^2}\,\delta(p^2 - m_k^2) - \frac{2m_k \partial_t m_k}{Z_k}\delta'(p^2 - m_k^2),$$

- ✓ Exact formal expression

$$\rho_k(p) = \delta(\Gamma_k^{(2)}(p) - R_k) = \delta(\tilde{\Gamma}_k^{(2)}(p)) \quad \Rightarrow \quad \partial_t \rho_k(p) = \partial_t \tilde{\Gamma}_k^{(2)}(p)\ \delta'(\tilde{\Gamma}_k^{(2)}(p)),$$

- ✓ Using expansion formula

$$g(x)\frac{d}{df(x)}\delta(f(x)) = \left(\frac{g(x_0)f''(x_0)}{(f'^{\,3}(x_0))} - \frac{g'(x_0)}{(f'^{\,2}(x_0))}\right)\delta(x - x_0) + \frac{g(x)}{(f'^2(x_0)^{\,2})}\ \delta'(x - x_0),$$

- ✓ Flow equations of parameters

$$\partial_t m_k = -\frac{\partial_t \Gamma^{(2)}(p = m_k) - \partial_t R_k}{2m_k Z_k},$$

$$\partial_t Z_k = \frac{\partial_t \Gamma^{(2)}(p = m_k) - \partial_t R_k}{2m_k^2 Z_k}\left(Z_k - \frac{1}{2}\partial_p^2 \Gamma^{(2)}(p = m_k)\right) + \frac{\partial_t \partial_p \Gamma^{(2)}(p = m_k)}{2m_k},$$

Potential flow

✓ LPA flow is divergent

$$\partial_t U_k(\varphi) = \frac{1}{2}\int d^4q \left[\frac{3\,\partial_t R_k}{q^2 + R_k + U_k'(\varphi)} + \frac{\partial_t R_k}{q^2 + R_k + U_k'(\varphi) + 2\rho U_k''(\varphi)}\right],$$

✓ Only the flow of field-derivatives is needed to derive flow of potential parameters

$$\partial_t U_k'(\varphi) = -\frac{1}{2}\int d^4q \left[\frac{3\,\partial_t R_k\, U_k''(\varphi)}{(q^2 + R_k + U_k'(\varphi))^2} + \frac{3\,\partial_t R_k\, U_k''(\varphi)}{(q^2 + R_k + U_k'(\varphi) + 2\rho U_k''(\varphi))^2}\right],$$

✓ $\partial_t \Gamma_k^{(2),E}(p=0)$ can be expressed through the potential field-derivatives

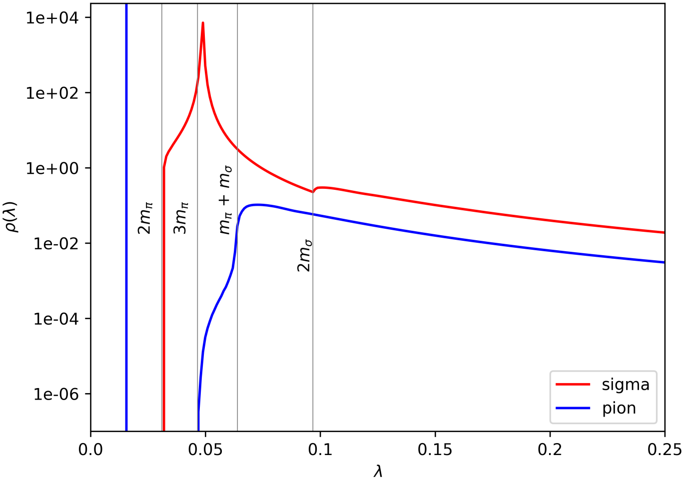
$$\partial_t \Gamma^{(2),E}_{k\,\sigma\sigma}(p=0) = \partial_t U_k'(\varphi) + 2\rho\,\partial_t U_k''(\varphi), \quad \partial_t \Gamma^{(2),E}_{k\,\pi\pi}(p=0) = \partial_t U_k'(\varphi),$$

✓ There are already preformed subtractions

$$\partial_t U_{k\ R}'(\varphi) + 2\rho\,\partial_t U_{k\ R}''(\varphi) = \partial_t U_k'(\varphi) + 2\rho\,\partial_t U_k''(\varphi) - \partial_t \Gamma^{(2),E}_{k\,\sigma\sigma}(p_E = \mu),$$

$$\partial_t U_{k\ R}'(\varphi) = \partial_t U_k'(\varphi) - \partial_t \Gamma^{(2),E}_{k\,\pi\pi}(p_E = \mu).$$

Solution for the spectral functions of $O(4)$ model



Dependence of the spectral functions on the subtraction point μ

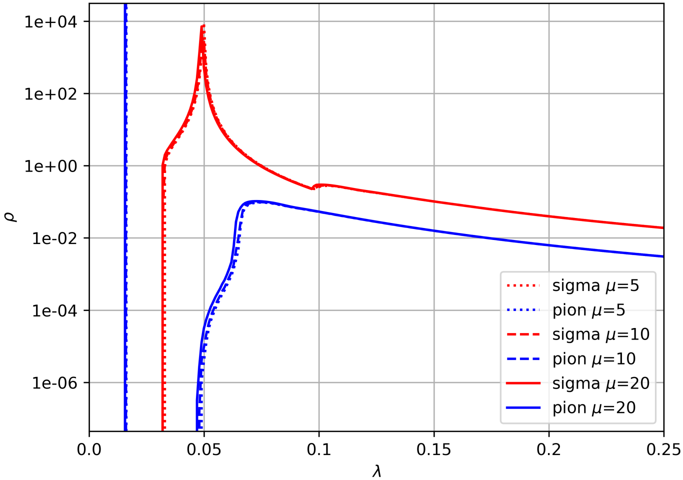


Figure 5: μ in the legend is given as a percentage of Λ

Thank you for attention!