Applications of dual symmetries of QCD

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International Workshop "Infinite and Finite Nuclear Matter"
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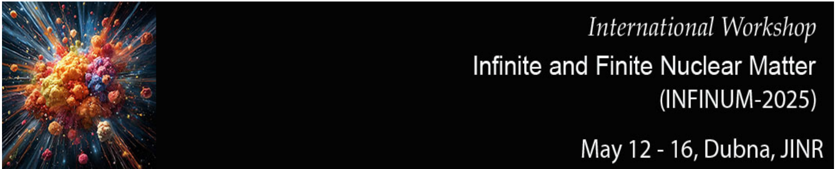

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RSF | Russian Science Foundation

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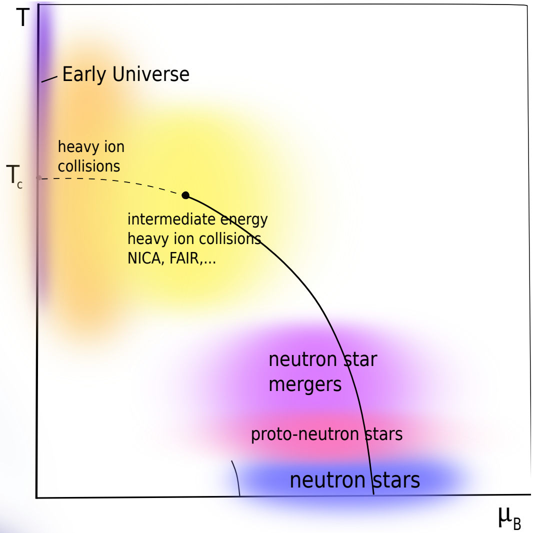
БАЗИС
Фонд развития теоретической физики и математики

QCD Dhase Diagram

QCD at T and μ
(QCD at extreme conditions)

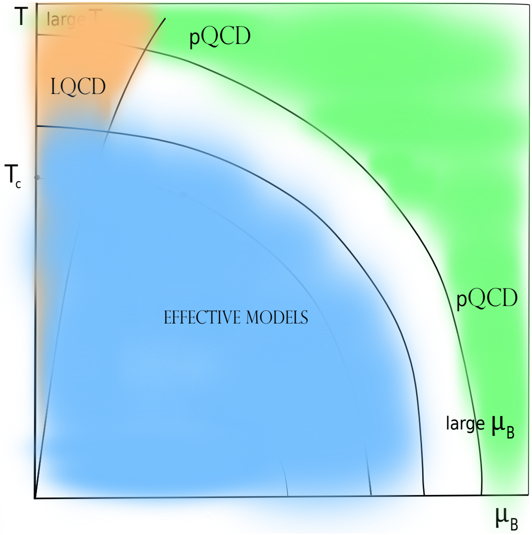
- Early Universe
- heavy ion collisions
- neutron stars
- proto- neutron stars
- neutron star mergers



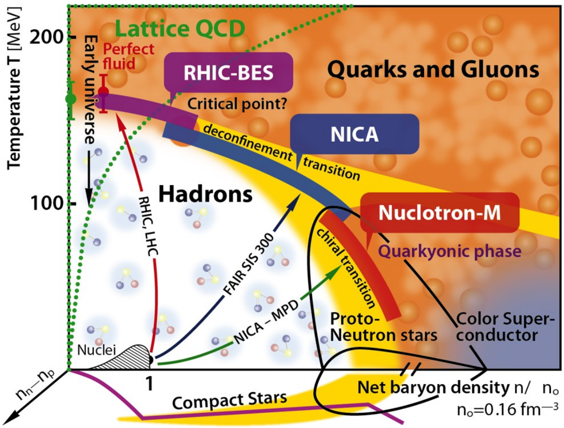
QCD Dhase Diagram and Methods

Methods of dealing with QCD

- Perturbative QCD
- First principle calculation – lattice QCD *(see talk by V. Braguta, A. Roenko and D. Stepanov)*
- Effective models
- DSE, FRG
- Gauge/Gravity duality *(see talk by Nguyen Hoang Vu)*
- .....



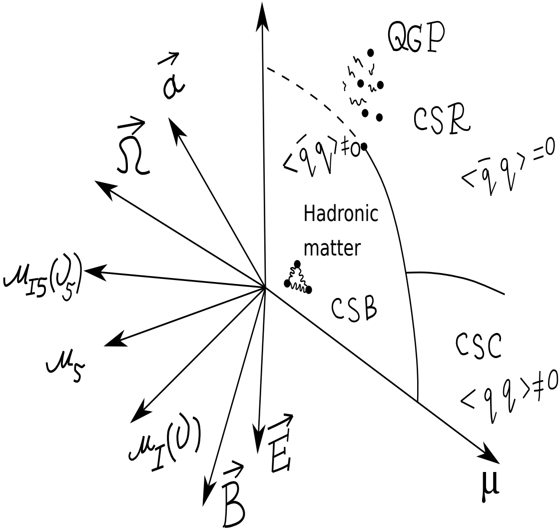
QCD Phase Diagram



More external conditions to QCD

More than just QCD at (μ, T)

- more chemical potentials μ_i
- magnetic fields
- rotation of the system $\vec{\Omega}$ *(see talk by V. Braguta)*
- acceleration $\vec{a}$ *(see talks by V. Zakharov, G. Prokhorov and D. Stepanov)*
- finite size effects (finite volume and boundary conditions)



More external conditions to QCD

- **more chemical potentials** μ_i
- magnetic fields
- rotation of the system $\vec{\Omega}$
  (see talk by V. Braguta)
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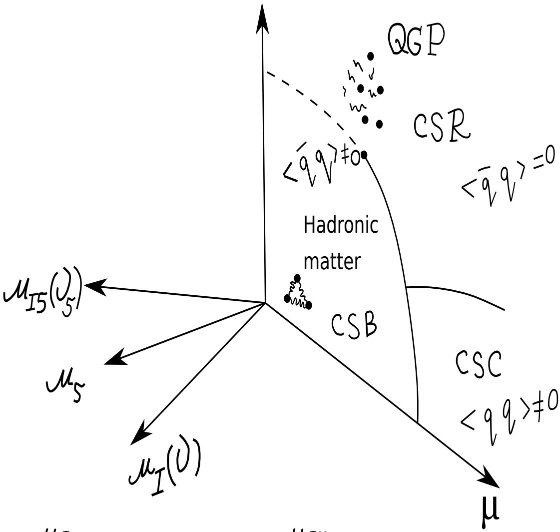


$$\mu = \frac{\mu_B}{3}, \qquad \nu = \frac{\mu_I}{2}, \qquad \mu_5, \qquad \nu_5 = \frac{\mu_{I5}}{2}$$

- **Isotopic chemical potential μ_I**

  Allow to consider systems with isospin imbalance ($n_n \neq n_p$).

- Neutron stars, intermediate energy heavy-ion collisions, neutron star mergers

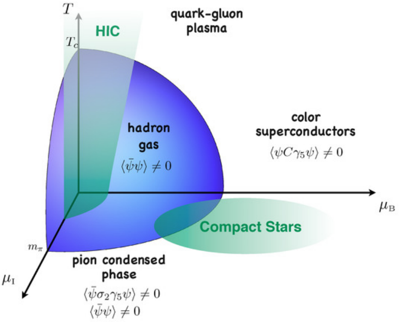


Figure: *taken from Massimo Mannarelli*

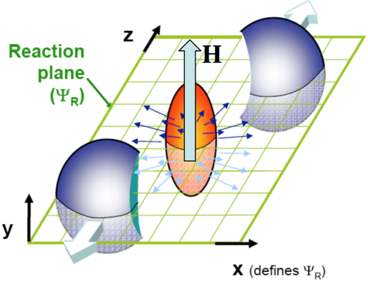
$$\frac{\mu_I}{2}\bar{q}\gamma^0\tau_3 q = \nu\left(\bar{q}\gamma^0\tau_3 q\right) \qquad n_I = n_u - n_d \quad \longleftrightarrow \quad \mu_I = \mu_u - \mu_d$$

Chiral imbalance

- **Chiral (axial) chemical potential**

Allow to consider systems with chiral imbalance (difference between densities of left-handed and right-handed quarks).

$$n_5 = n_R - n_L$$
$$\mu_5 = \mu_R - \mu_L$$

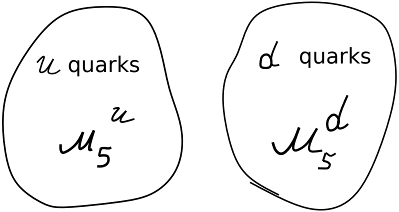


$$\vec{J} \sim \mu_5 \vec{B},$$

The corresponding term in the Lagrangian is

$$\mu_5 \bar{q} \gamma^0 \gamma^5 q$$

Chiral isospin imbalance



$$\mu_5^u \neq \mu_5^d \qquad \text{and} \qquad \mu_{I5} = \mu_5^u - \mu_5^d$$

Term in the Lagrangian — $\frac{\mu_{I5}}{2}\bar{q}\tau_3\gamma^0\gamma^5 q = \nu_5(\bar{q}\tau_3\gamma^0\gamma^5 q)$

$$n_{I5} = n_{u5} - n_{d5}, \qquad n_{I5} \quad \longleftrightarrow \quad \nu_5$$

- **Recalling the dualities of phase diagram**
- **Dualities in QCD and $\mathbf{QC_2D}$ from first principles**
- **Wide swathes of application of dual**
  - **Speed of sound in quark matter with different properties**
  - **Inhomogeneous phase shown in functional approach**

Recall that in NJL model **in** $1/N_c$ **approximation or in the mean field** there have been found **dualities**

(*It is not related to holography or gauge/gravity duality*)

Chiral symmetry breaking $\Longleftrightarrow$ pion condensation

Isospin imbalance $\Longleftrightarrow$ Chiral imbalance

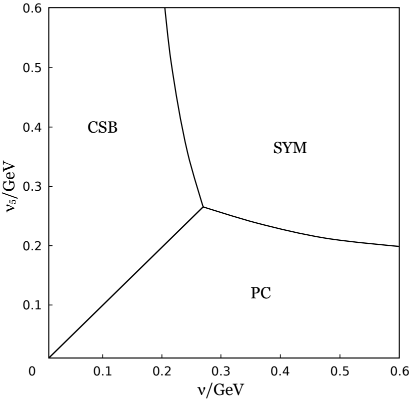
The TDP

$$\Omega(T, \mu, \mu_i, ..., \langle \bar{q}q \rangle, ...) \qquad \Omega(T, \mu, \nu, \nu_5, ..., M, \pi, ...)$$

Duality in phase diagram

The TDP

$$\Omega(T, \mu, \mu_i, ..., \langle \bar{q}q \rangle, ...) \qquad \Omega(T, \mu, \nu, \nu_5, ..., M, \pi, ...)$$



$$\mathcal{D}: \; M \longleftrightarrow \pi, \quad \nu \longleftrightarrow \nu_5$$

Duality between chiral symmetry breaking and pion condensation

$$\text{PC} \longleftrightarrow \text{CSB} \quad \nu \longleftrightarrow \nu_5$$

- A lot of densities and imbalances
  baryon, isospin, chiral, chiral isospin imbalances
- Finite temperature $T \neq 0$
- Physical pion mass $m_\pi \approx 140$ MeV
- Inhomogeneous phases (case)

$$\langle \sigma(x) \rangle = M(x), \quad \langle \pi_\pm(x) \rangle = \pi(x), \quad \langle \pi_3(x) \rangle = 0.$$

- Inclusion of color superconductivity phenomenon

Dualities in QC_2D

Much richer duality picture was found in the phase diagram of **two colour QCD**

$$L_{NJL} = i\overline{\psi}\gamma^\mu\partial_\mu\psi + \overline{\psi}\mathcal{M}\psi +$$

$$G\Big\{(\bar{\psi}\psi)^2 + (i\bar{\psi}\vec{\tau}\gamma^5\psi)^2 + (i\bar{\psi}\sigma_2\tau_2\gamma^5\psi^C)(i\bar{\psi}^C\sigma_2\tau_2\gamma^5\psi)\Big\}$$

where

$$\mathcal{M} = \frac{\mu_B}{3}\overline{\psi}\gamma^0\psi + \frac{\mu_I}{2}\overline{\psi}\gamma^0\tau_3\psi + \frac{\mu_{I5}}{2}\overline{\psi}\gamma^0\gamma^5\tau_3\psi + \mu_5\overline{\psi}\gamma^0\gamma^5\psi$$

$$\sigma(x) = -2H(\bar{q}q), \qquad \Delta(x) = -2H\left[\overline{q^c} i\gamma^5 \sigma_2 \tau_2 q\right]$$

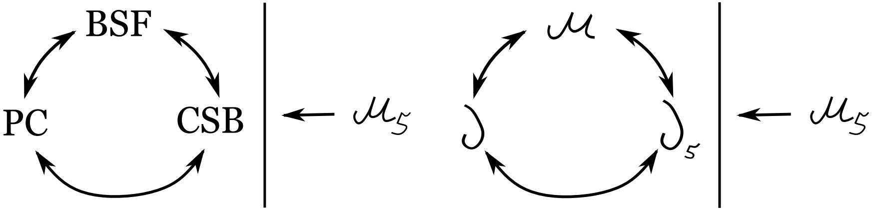
$$\vec{\pi}(x) = -2H(\bar{q}i\gamma^5\vec{\tau}q), \qquad \Delta^*(x) = -2H\left[\bar{q}i\gamma^5 \sigma_2 \tau_2 q^c\right]$$

Condensates and phases

$M = \langle \sigma(x) \rangle \sim \langle \bar{q}q \rangle,$ CSB phase: $M \neq 0,$

$\pi_1 = \langle \pi_1(x) \rangle = \langle \bar{q}\gamma^5\tau_1 q \rangle,$ PC phase: $\pi_1 \neq 0,$

$\Delta = \langle \Delta(x) \rangle = \langle qq \rangle = \langle q^T C\gamma^5 \sigma_2 \tau_2 q \rangle,$ BSF phase: $\Delta \neq 0.$



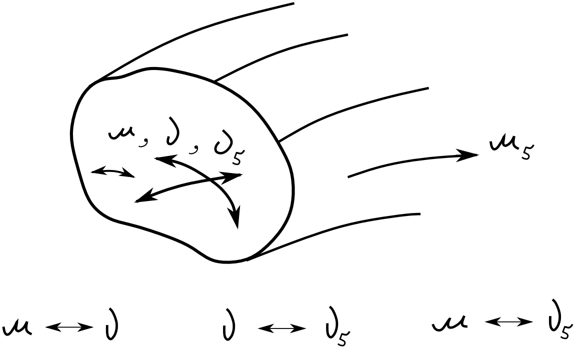
$$\text{(I)} \qquad \mathcal{D}_1: \quad \mu \longleftrightarrow \nu, \quad \pi_1 \longleftrightarrow \Delta, \qquad \text{PC} \longleftrightarrow \text{BSF}$$

$$\text{(II)} \qquad \mathcal{D}_3: \quad \nu \longleftrightarrow \nu_5, \quad M \longleftrightarrow \pi_1, \qquad \text{PC} \longleftrightarrow \text{CSB}$$

$$\text{(III)} \qquad \mathcal{D}_2: \quad \mu \longleftrightarrow \nu_5, \quad M \longleftrightarrow \Delta, \qquad \text{CSB} \longleftrightarrow \text{BSF}$$

The phase diagram of (μ, ν, μ_5, ν_5)

The phase diagram is foliation of dually connected cross-section of (μ, ν, ν_5) along the μ_5 direction



Lagrangian of two colour QCD can be written in the form

$$\mathcal{L} = i\bar{\Psi}\gamma^{\mu}D_{\mu}\Psi$$

where $D_\mu = \partial_\mu + igA_\mu = \partial_\mu + ie\sigma_a A^a_\mu$

$$\Psi^T = \left(\ \psi^u_L,\ \ \psi^d_L,\ \ \sigma_2(\psi^u_R)^C,\ \ \sigma_2(\psi^d_R)^C\ \right)$$

Flavour symmetry is $SU(4)$

Pauli-Gursoy symmetry

Two colour

effective NJL model

$$Q \in so^{+}(4,C), \quad Q^T = -Q$$

$$so^{\pm}(4,C) = \left\{ Q \in so(4,C) : *Q = \pm Q^* \right\}$$

$$A \in SU(4) : \; A^\dagger A = 1$$

$$\rho(A) : \qquad \rho(A)Q = A^T Q A \in so^{+}(4,C)$$

$$N(Q) = \frac{1}{4} tr(Q^\dagger Q), \qquad N\left(\rho(A)Q\right) = N(Q)$$

$N(Q)$ is invariant with respect to $SU(4)$

$$Q = \xi^i \Sigma_i$$

$$\Sigma_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ \Sigma_2 = \begin{pmatrix} \tau_2 & 0 \\ 0 & \tau_2 \end{pmatrix},\ \Sigma_3 = \begin{pmatrix} 0 & i\tau_1 \\ -i\tau_1 & 0 \end{pmatrix},$$

$$\Sigma_4 = \begin{pmatrix} i\tau_2 & 0 \\ 0 & -i\tau_2 \end{pmatrix},\ \Sigma_5 = \begin{pmatrix} 0 & i\tau_2 \\ -i\tau_2 & 0 \end{pmatrix},\ \Sigma_6 = \begin{pmatrix} 0 & i\tau_3 \\ -i\tau_3 & 0 \end{pmatrix}$$

$N(Q)$ is invariant with respect to $SU(4)$

$$\sum_{i=1}^{6} \xi_i'^2 = \sum_{i=1}^{6} \xi_i^2$$

$$SU(4)/Z_2 \approx SO(6)$$

$$\bar{\Psi}^C \sim \Psi^T \to \Psi^T \omega^T$$

$$\omega \in SU(4), \qquad \Psi \to \omega\Psi, \qquad \bar{\Psi}^C \to \bar{\Psi}^C \omega^T$$

$$\bar{\Psi}^C \vec{\Sigma}\, \Psi \to \bar{\Psi}^C \omega^T \vec{\Sigma}\, \omega\, \Psi$$

$$\bar{\Psi}^C \xi^i \Sigma_i\, \Psi \to \bar{\Psi}^C \omega^T \xi^i \Sigma_i\, \omega\, \Psi = \bar{\Psi}^C \xi'^i \Sigma_i \Psi = \bar{\Psi}^C \xi^i \Sigma'_i \Psi$$

$$\xi'^i = \Omega^i_j \xi^j,\ \Omega \in SO(6) \quad \Psi^C \Sigma_i \Psi \to \Omega_{ij} \Psi^C \Sigma_j \Psi,\ \Omega \in SO(6)$$

$$\bar{\Psi}^C \vec{\Sigma} \Psi \in SO(6)$$

$$|\bar{\Psi}^C \vec{\Sigma} \Psi|^2 \quad \text{and} \quad (\bar{\Psi}^C \vec{\Sigma} \Psi)^2 + \ h.\ c.$$

$$\mathcal{L} = i\bar{\Psi}\gamma^{\mu}\partial_{\mu}\Psi + \tilde{G}_1|\bar{\Psi}^C\vec{\Sigma}\Psi|^2$$

$$+\tilde{G}_2\Big[(\bar{\Psi}^C\vec{\Sigma}\Psi)^2 + \ h.\ c.\ \Big]$$

Symmetric under $SU(4)$

$$\frac{\mu_B}{3}\overline{\psi}\gamma^0\psi + \frac{\mu_I}{2}\overline{\psi}\gamma^0\tau_3\psi + \frac{\mu_{I5}}{2}\overline{\psi}\gamma^0\gamma^5\tau_3\psi + \mu_5\overline{\psi}\gamma^0\gamma^5\psi$$

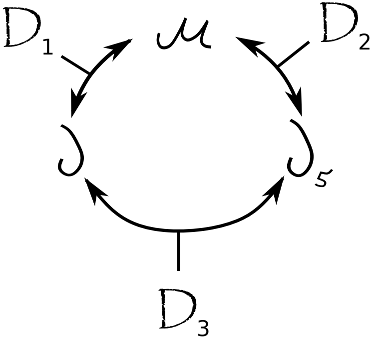
$$\mathcal{M} = \mu \Psi^\dagger \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Psi + \frac{\mu_I}{2} \Psi^\dagger \begin{pmatrix} \tau_3 & 0 \\ 0 & -\tau_3 \end{pmatrix} \Psi +$$

$$\frac{\mu_{I5}}{2} \Psi^\dagger \begin{pmatrix} \tau_3 & 0 \\ 0 & \tau_3 \end{pmatrix} \Psi + \mu_5 \Psi^\dagger \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Psi$$

$$\mu \Psi^\dagger \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Psi \qquad \longleftrightarrow \qquad \frac{\mu_I}{2} \Psi^\dagger \begin{pmatrix} \tau_3 & 0 \\ 0 & -\tau_3 \end{pmatrix} \Psi$$

$$\mathcal{D}_{\mathrm{I}} : \qquad \begin{pmatrix} \psi_L^d \\ \sigma_2 (\psi_R^C)^d \end{pmatrix} \to i\tau_1 \begin{pmatrix} \psi_L^d \\ \sigma_2 (\psi_R^C)^d \end{pmatrix} \qquad \mu \leftrightarrow \nu$$

Duality structure in QC_2D



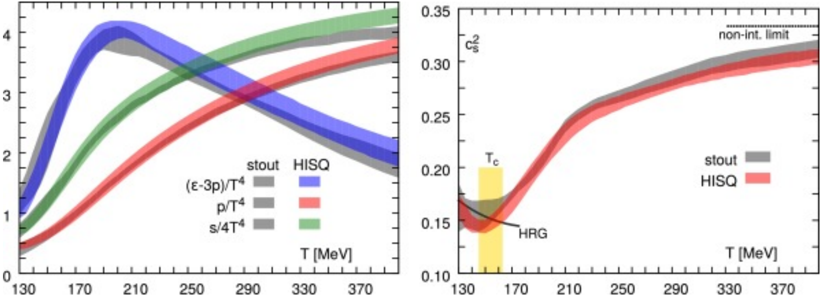
$\boxed{D_1: \; \mu \longleftrightarrow \nu}$ $\boxed{D_2: \; \mu \longleftrightarrow \nu_5}$ $\boxed{D_3: \; \nu \longleftrightarrow \nu_5}$

$$\mathcal{D}_{\mathrm{II}}: \qquad \langle\bar{\psi}\psi\rangle \longleftrightarrow \langle i\bar{\psi}\gamma^5\tau_1\psi\rangle, \quad M \longleftrightarrow \pi, \quad \nu \leftrightarrow \nu_5$$

From first principles

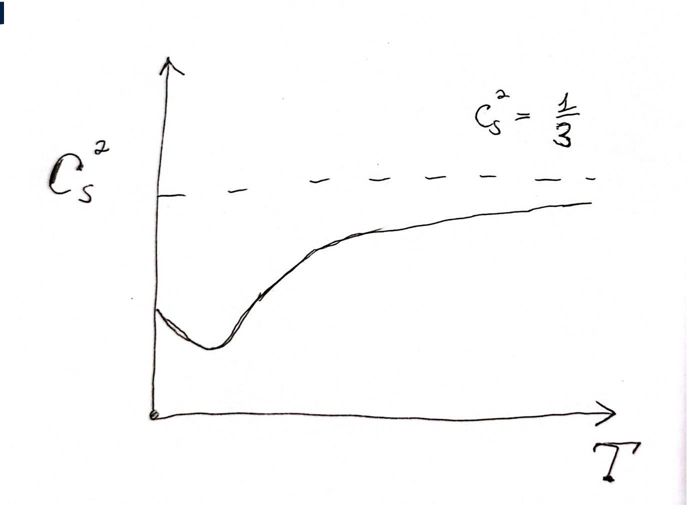
Speed of sound c_s^2

Thermodynamic properties could be calculated in lattice QCD



A. Bazavov et al. [HotQCD], Phys. Rev. D ***90*** *(2014), 094503*

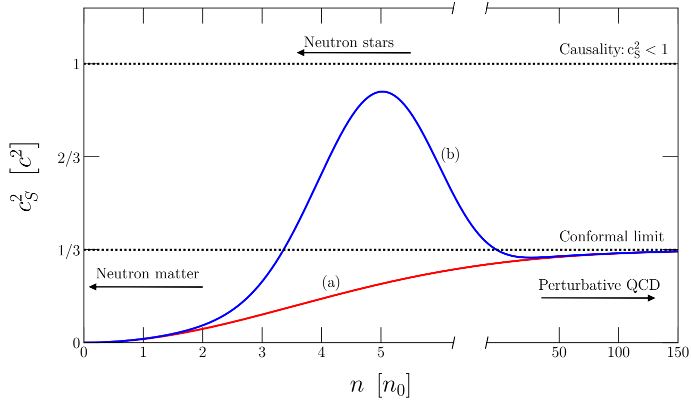
Sound speed in QCD at finite T: skematic



There was discussed bound from holography

A. Cherman, T. D. Cohen and A. Nellore, Phys. Rev. D 80 (2009), 066003

Two possible scenario of **speed of sound** at **non-zero baryon density**

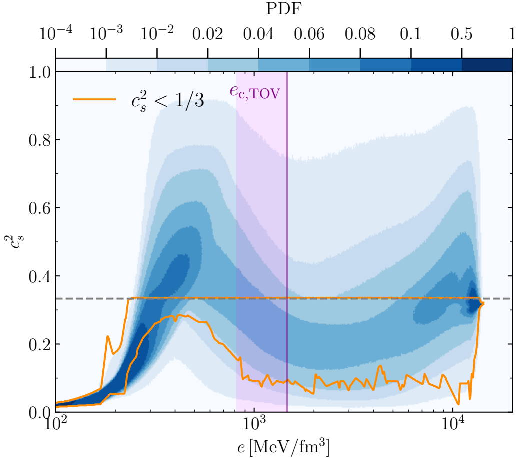


taken from S. Reddy et al, Astrophys. J. ***860*** *(2018) no.2, 149*

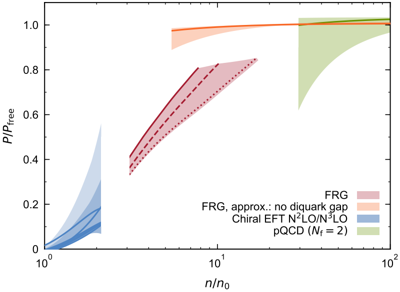
EOS with continuous c_s^2 consistent not only with nuclear theory and perturbative QCD, but also with astrophysical observations.

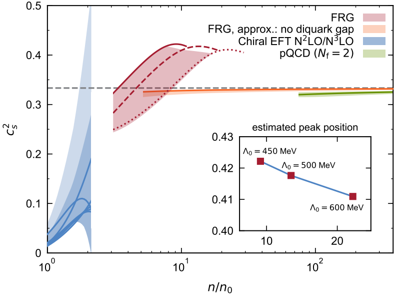
EOS with sub-conformal sound speeds, i.e.,$c_s^2 < 1/3$ are **possible in principle but very unlikely in practice**

L. Rezzolla et al, Astrophys.J.Lett. 939 (2022) 2, L34



Sound speed in FRG approach





- **Sound speed** squared has been obtained from **FRG approach**

Phys.Rev.Lett. 125 (2020) 14, 142502

$$Z = \int D[\text{gluons}]\, D[\text{quarks}]\, e^{-S^{E}_{QCD}}$$

$$Z = \int D[\text{gluons}]\, \mathrm{Det}\, \mathbb{D}(\mu)\, e^{-S^{E}_{\text{gluons}}}$$

It is well known that **at non-zero baryon chemical potential μ_B lattice simulation** is quite challenging due to the **sign problem**
complex determinant

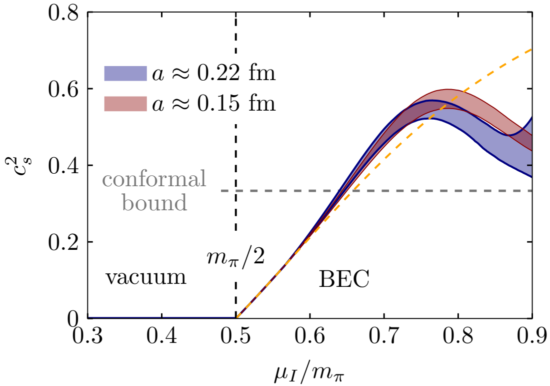
$$Det(D(\mu))^{\dagger} = Det(D(-\mu))$$

For isospin chemical potential μ_I

$$Det(D(\mu_I))^{\dagger} = Det(D(\mu_I))$$

- **Sound speed** squared has been obtained from **lattice QCD simulations** for **QCD with non-zero isospin** μ_I

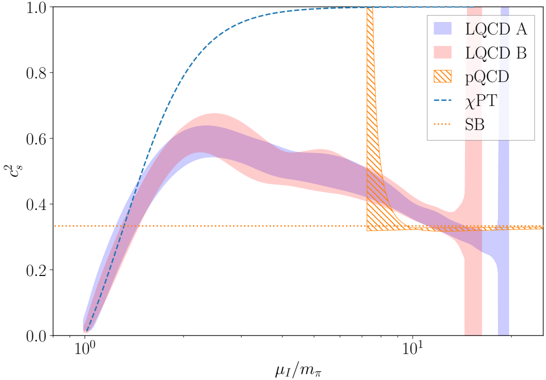
  B. B. Brandt, F. Cuteri and G. Endrodi, JHEP 07, 055 (2023)



Sound speed in QCD with non-zero isospin density

- **Sound speed** squared has been obtained from **lattice QCD simulations** for **QCD with non-zero isospin** μ_I for values of μ_I up to $10 m_\pi$

  R. Abbott et al. [NPLQCD], Phys. Rev. D 108, no.11, 114506 (2023)



Duality between chiral symmetry breaking and pion condensation

$$\mathcal{D}: \;\; M \longleftrightarrow \pi, \quad \nu \longleftrightarrow \nu_5$$

The TDP of the quark matter

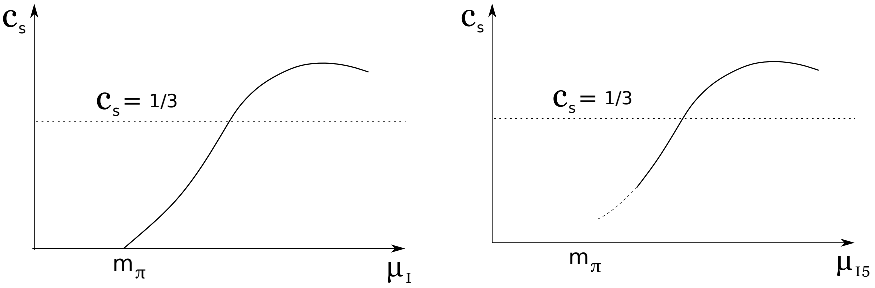
$$\Omega(T,\, \mu,\, \nu,\, \nu_5,\, \mu_5,\, \mid M,\, \pi) = \mathrm{inv}$$

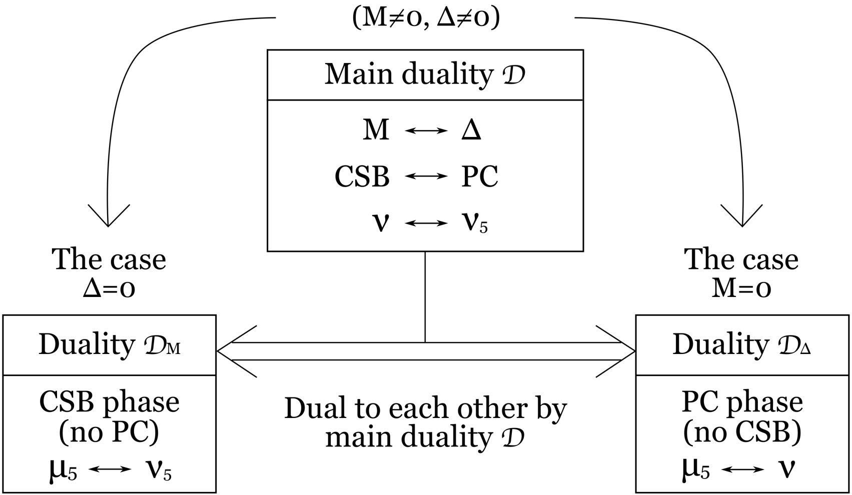
The speed of sound $\qquad c_s^2 = \dfrac{dp}{d\epsilon}$

$$\Omega(T, ...) \Longrightarrow c_s^2(T, ...)$$

The speed of sound $\quad c_s^2 = \dfrac{dp}{d\epsilon}, \qquad \Omega(T, ...) \Longrightarrow c_s^2(T, ...)$

$$\Omega(T, ..., \nu) = \Omega(T, ..., \nu_5) \Longrightarrow c_s^2(T, ..., \nu) = c_s^2(T, ..., \nu_5)$$

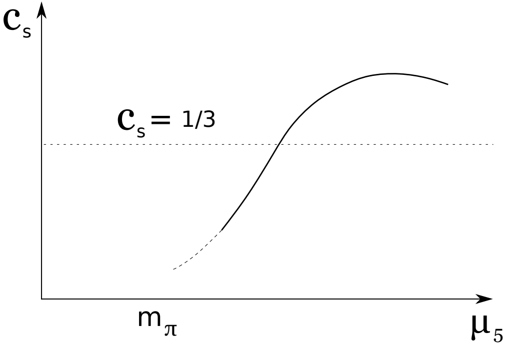



(M≠0, Δ≠0)
Main duality $\mathcal{D}$
M ⟷ Δ
CSB ⟷ PC
$\nu \longleftrightarrow \nu_5$
The case Δ=0
The case M=0
Duality $\mathcal{D}_M$
CSB phase (no PC)
$\mu_5 \longleftrightarrow \nu_5$
Dual to each other by main duality $\mathcal{D}$
Duality $\mathcal{D}_\Delta$
PC phase (no CSB)
$\mu_5 \longleftrightarrow \nu$

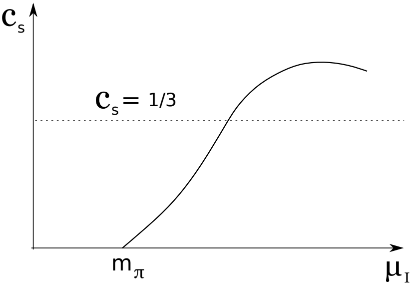
Duality

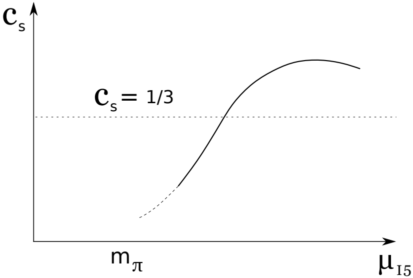
$$\nu_5 \longleftrightarrow \mu_5, \qquad M \neq 0, \quad \langle \pi \rangle = \langle \Delta \rangle = 0$$

- **Sound speed** squared for **QCD with non-zero chiral imbalance** μ_5 **only in the framework of effective model**

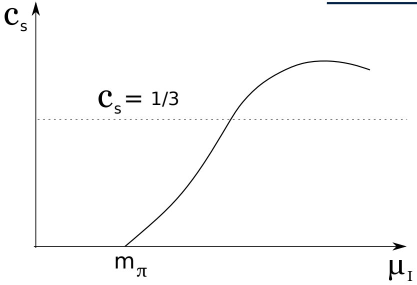


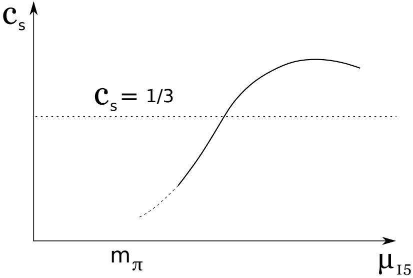
Speed of sound in QCD: First principles

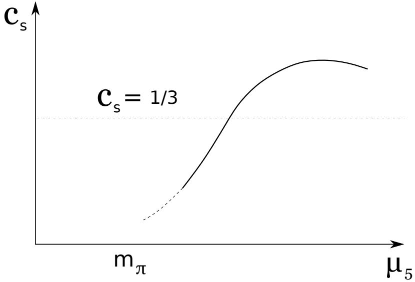





C_s^2
$C_s = \frac{1}{3}$
μ


c_s
$c_s = 1/3$
m_π
μ_I


c_s
$c_s = 1/3$
m_π
μ_{I5}


c_s
$c_s = 1/3$
m_π
μ_5

Two colour QCD case

QC_2D

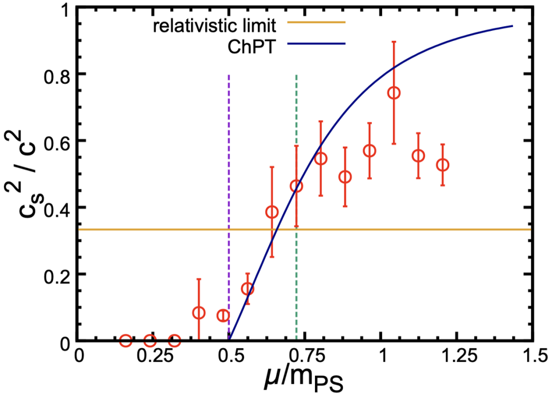
No sign problem in SU(2) case at $\mu_B \neq 0$

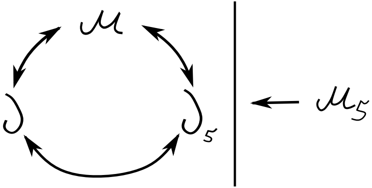
$$(Det(D(\mu)))^\dagger = Det(D(\mu))$$

Sound speed in two color QCD

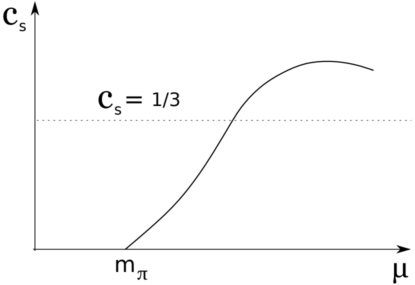
- **Sound speed** squared has been obtained from **lattice QCD simulations** for **two color QCD**

  E. Itou and K. Iida, PoS LATTICE2023, 111 (2024); PTEP 2022 (2022) no.11, 111B01

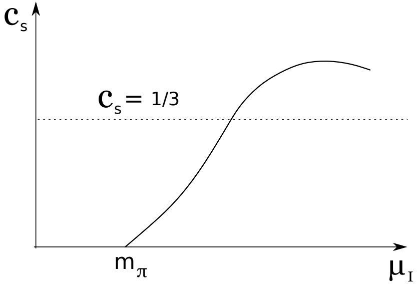




μ
μ_5

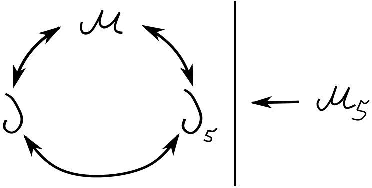


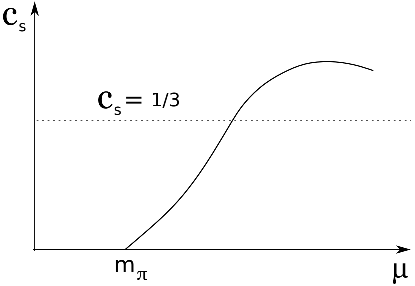
c_s
$c_s = 1/3$
m_π
μ

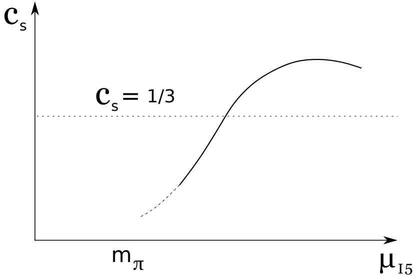


c_s
$c_s = 1/3$
m_π
μ_I

Duality structure in QC_2D



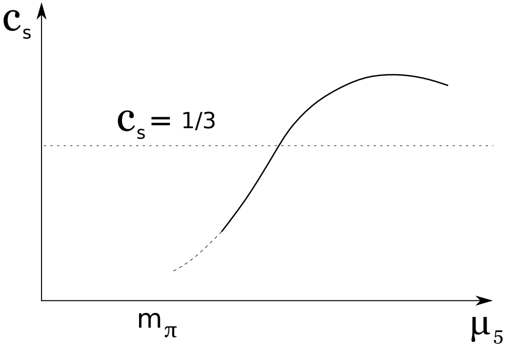




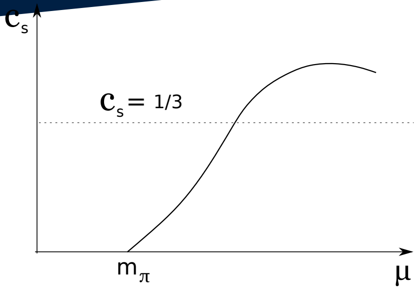
$$\text{Duality} \quad \nu_5 \longleftrightarrow \mu_5$$

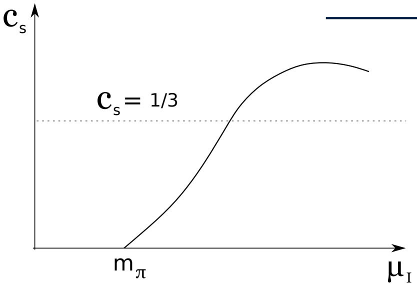
was shown in two color effective model as well

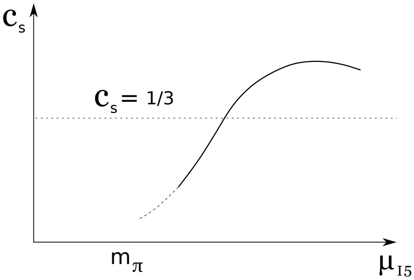
- **Sound speed** squared for **QCD with non-zero chiral imbalance** μ_5 **only in the framework of effective model**

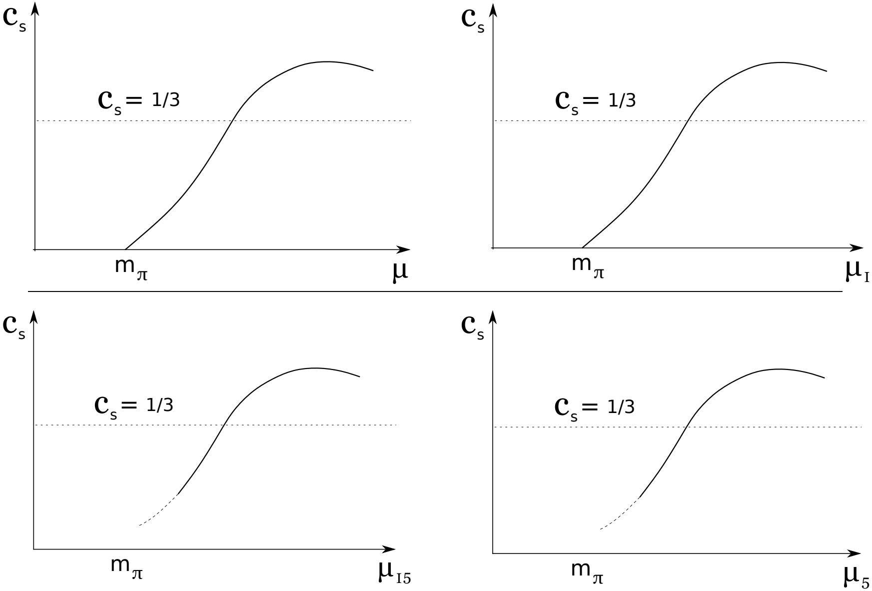


Speed of sound in QC$_2$D: First principle



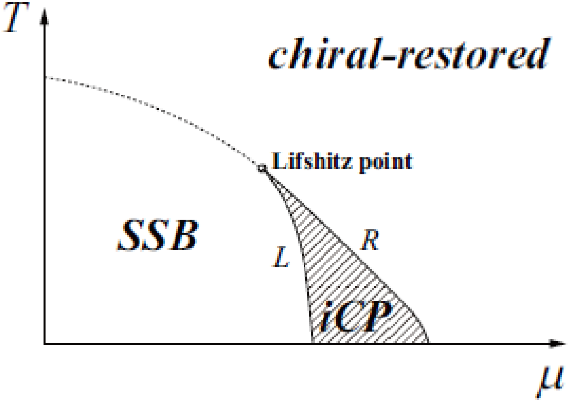





c_s
$c_s = 1/3$
m_π
μ
c_s
$c_s = 1/3$
m_π
μ_I
c_s
$c_s = 1/3$
m_π
μ_{I5}
c_s
$c_s = 1/3$
m_π
μ_5

Inhomogeneous phases in QCD and QC_2D

It is open question if there is inhomogeneous chiral symmetry breaking phase at $\mu_B \neq 0$



$$\langle \bar{q}q \rangle \sim M(x)$$

- **Inhomogeneous phase was predicted in: (1+1)-dimensional Gross-Neveu (GN) model**

  M. Thies, ...

  A. Wipf, M. Wagner, M. Winstel, L. Pannullo etc.

- **Inhomogeneous phase in (3+1)-dimensional effective models**

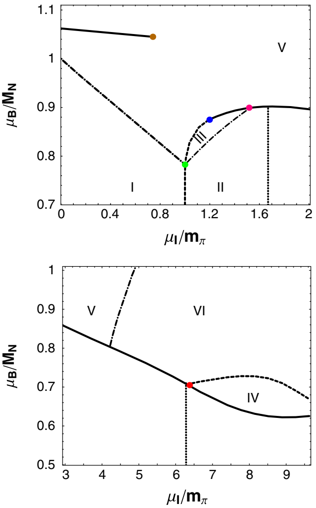
- **Inhomogeneous phase in effective models: dependence on the chosen regularization scheme**

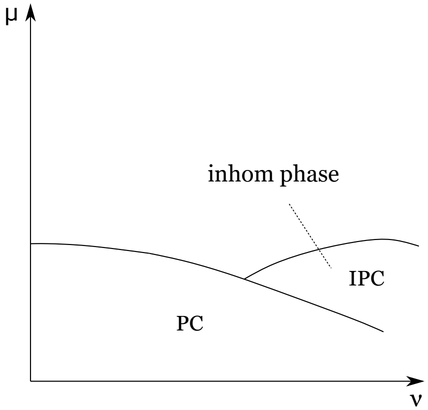
  M. Wagner et al, Phys. Rev. D 110 (2024) 7, 076006

- **Inhomogeneous phase shown in functional approach**

  C. Fischer et al, Phys. Rev. D 108 (2023) 11, 114019, Phys.Rev.D 110 (2024) 7, 074014

Inhomogeneous PC at μ_I: skematic

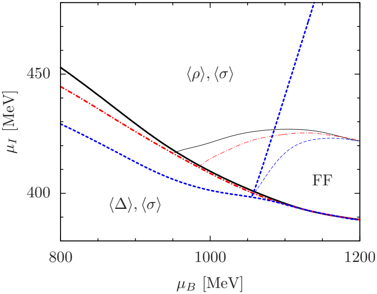




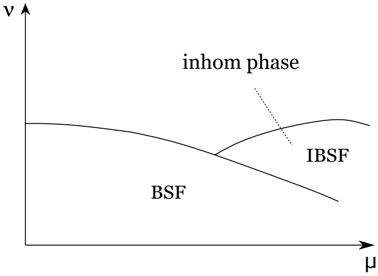
Lianyi He et al, Phys.Rev.D 82 (2010) 056006

Inhomogeneous phase at μ_I: skematic

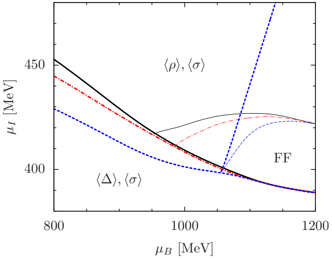
Inhomogeneous diquark condensation found in two color case in the framework of effective models

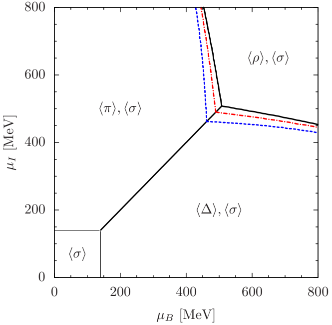


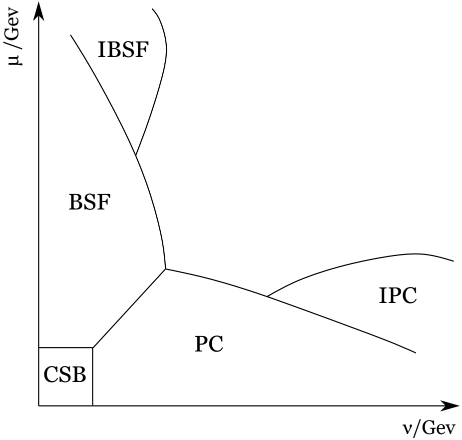
J. Andersen et al Phys.Rev.D 81 (2010) 096004



Inhomogeneous phase at μ_I: skematic





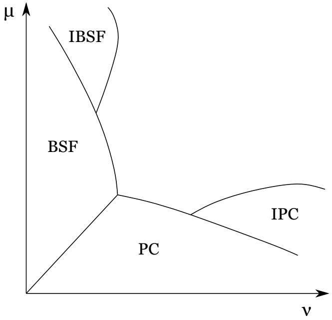


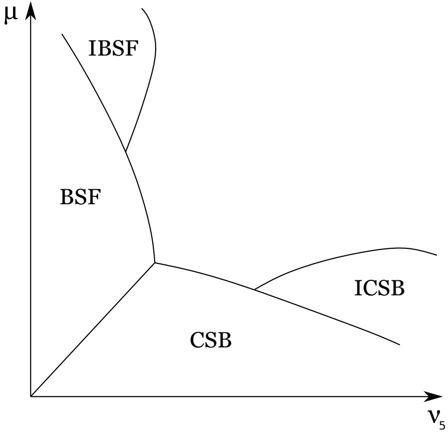
$$\mu \longleftrightarrow \nu, \qquad \pi_1 \longleftrightarrow \Delta$$

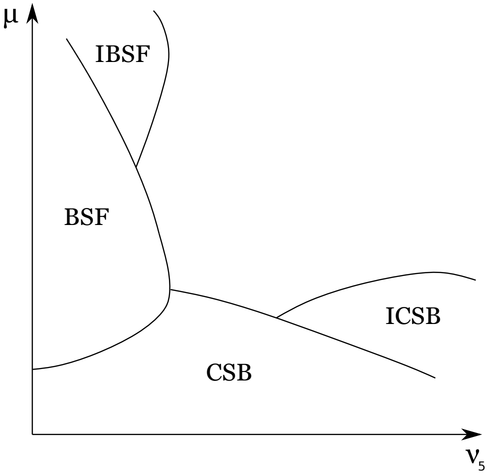
$$\text{PC} \longleftrightarrow \text{BSF}, \qquad \text{IPC} \longleftrightarrow \text{IBSF}$$

Inhomogeneous phase: skematic

$$\nu \longleftrightarrow \nu_5, \qquad M \longleftrightarrow \pi_1, \qquad \text{CSB} \longleftrightarrow \text{PC}, \qquad \text{ICSB} \longleftrightarrow \text{IPC}$$







$$\mu \longleftrightarrow \nu_5, \quad M \longleftrightarrow \Delta, \qquad \text{CSB} \longleftrightarrow \text{BSF}, \qquad \text{ICSB} \longleftrightarrow \text{IBSF}$$

Inhomogeneous phases

Homogeneous case

$$\langle \sigma(x) \rangle = M, \quad \langle \pi_{\pm}(x) \rangle = \pi, \quad \langle \pi_3(x) \rangle = 0.$$

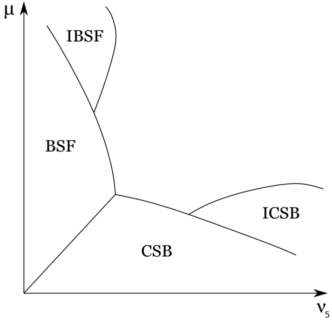
Inhomogeneous phases (three color case)

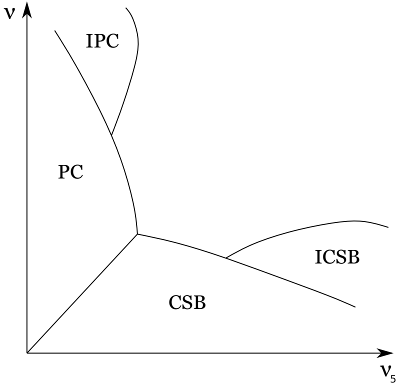
$$\langle \sigma(x) \rangle = M(x), \quad \langle \pi_{\pm}(x) \rangle = \pi(x), \quad \langle \pi_3(x) \rangle = 0.$$

$$\mathcal{D}: \; M(x) \longleftrightarrow \pi(x), \quad \nu \longleftrightarrow \nu_5$$

$$\text{ICSB} \longleftrightarrow \text{IPC} \quad \nu \longleftrightarrow \nu_5$$

$$\mu \longleftrightarrow \nu, \quad \pi_1 \longleftrightarrow \Delta, \qquad \mathrm{PC} \longleftrightarrow \mathrm{BSF}, \qquad \mathrm{IPC} \longleftrightarrow \mathrm{IBSF}$$

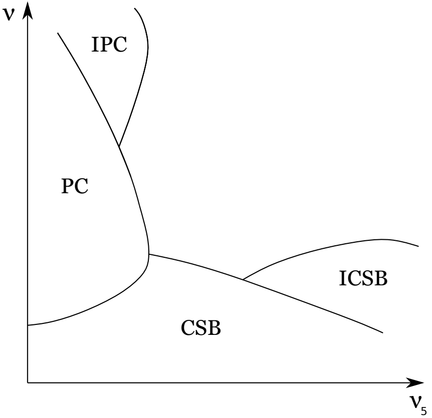




Inhomogeneous phase at zero μ_B: skematic

Inhomogeneous phases exist usually at $\mu_B \neq 0$

Inhomogeneous phase in two color case exist at $\mu_B = 0$



$$\nu \longleftrightarrow \nu_5, \qquad M \longleftrightarrow \pi_1, \qquad \text{CSB} \longleftrightarrow \text{PC}, \qquad \text{ICSB} \longleftrightarrow \text{IPC}$$

Dualities has been proven from first principles

Speed of sound exceeding the conformal limit is rather **natural** and taking place in a lot of systems, **with various chemical potentials**

And it is natural if it has similar structure in QCD at non-zero baryon density, the most interesting case

Inhomogeneous phases in two and three color case have been studied, in two color case exist at $\mu_B = 0$