Вычисления характеристик неклассичности кудитов

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ЛИТ, ОИЯИ

11 февраля, 2025

Содержание

Основная идея:

(A) Классическая механика в фазовом пространстве;

(B) Квантовая механика в фазовом пространстве;

(C) Отклонения (A) - (B) $:=$ "неклассичность".

Physical motivation

Classically, a particle in one dimension with its position q and momentum p is described by a phase space distribution $P_{Cl}(q,p)$. The average of a function of the position and momentum $A(q,p)$ can then be expressed as

$$\langle A\rangle_{Cl} = \int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} dp\, A(q,p)\, P_{Cl}(q,p)\,.$$

A **quantum mechanical** particle is described by a density matrix $\hat{\varrho}$, and the average of a function of the position and momentum operators $\hat{A}(\hat{q},\hat{p})$ is

$$\langle A\rangle_{QM} = \mathrm{tr}\left(\hat{A}\,\hat{\varrho}\right)\,.$$

A quantum mechanical average can be expressed using a quasiprobability distribution $P_{QM}(q,p)$ as

$$\langle A\rangle_{QM} = \int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} dp\, A(q,p)\, P_{QM}(q,p)\,.$$

Objective

Because of Heisenberg's uncertainty principle, the function $P_{\boldsymbol{QM}}(q,p)$ has **negative values** for certain quantum states. Hence, it is not a true probability density and is referred to as a quasiprobability distribution.

Due to this negativity property, quasiprobability distributions may serve as a tool for understanding the interrelations between quantum and classical statistical descriptions.

Aim:

To consider the Wigner quasiprobability distribution $W(q,p)$ and, specifying the notion of "classical states" as the states whose Wigner function is non-negative everywhere in the phase space, to quantify a state classicality.

1. A. Khvedelidze, A. Torosyan, *Comparing classicality of qutrits from Hilbert-Schmidt, Bures and Bogoliubov-Kubo-Mori ensembles*, Zap. Nauch. Sem. POMI 517, 250-267, 2022

Global indicator of classicality

Quantum state space $\mathfrak{P}_N = \{X \in M_N(\mathbb{C}) \mid X = X^\dagger\,, \quad X \geq 0\,, \quad \operatorname{tr}(X) = 1\}\,.$

Dual space $\mathfrak{P}_N^* = \{X \in M_N(\mathbb{C}) \mid X = X^\dagger\,, \quad \operatorname{tr}(X) = 1\,, \quad \operatorname{tr}\left(X^2\right) = N\}\,.$

Global indicator of classicality:

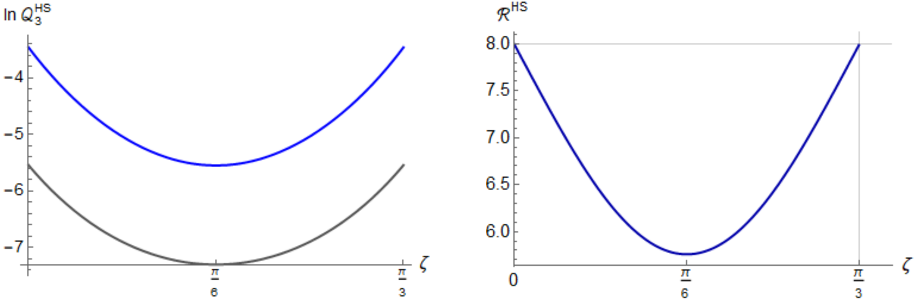
$$\mathcal{Q}_N = \frac{\text{Volume of orbit subspace } \mathcal{O}[\mathfrak{P}_N^{(+)}]}{\text{Volume of orbit space } \mathcal{O}[\mathfrak{P}_N]}\,,$$

where $\mathcal{O}[\mathfrak{P}_N^{(+)}]$ is the unitary orbit space of states with non-negative Wigner quasiprobability distribution $W(\Omega_N) = \operatorname{tr}[\varrho\, \Delta(\Omega_N)]$:

- density matrix $\varrho \in \mathfrak{P}_N$;
- Stratonovich-Weyl kernel $\Delta(\Omega_N) \in \mathfrak{P}_N^*$.

$\mathcal{Q}_3$−indicator for Hilbert-Schmidt ensemble of qutrits

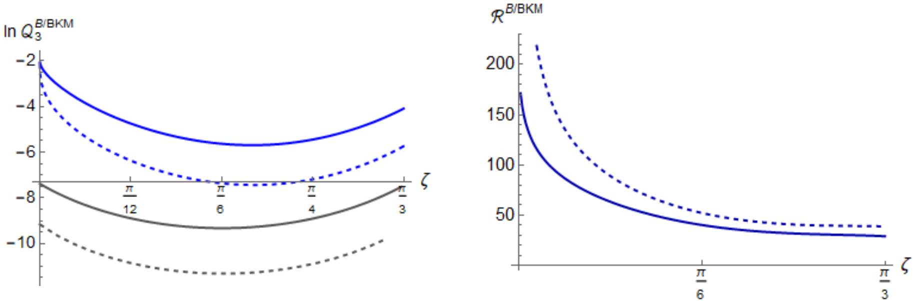
- from regular stratum: $\mathcal{Q}^{\mathrm{HS}}_{[T^3]}(\zeta) = \frac{20\cos^2(\zeta-\pi/6)+1}{128(4\cos^2(\zeta-\pi/6)-1)^5}$
- from degenerate stratum: $\mathcal{Q}^{\mathrm{HS}}_{[S(U(2)\times U(1))]}(\zeta) = \frac{1}{1056}\left(\csc^5\left(\zeta+\frac{\pi}{6}\right)+\sec^5(\zeta)\right)$



(a) $\mathcal{Q}_3$−indicators of a H-S qutrit as functions of ζ for the regular (gray curve) and degenerate (blue curve) strata. The absolute minimum of both indicators is attained at $\zeta = \pi/6$. (b) The ratio of degenerate to regular $\mathcal{Q}_3$−indicators.

The ratio $\mathrm{R}^{\mathrm{HS}}(\zeta) = \mathcal{Q}^{\mathrm{HS}}_{[S(U(2)\times U(1))]}(\zeta)/\mathcal{Q}^{\mathrm{HS}}_{[T^3]}(\zeta)$ is a certain measure of relation between the symmetry of a state and its classicality.

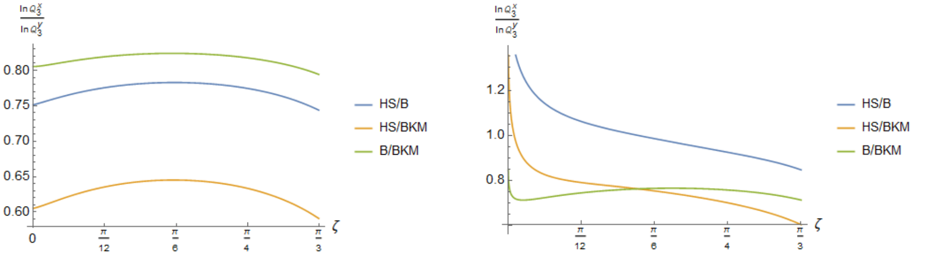
$\mathcal{Q}_3$−indicator for Bures and BKM ensemble of qutrits



(a) The plot of $\mathcal{Q}_\mathbf{3}$ for the Bures (solid curves) and BKM (dashed curves) ensembles of qutrits from the regular (gray curves) and degenerate (blue curves) strata. (b) The ratio R of degenerate to regular $\mathcal{Q}_\mathbf{3}$−indicators for the Bures (solid blue) and the BKM (dashed blue) ensembles.

Summary

- There is a certain coherence between the classification of states according to their classicality and their symmetry properties. In particular, it turns out that the states with a “larger” symmetry are more classical.
- The character of the dependence of $\mathcal{Q}_3$ on the type of the ensemble is monotone, i.e., the values of $\mathcal{Q}_3$ for all strata are ordered in correspondence with the order of the ensembles.

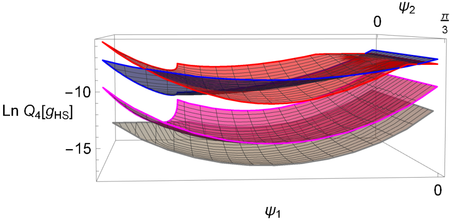


Pairwise ratios of $\mathcal{Q}_3$ −indicators of different ensembles for the regular (a) and for the degenerate (b) stratum.

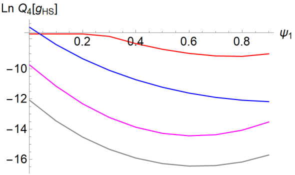
2. A. Khvedelidze, A. Torosyan, *On the hierarchy of classicality and symmetry of quantum states*, Zap. Nauch. Sem. POMI 528, 238-260, 2023

$\mathcal{Q}$-indicators of a qubit-qubit system

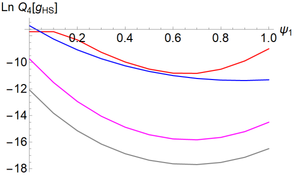
$\mathcal{Q}_4$−indicators for strata of different isotropy types: $\mathcal{Q}_{[S(U(3)\times U(1))]}$; $\mathcal{Q}_{[U(1)\times SU(2)\times U(1)]}$; $\mathcal{Q}_{[SU(2)\times U(1)^2]}$; and regular $\mathcal{Q}_{[T^4]}$-indicator.



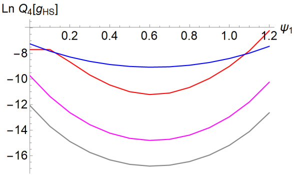
Slices of global indicators of classicality for different types of orbits of a qubit-qubit system for Hilbert-Schmidt metric:



$\psi_2 = 0$



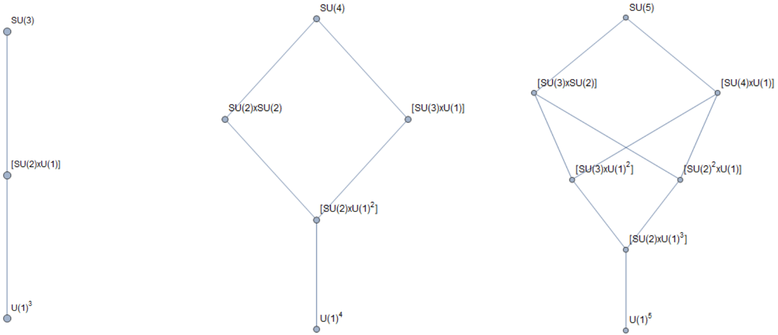
$\psi_2 = \pi/6$



$\psi_2 = \pi/3$

Results

The classicality indicators of qutrit and quatrit for the regular and degenerate strata respect the order of the corresponding isotropy groups in agreement with their Hasse diagram for partially ordered subgroups of unitary groups: $\mathcal{Q}_N[T_N] < \mathcal{Q}_N[H_1] < \cdots < \mathcal{Q}_N[SU(N)] = 1$.



$N = 3$ $N = 4$ $N = 5$

Hasse diagram as a graphical representation of the relation of elements of a partially ordered set with an implied upward orientation.

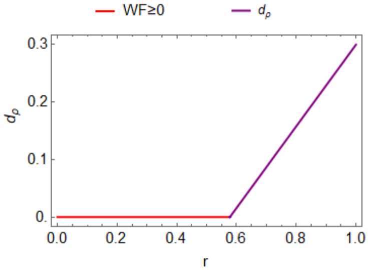
3. A. Khvedelidze, A. Torosyan, *On the nonclassicality distance indicator of qudits*, Phys. Part. Nuclei 55 (3), 591-593, 2024

Nonclassicality distance is based on a **distance** of a state from the "classical states" $\mathfrak{P}_N^{(+)}$ (states with positive Wigner functions):

$$d_\varrho = \inf_{x \in \mathfrak{P}_N^{(+)}} D(\varrho, x) = \sqrt{\inf_{x_{diag} \in \mathcal{O}[\mathfrak{P}_N^{(+)}]} \sum_{i=1}^{N} (r_i - x_i)^2} \,.$$

Qubit nonclassicality distance for Hilbert-Schmidt metric:

$$d_\varrho = \theta[\mathrm{r} - \frac{1}{\sqrt{3}}] \left(\frac{\mathrm{r}}{\sqrt{2}} - \frac{1}{\sqrt{6}} \right) .$$

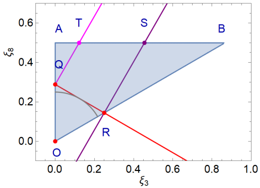


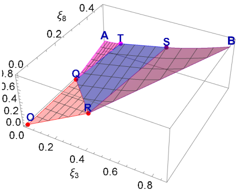
Qutrit nonclassicality distance

Qutrit nonclassicality distance for Hilbert-Schmidt metric:

$$d_\varrho = \begin{cases} 0\,, & \text{if} \quad \xi_3, \xi_8 \in \triangle OQR\,, \\ \sqrt{\xi_3^2 + \left(\xi_8 - \frac{1}{4\cos\left(\zeta - \frac{\pi}{3}\right)}\right)^2}\,, & \text{if} \quad \xi_3, \xi_8 \in \triangle AQT\,, \\ \xi_3 \cos\left(\zeta + \frac{\pi}{6}\right) + \xi_8 \sin\left(\zeta + \frac{\pi}{6}\right) - \frac{1}{4}\,, & \text{if} \quad \xi_3, \xi_8 \in \square QRST\,, \\ \sqrt{\left(\xi_3 - \frac{\sqrt{3}}{8}\sec(\zeta)\right)^2 + \left(\xi_8 - \frac{\sec(\zeta)}{8}\right)^2}\,, & \text{if} \quad \xi_3, \xi_8 \in \triangle BRS\,. \end{cases}$$

Qutrit $\underline{\Delta}_2$-simplex with WF positivity boundary and nonclassicality distance ($\zeta = \frac{\pi}{6}$):





4. A. Khvedelidze, D. Mladenov, A. Torosyan, *One other parameterization of SU(4) group*, Preprint, 2024

From group to coset

Equivalence relations. Let g and $g' \in G$ be elements of the Lie group, G and $K \subset G \supset H$. Then the equivalence relations

$$g \sim g' \,, \quad \text{if} \quad g' = g_1 g g_2^{-1} \,, \quad g_1 \in K \,, \quad g_2 \in H$$

define the double coset $K\backslash G/H$.

Coordinatizing double coset

Starting with the coordinates on the group manifold G and restricting to an appropriate subset, to describe $K\backslash G/H$.

Task: Find a parameterization of G adapted to its subgroups structure resulting in effective description of the corresponding double coset.

KAT -decomposition of $SU(4)$

The decomposition of $\mathfrak{su}(4) = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{a}' \oplus \mathfrak{h}$, where $\mathfrak{h}$ is Cartan subalgebra of $\mathfrak{su}(4)$, $\mathfrak{k} := \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, and components $\mathfrak{a}$ and $\mathfrak{a}'$ are 3-dimensional Abelian subalgebras, such that

$$[\mathfrak{a}', \mathfrak{a}] \subseteq \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}, \quad [\mathfrak{h}, \mathfrak{a}] \subseteq \mathfrak{k}, \quad [\mathfrak{h}, \mathfrak{a}'] \subseteq \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{h}] \subseteq \mathfrak{a} \oplus \mathfrak{a}',$$

determines the canonical coordinates in the exponential map $\exp : \mathfrak{su}(4) \to \mathrm{SU}(4)$, of the following form:

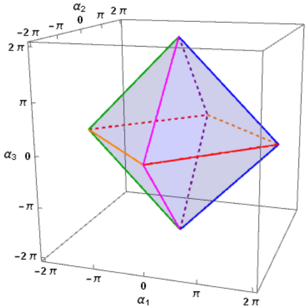
$$g := K\,\mathcal{A}\,T^3, \qquad g \in SU(4),$$

where $K := \exp(\mathfrak{k}) = \exp(\mathfrak{su}(2)) \times \exp(\mathfrak{su}(2))$, T^3 is the maximal torus in $SU(4)$, and factor $\mathcal{A}$ is product of two conjugated copies of the maximal Abelian subgroup of $SU(4)$: $\mathcal{A} := \exp(\mathfrak{a}) \exp(\mathfrak{a}')$.

Analyticity domain of $\mathcal{A}$-factor

$\mathcal{A}$-factor is identified to $\mathbb{R}^6$ by introducing coordinates α_i and β_i:

$$\mathcal{A} = \mathcal{A}_1\mathcal{A}_2 \quad \text{with} \quad \mathcal{A}_1 = \exp(\boldsymbol{\alpha}, \boldsymbol{\Lambda}_1)\,, \ \mathcal{A}_2 = \exp(\boldsymbol{\beta}, \boldsymbol{\Lambda}_2)\,,$$
$$\boldsymbol{\Lambda}_1 = \{\lambda_1, \lambda_4, \lambda_7\}\,, \quad \boldsymbol{\Lambda}_2 = \{\lambda_9, \lambda_{11}, \lambda_{13}\}\,.$$



The regular octahedron as the domain of the analiticity of $\mathcal{A}_1$-factor in KAT-decomposition.

5. A. Khvedelidze, D. Mladenov, A. Torosyan, *Towards parameterizing the entanglement body of a qubit pair*, Preprint, 2024

Describing 2-qubit entanglement space

Task: to describe the entanglement space $\mathcal{E} = \mathfrak{P}_4/SU(2) \times SU(2)$ of 2-qubits in terms of two octahedron coordinates and simplex of the density matrix eigenvalues.

Local structure of 2-qubit entanglement body. In the vicinity of a state ϱ consisting of maximal rank states with a non-degenerate spectrum, the entanglement space admits representation:

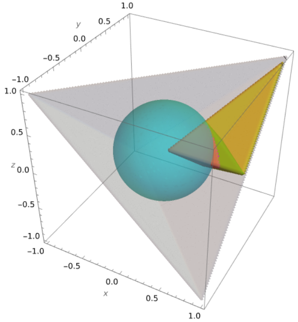
$$\mathcal{E}_{2\times 2}[\mathrm{T}^3] = \mathrm{Int}\Delta_3 \times \mathbb{B}\,,$$

where T^3 is maximal torus of $SU(4)$, the factor $\mathrm{Int}\Delta_3$ denotes the interior of the ordered 3-simplex and $\mathbb{B}$ is 6-dimensional double coset:

$$\mathbb{B} = SU(2) \times SU(2)\backslash SU(4)/T^3\,.$$

Visualization of domains of separable and absolutely separable states

Using $\mathcal{KAT}$ representation for $SU(4)$ group, we derived two inequalities in 3rd and 4th ordered polynomials in 2-qubit density matrix eigenvalues describing the convex body of separable states.



Intersection of the separability ball of radius $1/\sqrt{3}$, absolute separable states and maximally mixed marginals separable states inside the ordered simplex.

Конференции

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2. A. Khvedelidze, A. Torosyan, *Hierarchy of classicality indicators for N-level systems*, Polynomial Computer Algebra, Russia, 2022
3. V. Abgaryan, A. Khvedelidze, A. Torosyan, *On the interplay between 2-qubit X-states separability and Wigner functions positivity*, 9th International Symposium on Optics & its applications, Armenia, 2022
4. А. Торосян, *От квантовой механики до квантовых компьютеров*, Всероссийский фестиваль наука 0+, Россия, 2022
5. A. Khvedelidze, A. Torosyan, *On distance indicator of non-classicality of qudits*, Distributed Computing and Grid Technologies in Science and Education, Russia, 2023
6. A. Khvedelidze, A. Torosyan, *On the states of N-level quantum system with positive Wigner function*, 5th International Conference "Computer Algebra", Russia, 2023
7. A. Khvedelidze, A. Torosyan, *Describing classicality of states of a finite-dimensional quantum system via Wigner function positivity*, Polynomial Computer Algebra, Russia, 2023
8. A. Khvedelidze, D. Mladenov, A. Torosyan, *On the parameterizations of the special unitary group SU(4) and related double cosets*, Polynomial Computer Algebra, Russia, 2024
9. A. Khvedelidze, D. Mladenov, A. Torosyan, *Parameterizing the entanglement body of a qubit pair*, International Conference Mathematical Modeling and Computational Physics, Armenia, 2024

Задачи 2025-2028

- Построение функции Вигнера двух кубитов и ее сравнение с функцией Вигнера четырехуровневой квантовой системы в рамках обобщения формализма Стратоновича-Вейля.
- Исследование соотношения между (абсолютной) сепарабельностью 2-кубитных X-состояний и положительностью соответствующих им функций Вигнера.
- Исследование задачи перепутанности состояний кудитов с помощью введенной $K\mathcal{A}T$-параметризации.
- ...
- Написание кандидатской диссертации.

Спасибо!