Spin polarization in heavy ion collisions and relativistic spin hydrodynamics

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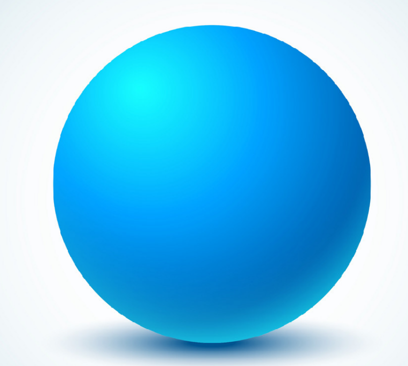
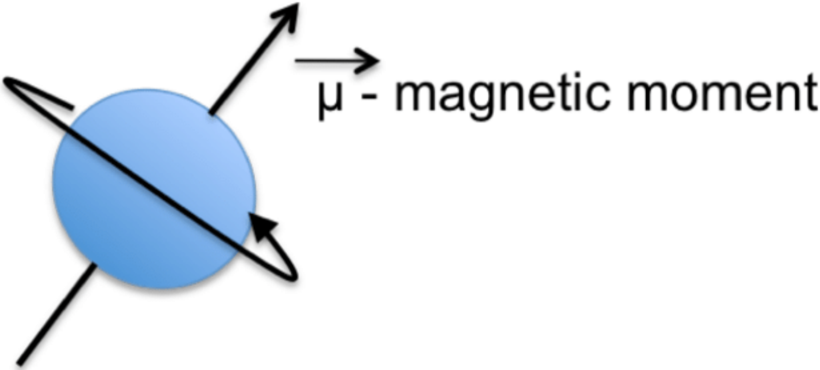
Decay of scalar particles



No anisotropy in the rest frame: isotropic decay products.

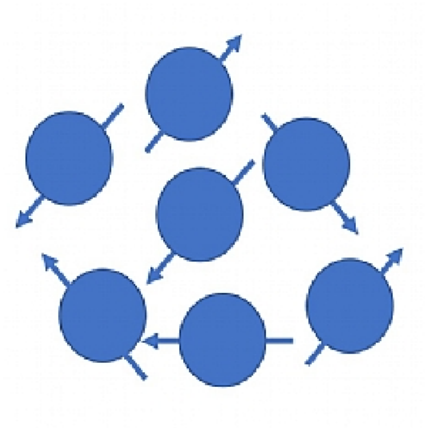
Decay of particles with spin



Preferred direction due to spin: anisotropic decay products

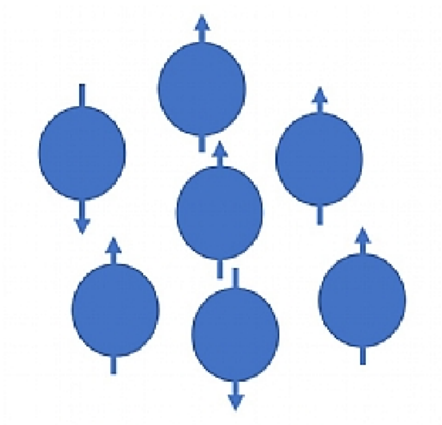
Basis for polarization observables.

Several random decays



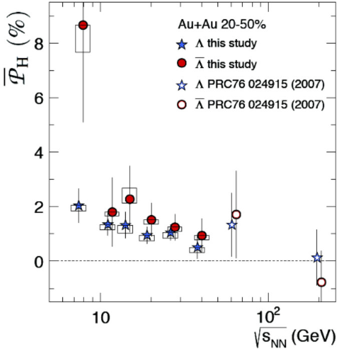
Averaging over random decays should lead to isotropic decay products.

Decay of spin polarized particles



Averaging over decay of spin-polarized particles should lead to anisotropic decay products.

STAR Collaboration, *Global Lambda hyperon polarization in nuclear collisions,* Nature 548 62-65, 2017



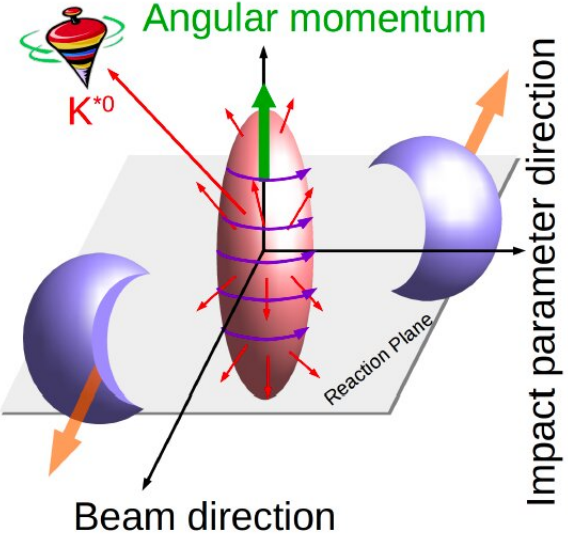
First evidence of a quantum effect in (relativistic) hydrodynamics

Adapted from F. Becattini
'Subatomic Vortices'

Spin polarization of hadrons in heavy-ion collisions

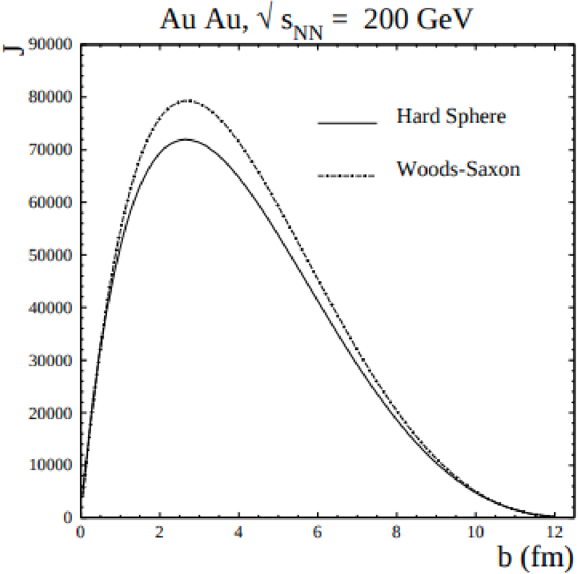
- Spin polarization is a relatively new topic in heavy ion collisions.
- Provides unique opportunity to probe QGP properties.
- Several measurements of spin polarization of hadrons.
- In baryon sector:
  - Λ (spin 1/2): STAR, Nature, 548, 62–65 (2017); HADES; ALICE.
  - Ω (spin 3/2): STAR, Phys. Rev. Lett. 126, 162301 (2021).
  - Ξ (spin 1/2): STAR, Phys. Rev. Lett. 126, 162301 (2021).
- In meson sector:
  - K^{*0} : ALICE, PRL 125, 012301 (2020); STAR, Nature, 614, 244-248 (2023).
  - ϕ : ALICE, PRL 125, 012301 (2020); STAR, Nature, 614, 244-248 (2023).
  - Heavy quarkonium, J/ψ and Υ(1S) : ALICE, PLB 815, 136146 (2021).
- Global and local polarization measurements.

Global angular momentum in heavy ion collisions

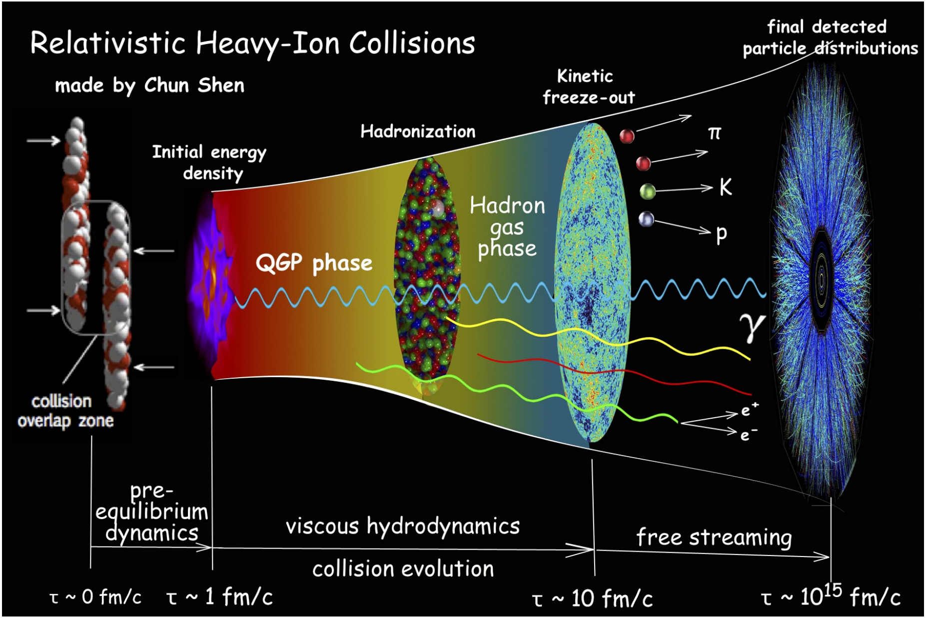


[B. Mohanty, ICTS News 6, 18-20 (2020).]

Angular momentum generation in non-central collisions



[F. Becattini, et al., Phys. Rev. C77, 024906 (2008)]


Relativistic Heavy-Ion Collisions
made by Chun Shen
final detected particle distributions
Kinetic freeze-out
Hadronization
Initial energy density
Hadron gas phase
QGP phase
π
K
p
γ
e+
e-
collision overlap zone
pre-equilibrium dynamics
viscous hydrodynamics
free streaming
collision evolution
τ ~ 0 fm/c
τ ~ 1 fm/c
τ ~ 10 fm/c
τ ~ 10^15 fm/c

Relativistic spin-hydrodynamics

Angular momentum conservation: particles

- Angular momentum of a particle with momentum $\vec{p}$:

$$\vec{L} = \vec{x} \times \vec{p} \quad \Rightarrow \quad L_i = \varepsilon_{ijk}\, x_i\, p_j$$

- One can obtain the dual tensor:

$$L_{ij} \equiv \varepsilon_{ijk}\, L_k \quad \Rightarrow \quad L_{ij} = x_i\, p_j - x_j\, p_i$$

- We know that both definitions are equivalent.
- In absence of external torque, $\dfrac{d\vec{L}}{dt} = 0$, we also have: $\partial_i L_{ij} = 0$.
- Relativistic generalization: $L^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu$ and $\partial_\mu L^{\mu\nu} = 0$.
- This treatment valid for point particles.
- For fluids, particle momenta $\rightarrow$ "generalized fluid momenta"

The energy-momentum tensor

Angular momentum conservation: fluid

- The orbital angular momentum for relativistic fluids is defined as

$$L^{\lambda,\mu\nu} = x^\mu T^{\lambda\nu} - x^\nu T^{\lambda\mu}$$

- Keeping in mind the energy-momentum conservation, $\partial_\mu T^{\mu\nu} = 0$:

$$\partial_\lambda L^{\lambda,\mu\nu} = T^{\mu\nu} - T^{\nu\mu}$$

- Obviously, for symmetric $T^{\mu\nu}$, orbital angular momentum is automatically conserved. Classically $T^{\mu\nu}$ symmetric.
- For medium constituent with intrinsic spin, different story

$$J^{\lambda,\mu\nu} = L^{\lambda,\mu\nu} + S^{\lambda,\mu\nu}$$

- Ensure total angular momentum conservation: $\partial_\lambda J^{\lambda,\mu\nu} = 0$.
- Basis for formulation of spin Hydrodynamics.
  [Florkowski et. al., Prog.Part.Nucl.Phys. 108 (2019) 103709; Bhadury et. al., Eur.Phys.J.ST 230 (2021) 3, 655-672]

Pseudo-gauge transformations

- Total angular momentum is

$$J^{\lambda,\mu\nu} = L^{\lambda,\mu\nu} + S^{\lambda,\mu\nu}$$

- With $\partial_\mu T^{\mu\nu} = 0$, and $\partial_\lambda L^{\lambda,\mu\nu} = T^{\mu\nu} - T^{\nu\mu}$,

$$\partial_\lambda J^{\lambda,\mu\nu} = 0 \quad \Longrightarrow \quad \partial_\lambda S^{\lambda,\mu\nu} = T^{\nu\mu} - T^{\mu\nu}$$

- Hence the final hydrodynamic equations can be written as

$$\partial_\mu T^{\mu\nu} = 0, \qquad \partial_\lambda S^{\lambda,\mu\nu} = T^{\nu\mu} - T^{\mu\nu}$$

- Also holds with the following redefinition

$$\tilde{T}^{\mu\nu} = T^{\mu\nu} + \frac{1}{2}\partial_\lambda \left(\Phi^{\lambda,\mu\nu} - \Phi^{\mu,\lambda\nu} - \Phi^{\nu,\lambda\mu}\right)$$

$$\tilde{S}^{\lambda,\mu\nu} = S^{\lambda,\mu\nu} - \Phi^{\lambda,\mu\nu} + \partial_\rho Z^{\mu\nu,\lambda\rho}$$

- Polarization observables are independent of pseudo-gauge freedom.

[Gallegos et. al., SciPost Phys. 11, 041 (2021); Hongo et. al., JHEP 11 (2021) 150]

Pseudo-gauge transformations and transport

- Different forms of conserved currents used:
  1. Canonical: $S^{[\lambda\mu\nu]},\ T^{(\mu\nu)} + T^{[\mu\nu]}$
  2. Belinfante: $S^{\lambda,\mu\nu} = 0,\ T^{(\mu\nu)}$
  3. de Groot, van Leuween and van Weert (GLW): $S^{\lambda,[\mu\nu]},\ T^{(\mu\nu)}$
  4. Hilgevoord and Wouthuysen (HW): $S^{\lambda,[\mu\nu]},\ T^{(\mu\nu)}$
  5. Phenomenological: $S^{\lambda,\mu\nu} \sim u^\lambda \omega^{\mu\nu},\ T^{(\mu\nu)} + T^{[\mu\nu]}$
- Belinfante does not retain information about evolution of spin.
- Canonical is not most general: anti-symmetry in all three indices.
- Phenomenological not related to canonical via PG transformation.
- Derivative terms are generated in conserved currents by PG trans.
- Redistribution of spin evolution between $S^{\lambda,\mu\nu}$ and $T^{[\mu\nu]}$.
- Issues in counting of transport coefficients for spin evolution.

Extended phase-space for spin degrees of freedom

- The phase-space for single particle distribution function gets extended $f(x, p, s)$.
- The equilibrium distribution for Fermions is given by

$$f_{eq}(x,p,s) = \frac{1}{\exp\left[\beta \cdot p - \alpha - \frac{1}{2}\,\omega : s\right] + 1} \qquad \begin{cases} \beta \cdot p \equiv \beta_\mu p^\mu \\ \omega : s \equiv \omega_{\mu\nu} s^{\mu\nu} \end{cases}$$

- Quantities $\beta^\mu = u^\mu/T,\ \alpha = \mu/T,\ \omega_{\mu\nu}$ are functions of x.
- $\alpha,\ \beta^\mu,\ \omega^{\mu\nu}$: Lagrange multipliers for conserved quantities.
- $s^{\mu\nu}$: Particle spin, on equal footing with particle momenta p^μ.
- Hydrodynamics: average over particle momenta and spin.
- Like T, μ, u^μ, solve for $\omega^{\mu\nu}$ with appropriate initial conditions.
- Current state-of-art: Thermal vorticity used as a proxy for $\omega^{\mu\nu}$.

Boltzmann equation and global equilibrium

- Boltzmann equation for distribution function is

$$p^\mu \partial_\mu f = C[f]$$

- In equilibrium, $C[f] = 0$. Global equilibrium condition:

$$p^\mu \partial_\mu f_{eq} = 0$$

- For $f_{eq} = \left[\exp\left(\beta \cdot p - \alpha - \frac{1}{2}\,\omega : s\right) + 1\right]^{-1}$, one obtains

$$\partial_\mu \alpha = 0; \quad \partial^\mu \beta^\nu + \partial^\nu \beta^\mu = 0; \quad \partial_\mu \omega_{\rho\sigma} = 0$$

- A solution can be obtained as

$$\alpha = \text{const.}; \quad \beta^\mu = \beta_0^\mu + x_\lambda\, \omega_0^{\mu\lambda}; \quad \omega_{\rho\sigma} = \text{const.}$$

- The last two solutions leads to

$$\omega_0^{\mu\nu} = -\frac{1}{2}\left(\partial^\mu \beta^\nu - \partial^\nu \beta^\mu\right); \quad \omega_{\mu\nu} \to \omega_0^{\mu\nu} \equiv \varpi_{\mu\nu}$$

- This assumption avoids solving spin-hydro equations.

Pauli-Lubanski and Polarization

- On freeze-out hypersurface: $\langle P(\phi_p)\rangle = \dfrac{\int p_T dp_T E_p \frac{d\Pi^z(p)}{d^3p}}{\int d\phi_p p_T dp_T E_p \frac{dN(p)}{d^3p}}$

- $$E_p \frac{dN(p)}{d^3p} = \frac{4\cosh\xi}{(2\pi)^3}\int \Delta\Sigma_\lambda p^\lambda e^{-\beta\cdot p}, \qquad \xi = \mu/T,\ \beta^\mu = u^\mu/T$$

- $$E_p \frac{d\Delta\Pi_\tau(x,p)}{d^3p} = -\frac{1}{2}\epsilon_{\tau\mu\nu\beta}\,\Delta\Sigma_\lambda E_p \frac{dS^{\lambda,\mu\nu}(\omega)}{d^3p}\frac{p^\beta}{m}$$

  [Florkowski et. al., Prog.Part.Nucl.Phys. 108 (2019) 103709]

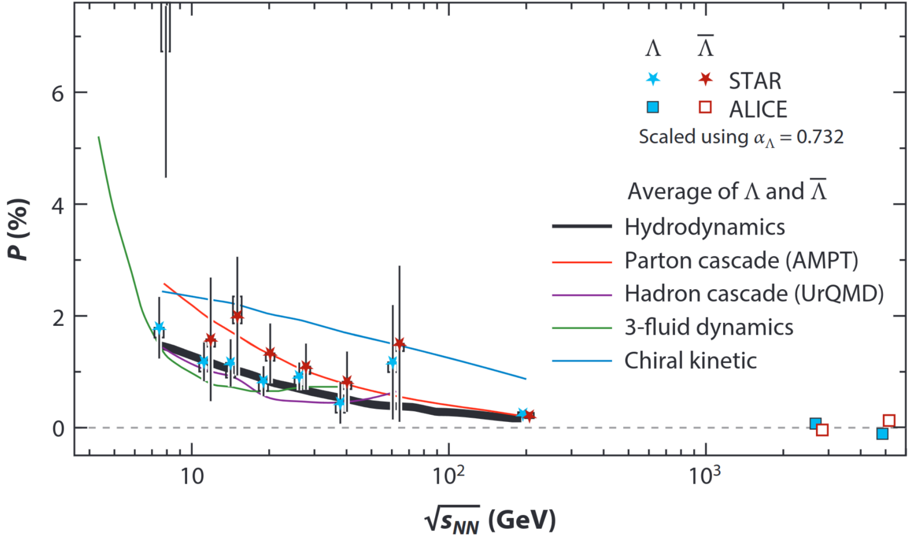
- The spin tensor can be defined as

$$S^{\lambda,\mu\nu}(\omega) = \int dP dS\ p^\lambda s^{\mu\nu}\left[f(x,p,s) + \bar{f}(x,p,s)\right]$$

- In absence of hydrodynamic evolution, one uses the ansatz:

$$\omega_{\mu\nu} \to \varpi_{\mu\nu} = -\frac{1}{2}(\partial_\mu\beta_\nu - \partial_\nu\beta_\mu)$$

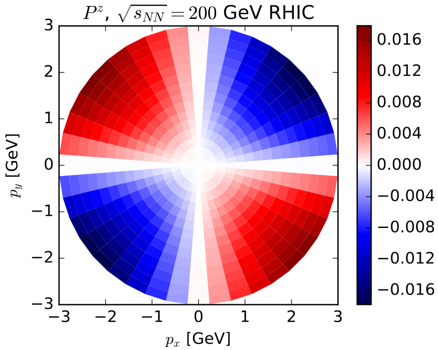
Success of thermal vorticity: Global polarization



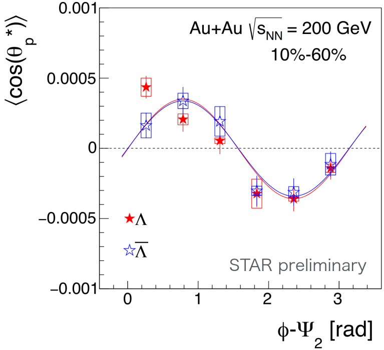
[F. Becattini and M. Lisa, Annu. Rev. Nucl. Part. Sci. 2020. 70:395-423]

Longitudinal/local polarization and sign problem

vHLLE+Glissando IS



Preliminary STAR data: Takafumi Niida, talk at Chirality workshop 2018

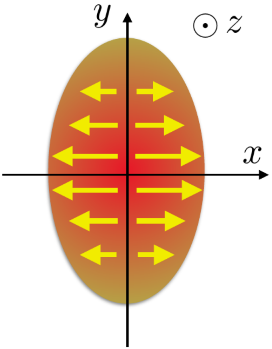


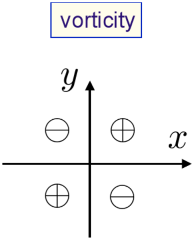
Similar $\sin(2\phi)$ structure is observed, with opposite sign!

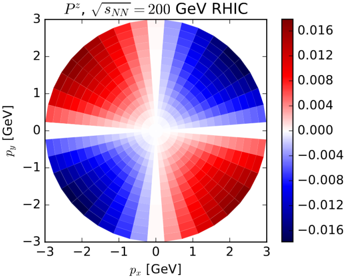
[Iurii Karpenko, Lambda polarization from RHIC BES to LHC]

Simplified explanation of the quadrupole structure

(c) Sergei Voloshin, SQM2017







Polarization depends on the the thermal vorticity:

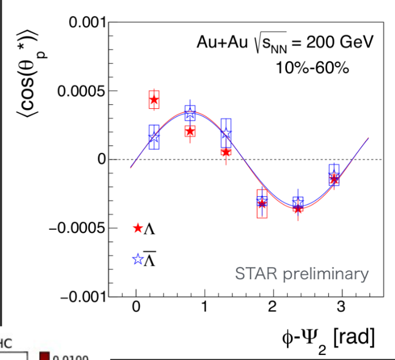
$$\varpi_{\mu\nu} = -\frac{1}{2}\left(\partial_\mu \beta_\nu - \partial_\nu \beta_\mu\right)$$

[Iurii Karpenko, Lambda polarization from RHIC BES to LHC]

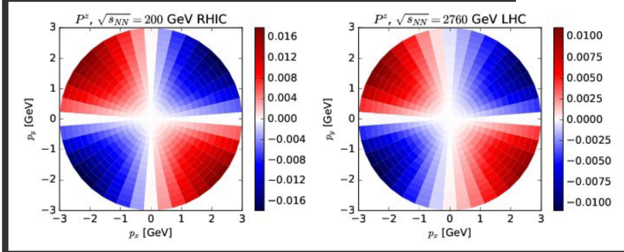
A *sign problem* for the longitudinal component

Quadrupolar structure of longitudinal polarization in the transverse momentum plane, as predicted.
Spectacular confirmation of hydro predictions... yet with a flipped sign!

- Hydro initial conditions? (polarization is a sensitive probe of the initial flow)
- Incomplete local thermodynamic equilibrium for the spin degrees of freedom (spin kinetic theory)?
- Effect of spin dissipative transport coefficients?
- Effect of initial state fluctuations?
- Effect of decays?
- Error in the calculation



Z. Ye, T. Niida, Quark Matter 2018



F. B., I. Karpenko, Phys. Rev. Lett. 120 (2018) 012302
S. Voloshin, in SQM 2017

Same pattern found in AMPT+thermal vorticity calculation X. L. Xia, H. Li, Z. B. Tang and Q. Wang, 1803.00867

Global equilibrium and thermal vorticity

- Global equilibrium may not be achievable: short fireball lifetime.
- Large spin equilibration time [1907.10750, 2405.00533, 2405.05089, ...].
- Spin hydrodynamic evolution necessary with appropriate initial conditions [Singh et. al., 2411.08223].
- Thermal vorticity is a robust prediction of spin-hydrodynamics.
- Alternate systems for signature of thermal vorticity solution.
- Electrons in graphene near Dirac point: "relativistic" dispersion.
- No issues with short lifetime for graphene: global equilibrium.
- Analog of Barnett effect: *Thermovortical magnetization* [2409.07764].

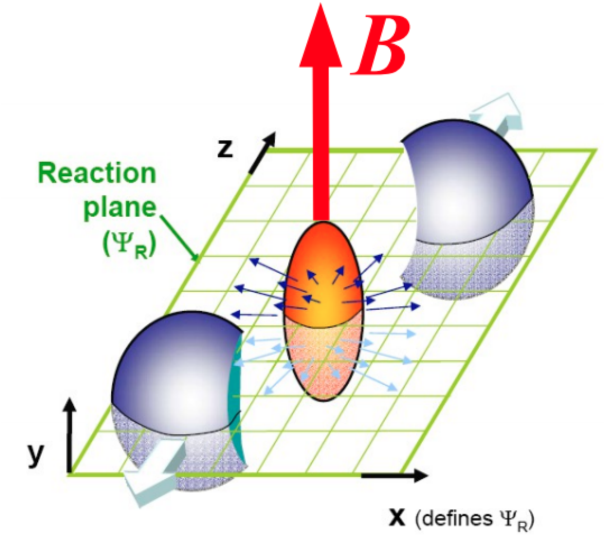
Our work on spin hydrodynamics within kinetic theory

- Non-dissipative spin-hydrodynamics:
  - W. Flokowski, B. Friman, A. Jaiswal and E. Speranza, Physical Review C 97, 041901 (2018).
  - W. Flokowski, B. Friman, A. Jaiswal, R. Ryblewski and E. Speranza, Physical Review D 97, 116017 (2018).
- Dissipative spin-hydrodynamics:
  - S. Bhadury, W. Flokowski, A. Jaiswal, A. Kumar, and R. Ryblewski, Physics Letters B 814, 136096 (2021).
  - S. Bhadury, W. Flokowski, A. Jaiswal, A. Kumar, and R. Ryblewski, Physical Review D 103, 014030 (2021).

- Relativistic Spin Magnetohydrodynamics: S. Bhadury, W. Flokowski, A. Jaiswal, A. Kumar, and R. Ryblewski, Phys. Rev. Lett., 129, 192301 (2022).

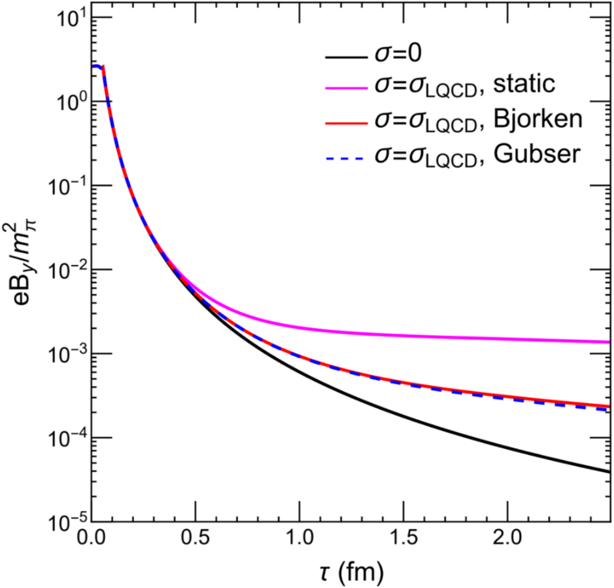
Heavy quark spin polarization

Generation of magnetic field in heavy ion collisions



[Adapted from D. Kharzeev @ CPOD 2013.]

Magnetic field time evolution

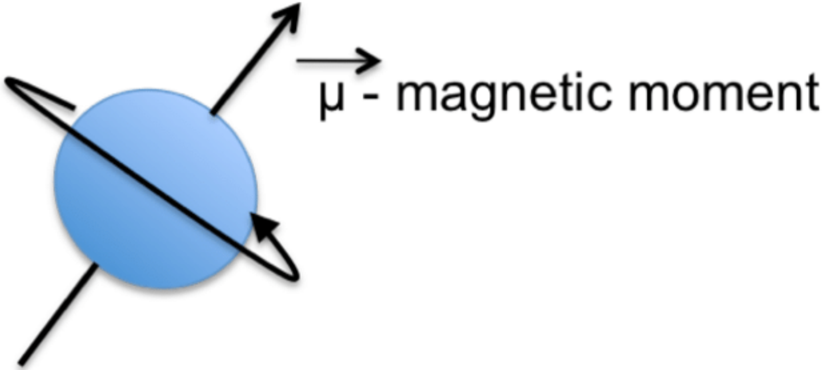


[A. Huang, D. She, S. Shi, M. Huang and J. Liao, Phys. Rev. C 107, 034901 (2023).]

Heavy quarks in relativistic heavy-ion collisions

- Heavy quarks (charm and bottom) has long been recognized as an excellent probe of transport properties of QCD medium. [Moore and Teaney PRC 71 (2005) 064904; Rapp and van Hees, Quark Gluon Plasma 4; Banerjee et. al., PRD 85 (2012) 014510; Das et. al. PLB 768 (2017) 260-264; $\cdots$]
- Heavy quarks are primarily generated in the initial hard scatterings of partons.
- Clean probe of the early-stage properties of heavy-ion collisions.
- Strong transient magnetic fields produced which are significant only during the early stages of the collision.
- Heavy quarks: ideal for observable signals of initial magnetic field.
- Our proposal:
  - Strong magnetic fields induce spin polarization of heavy quarks.
  - These induced spin polarization of heavy quarks can be observed in the polarization of open heavy-flavor hadrons.
  - Transverse momentum dependence of open heavy-flavor hadron polarization: distinctive signal for initial strong magnetic field.

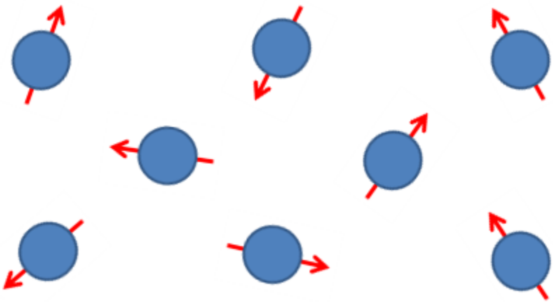
Heavy quarks: charged, spin-1/2 particles



Interaction with magnetic field: $\mathcal{H} = -\vec{\mu} \cdot \vec{B}$

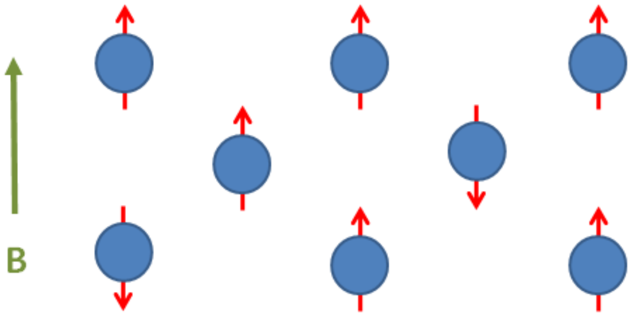
Magnetic moment and spin: $\vec{\mu} = \gamma \vec{s}$

No external magnetic field



Un-aligned spins of heavy quarks.

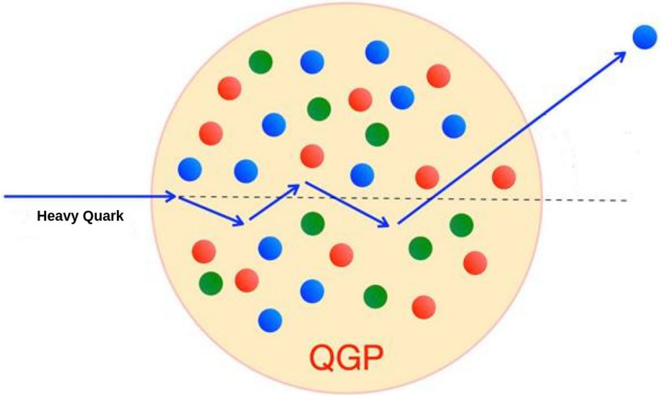
Heavy quarks in magnetic field



Aligned spins in presence of magnetic field.

Spin-polarization of heavy quarks.

Heavy quarks in QGP



Polarized heavy quarks propagates through QGP.

Rotational Brownian motion

- Random rotational motion (orientation and angular velocity) of a microscopic particle due to thermal fluctuations caused by collisions with surrounding medium particles.
- Rotational Brownian motion problem: first considered by Debye.
- For classical spins, the Langevin equation corresponds to the stochastic Landau–Lifshitz-Gilbert equation

$$\frac{d\mathbf{s}}{d\tau} = \mathbf{s} \times \left[\tilde{\mathbf{B}} + \boldsymbol{\xi}(\tau)\right] - \lambda\,\mathbf{s} \times \left(\mathbf{s} \times \tilde{\mathbf{B}}\right)$$

- Here $\tilde{\mathbf{B}} \equiv \gamma\mathbf{B} = -\frac{\partial\mathcal{H}}{\partial\mathbf{s}}$ and γ is the gyromagnetic ratio $\boldsymbol{\mu} = \gamma\,\mathbf{s}$.
- $\mathbf{s} \times \tilde{\mathbf{B}}$ represents precession dynamics of the system.
- $\boldsymbol{\xi}(\tau)$ is the random torque on the particle by the medium.
- λ is the damping coefficient.

Fokker-Planck equation for spin [S. Dey and AJ, arXiv:2502.20352]

- Fokker–Planck equation corresponding to the stochastic Landau–Lifshitz-Gilbert equation

$$\frac{\partial \mathcal{P}}{\partial \tau} = \lambda \frac{\partial}{\partial \mathbf{s}} \cdot \left[\mathbf{s} \times \left(\mathbf{s} \times \left(\tilde{\mathbf{B}} - T \frac{\partial}{\partial \mathbf{s}} \right) \right) \right] \mathcal{P}$$

- To find: Probability of a spin-polarized particle having an instantaneous orientation in the direction (θ, ϕ).
- Consider a sphere in spin-space of fixed radius s, i.e., $\mathbf{s} = (s, \theta, \phi)$: each point on the sphere represents a different spin orientation.
- Choose z-axis to be along $\tilde{\mathbf{B}}$. With Hamiltonian $\mathcal{H} = -\mathbf{s} \cdot \tilde{\mathbf{B}}$, only θ is relevant for polarization.
- Assuming all heavy quarks are initially spin polarized along $\theta = \theta_0$ direction, i.e., for the initial condition $\mathcal{P}(\theta, 0) = \delta(\cos\theta - \cos\theta_0)$,

$$\langle \cos\theta \rangle = \cos\theta_0 \, e^{-2\tau/\tau_s}, \qquad \langle \cos^2\theta \rangle = \frac{1}{3} + \frac{2}{3} e^{-6\tau/\tau_s}.$$

Heavy baryon and meson polarization

- For baryons, the angular distribution of one of the decay daughter

$$\frac{dN}{d\cos\theta} = \frac{1}{2}\left(1 + \alpha_B|\vec{P}_B|\cos\theta\right)$$

- α_B is decay parameter. Using this distribution, one gets

$$\langle\cos\theta\rangle = \int \cos\theta \frac{dN}{d\cos\theta} d\cos\theta \implies |\vec{P}_B| = \frac{3}{\alpha_B}\langle\cos\theta\rangle$$

- Similarly, for mesons, the angular distribution is

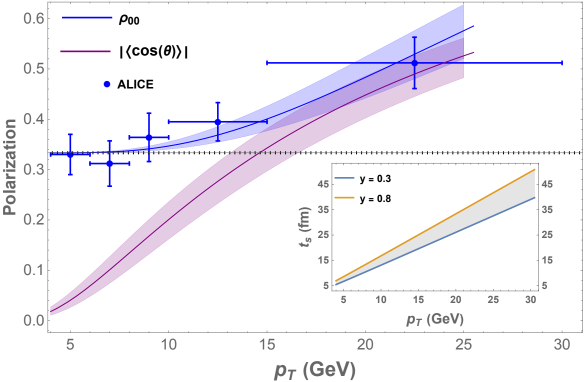
$$\frac{dN}{d\cos\theta} = \frac{3}{4}\left[1 - \rho_{00} + \left(3\rho_{00} - 1\right)\cos^2\theta\right]$$

- ρ_{00} is element of spin density matrix; unpolarized $\implies \rho_{00} = 1/3$.
- Using this distribution, one gets

$$\langle\cos^2\theta\rangle = \int \cos^2\theta \frac{dN}{d\cos\theta} d\cos\theta \implies \Delta\rho_{00} = \frac{5}{2}\left[\langle\cos^2\theta\rangle - \frac{1}{3}\right]$$

- Here $\Delta\rho_{00} \equiv \rho_{00} - 1/3$

D^{*+} meson spin alignment [S. Dey and AJ, arXiv:2502.20352]



- Duration for which heavy quark undergoes Brownian motion:

$$\tau = \frac{R\, m_Q}{p_T}, \quad R \approx 10 \text{ fm av. trans. size of fireball}$$

- With $\gamma_v = E/m_Q$, The quantities in the plot are:

$$\rho_{00} = \frac{1}{3} + \frac{5}{3} e^{-6\tau/\tau_s}, \quad |\langle \cos\theta \rangle| = e^{-2\tau/\tau_s}, \quad t_s = \gamma_v \tau_s$$

Ongoing and future works in this direction

- Fireball assumed to be static with constant average temperature.
- More realistic space-time evolution of the fireball and external magnetic field necessary.
- Predictions at forward rapidities.
- Calculation of spin relaxation time τ_s for heavy quarks.
- Derivation of an Einstein-Stokes-like relation between the spin diffusion coefficient and the dissipative parameters in spin hydrodynamics.
- Derivation of rotational Fokker-Planck equation from Kinetic theory with non-local collision terms.

Summary

- Pseudogauge freedom in the formulation of spin hydrodynamics.
- Polarization observables independent of pseudogauge freedom.
- Pseudogauge freedom and counting of spin transport not settled.
- Sign problem in longitudinal component of spin polarization.
- Thermal vorticity ansatz for polarization tensor: not good.
- Evolution with spin-hydrodynamics necessary, some progress.
- Intersting progress in heavy quark polarization measurements.
- Stochastic spin dynamics is the way forward.
- Polarization and spin hydrodynamics: exciting times.
- Opportunities for exciting new physics.

Thank you!

Back-up Slides

Relativistic kinetic theory

- Kinetic theory: calculation of macroscopic quantities by means of statistical description in terms of distribution function.
- Let us consider a system of relativistic particles of rest mass m with momenta $\mathbf{p}$ and energy p^0

$$p^0 = \sqrt{\mathbf{p}^2 + m^2}$$

- For large no. of particles, $f(x,p)$ gives a distribution of the four-momenta $p = p^\mu = (p^0, \mathbf{p})$ at each space-time point.
- $f(x,p)\Delta^3 x\, \Delta^3 p$ gives average no. of particles in the volume element $\Delta^3 x$ at point x with momenta in the range $(\mathbf{p}, \mathbf{p} + \Delta\mathbf{p})$.
- Statistical assumptions:
  - No. of particles contained in $\Delta^3 x$ is large $(N \gg 1)$.
  - $\Delta^3 x$ is small compared to macroscopic volume $(\Delta^3 x / V \ll 1)$.
- The equilibrium distribution: $f_{eq}(x,p) = \left[\exp\left(\beta \cdot p - \xi\right) \pm 1\right]^{-1}$

Extended phase-space for spin degrees of freedom

- The phase-space for single particle distribution function gets extended $f(x, p, s)$.
- The equilibrium distribution for Fermions is given by

$$f_{eq}(x,p,s) = \frac{1}{\exp\left[\beta \cdot p - \alpha - \frac{1}{2}\,\omega : s\right] + 1} \qquad \begin{cases} \beta \cdot p \equiv \beta_\mu p^\mu \\ \omega : s \equiv \omega_{\mu\nu} s^{\mu\nu} \end{cases}$$

- Quantities $\beta^\mu = u^\mu/T,\ \alpha = \mu/T,\ \omega_{\mu\nu}$ are functions of x.
- $\alpha,\ \beta^\mu,\ \omega^{\mu\nu}$: Lagrange multipliers for conserved quantities.
- $s^{\mu\nu}$: Particle spin, similar to particle momenta p^μ.
- Hydrodynamics: average over particle momenta and spin.
- Classical treatment of spin.

Bhadury et. al., PLB 814, 136096 (2021); PRD 103, 01430 (2021).

Conserved currents and spin-hydrodynamics

- Express hydrodynamic quantities in terms of $f(x,p,s)$.

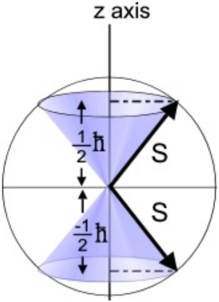
$$T^{\mu\nu}(x) = \int dP dS \; p^\mu p^\nu \left[f(x,p,s) + \bar{f}(x,p,s) \right]$$

$$N^{\mu}(x) = \int dP dS \; p^\mu \left[f(x,p,s) - \bar{f}(x,p,s) \right]$$

$$S^{\lambda,\mu\nu}(x) = \int dP dS \; p^\lambda s^{\mu\nu} \left[f(x,p,s) + \bar{f}(x,p,s) \right]$$

$$dP \equiv \frac{d^3 p}{E_p (2\pi)^3}, \quad dS \equiv m \frac{d^4 s}{\pi \mathfrak{s}} \, \delta(s\cdot s + \mathfrak{s}^2) \, \delta(p \cdot s)$$

$$\int dS = 2; \quad \mathfrak{s}^2 = \frac{1}{2}\left(\frac{1}{2}+1\right) = \frac{3}{4}; \quad s^\mu \equiv \frac{1}{2m} \epsilon^{\mu\nu\alpha\beta} p_\nu s_{\alpha\beta}$$



- Classical treatment of spin: internal angular momentum.
- Equations of motion: $\partial_\mu T^{\mu\nu} = 0, \quad \partial_\mu N^\mu = 0, \quad \partial_\lambda S^{\lambda,\mu\nu} = 0$.
- Non-dissipative spin hydrodynamics: $f(x,p,s) = f_{eq}(x,p,s)$.

Dissipative spin-hydrodynamics Bhadury et. al., PLB 814, 136096 (2021)

- Introduce out-of-equilibrium distribution function $f(x,p,s)$.
- Use Boltzmann equation for evolution of $f = f_{eq} + \delta f$.

$$p^\mu \partial_\mu f = C[f]$$

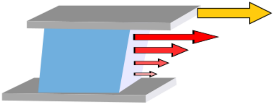
- Employ relaxation-time approximation for collision kernel.

$$C[f] = -(u \cdot p)\frac{f - f_{eq}}{\tau_{eq}}$$

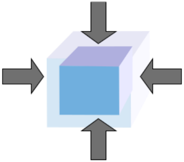
- Solve assuming small departure from equilibrium, $\delta f / f_{eq} \ll 1$.
- First order dissipative spin hydrodynamics for $\delta f = \delta f_1$.
- Relativistic Navier-Stokes analog of spin-hydrodynamics.

Dissipative effects

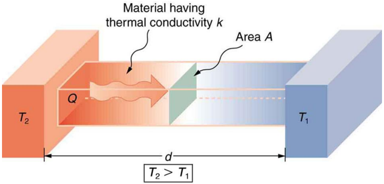
- Shear viscosity: fluid's resistance to shear forces



- Bulk viscosity: fluid's resistance to compression



- Charge/heat conductivity: fluid's resistance to flow of charge/heat.
- Dissipation to spin current new: Kubo formalism necessary.



Spin Magnetohydrodynamics Bhadury et. al., PRL 129, 192301 (2022)

- The particle four-current and its conservation is given by

$$N^{\mu} = n u^{\mu} + n^{\mu}, \qquad \partial_{\mu} N^{\mu} = 0$$

- Total stress-energy tensor of the system: $T^{\mu\nu} = T^{\mu\nu}_{\rm f} + T^{\mu\nu}_{\rm int} + T^{\mu\nu}_{\rm em}$

$$T^{\mu\nu}_{\rm f} = \epsilon u^{\mu} u^{\nu} - (P + \Pi)\,\Delta^{\mu\nu} + \pi^{\mu\nu},$$

$$T^{\mu\nu}_{\rm int} = -\Pi^{\mu} u^{\nu} - F^{\mu}{}_{\alpha} M^{\nu\alpha}$$

$$T^{\mu\nu}_{\rm em} = -F^{\mu\alpha} F^{\nu}{}_{\alpha} + \frac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}$$

- Maxwell's equation: $\partial_{\mu} H^{\mu\nu} = J^{\nu}$ and $H^{\mu\nu} = F^{\mu\nu} + M^{\mu\nu}$,

$$\partial_{\mu} T^{\mu\nu}_{\rm em} = F^{\nu}{}_{\alpha} J^{\alpha}$$

- Current generating external field, $J^{\mu} = J^{\mu}_{\rm f} + J^{\mu}_{\rm ext}$ where $J^{\mu}_{\rm f} = \mathfrak{q} N^{\mu}$,

$$\partial_{\mu} T^{\mu\nu} = -f^{\nu}_{\rm ext}, \qquad f^{\nu}_{\rm ext} = F^{\nu}{}_{\alpha} J^{\alpha}_{\rm ext}$$

Equations of motion

- Divergence of matter part of energy-momentum tensor,

$$\partial_\nu T_{\rm f}^{\mu\nu} = F^{\mu}{}_{\alpha} J_{\rm f}^{\alpha} + \frac{1}{2}\left(\partial^\mu F^{\nu\alpha}\right) M_{\nu\alpha}$$

- Next, consider total angular momentum conservation:

$$J^{\lambda,\mu\nu} = L^{\lambda,\mu\mu} + S^{\lambda,\mu\nu}$$

- In presence of external torque its divergence leads to,

$$\partial_\lambda J^{\lambda,\mu\nu} = -\tau_{\rm ext}^{\mu\nu}, \qquad \tau_{\rm ext}^{\mu\nu} = x^\mu f_{\rm ext}^{\nu} - x^\nu f_{\rm ext}^{\mu}$$

- Torque due to moment of external force; “pure” torque ignored.
- The orbital part of angular momentum and its divergence is

$$L^{\lambda,\mu\nu} = x^\mu T^{\lambda\nu} - x^\nu T^{\lambda\mu}, \qquad \partial_\lambda L^{\lambda,\mu\nu} = -\tau_{\rm ext}^{\mu\nu}$$

- Spin part of the total angular momentum is conserved

$$\partial_\lambda S^{\lambda,\mu\nu} = 0$$

- Along with particle four-current conservation, $\partial_\mu N^\mu = 0$.

Boltzmann equation

- Boltzmann equation (BE) in relaxation-time approximation (RTA)

$$\left(p^\alpha \frac{\partial}{\partial x^\alpha} + m\,\mathcal{F}^\alpha \frac{\partial}{\partial p^\alpha} + m\,\mathcal{S}^{\alpha\beta} \frac{\partial}{\partial s^{\alpha\beta}}\right) f = C[f] = -\,(u \cdot p)\,\frac{f - f_{\rm eq}}{\tau_{\rm eq}}$$

- The force term is:

$$\mathcal{F}^\alpha = \frac{\mathfrak{q}}{m}\,F^{\alpha\beta} p_\beta + \frac{1}{2}\left(\partial^\alpha F^{\beta\gamma}\right) m_{\beta\gamma}, \qquad m^{\alpha\beta} = \chi s^{\alpha\beta}$$

- There is a “pure” torque term:

$$\mathcal{S}^{\alpha\beta} = 2\,F^{\gamma[\alpha}\,m^{\beta]}{}_{\gamma} - \frac{2}{m^2}\left(\chi - \frac{\mathfrak{q}}{m}\right) F_{\phi\gamma}\,s^{\phi[\alpha}\,p^{\beta]} p^\gamma$$

- We ignore this “pure” torque term for now.
- Employ the Boltzmann equation to obtain $\delta f = \delta f_1$.
- Evolution equations for spin-magnetohydrodynamics.

Hydrodynamic equations from kinetic theory

- Impose Landau frame and extended matching conditions

$$u_\mu T^{\mu\nu} = \epsilon u^\nu, \quad \epsilon = \epsilon_{\rm eq}, \quad n = n_{\rm eq}, \quad u_\lambda \delta s^{\lambda,\mu\nu} = 0$$

- Zeroth, first and "spin" moment of the RTA collision vanishes

$$\int dP dS\, C[f] = \int dP dS\, p^\mu\, C[f] = \int dP dS\, s^{\mu\nu} C[f] = 0$$

- Using definitions of hydro quantities, these moments of BE gives

$$\partial_\mu N^\mu = 0, \quad \partial_\nu T^{\mu\nu}_{\rm f} = F^\mu{}_\alpha J^\alpha_{\rm f} + \frac{1}{2}\left(\partial^\mu F^{\nu\alpha}\right) M_{\nu\alpha}, \quad \partial_\lambda S^{\lambda,\mu\nu} = 0$$

- Same equations as obtained from macroscopic arguments.
- Polarization/magnetization emerge naturally at gradient order.
- Boltzmann equation → dissipative spin-magnetohydodynamics.

Einstein-de Haas and Barnett effects

- One can define the polarization-magnetization tensor as

$$M^{\mu\nu} = m \int dP dS \, m^{\mu\nu} \left(f - \bar{f}\right)$$

- The equilibrium polarization-magnetization tensor is

$$M_{eq}^{\mu\nu} = m \int dP dS \, m^{\mu\nu} \left(f_{eq} - \bar{f}_{eq}\right)$$

- Magnetic dipole moment $m^{\mu\nu} = \chi s^{\mu\nu}$.
- χ: resembles the gyromagnetic ratio.
- Integrating over the momentum and spin degrees of freedom,

$$M_{eq}^{\mu\nu} = a_1 \, \omega^{\mu\nu} + a_2 \, u^{[\mu} u_\gamma \omega^{\nu]\gamma}$$

- In global equilibrium, $\omega^{\mu\nu}$ corresponds to rotation of the fluid.
- Rotation produces magnetization (Barnett effect) and vice versa (Einstein-de Hass effect).