TSVD-based neutron spectra unfolding by Bonner multi-sphere spectrometer readings with iteration procedure

A.V. Chizhov [1], K.A. Chizhov [2,3]

[1] *Laboratory of Radiation Biology*

[2] *Meshcheryakov Laboratory of Information Technologies*

Joint Institute for Nuclear Research, Dubna

[3] *Dubna State University*

Outline

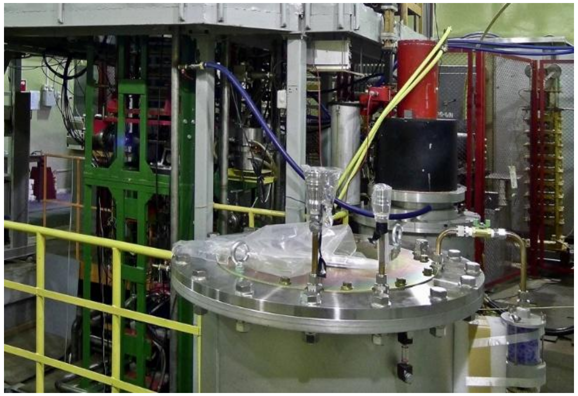
- Motivation
- Measurement of neutron spectra with Bonner spectrometers
- Truncated Singular Value Decomposition Method as regularization
- Iterative TSVD Method
- Examples of neutron spectra unfolding at JINR facilities:

  different sets of moderator spheres due to truncation
- Conclusion

JINR Acceleration Facilities

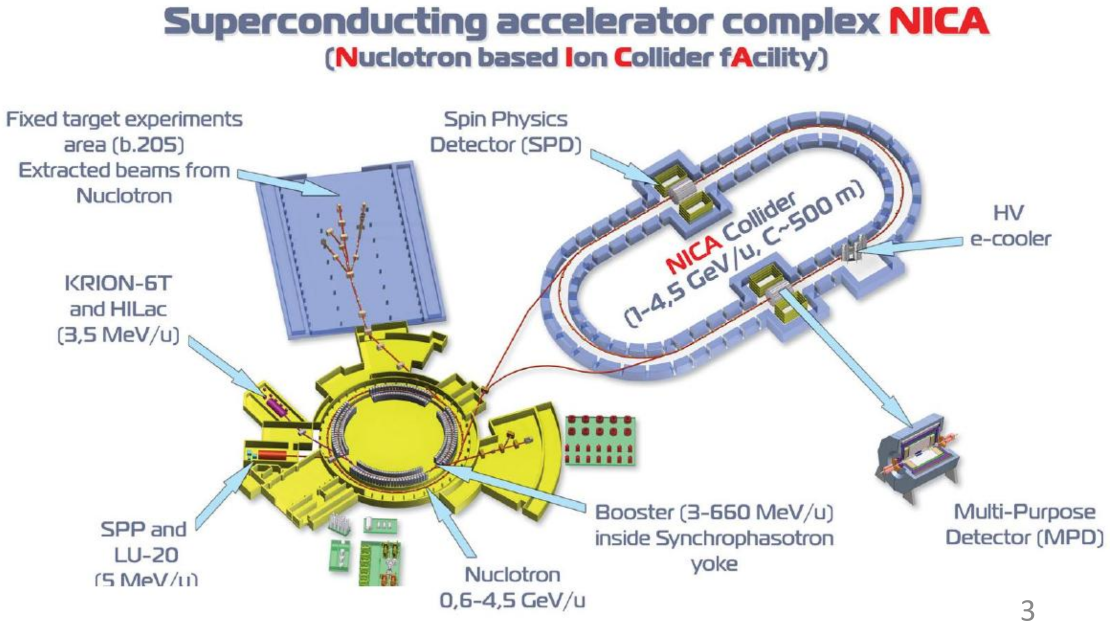
Phasotron: proton beam 640 MeV

Neutrons: IREN



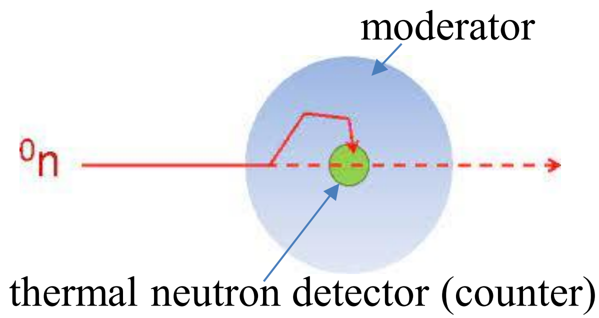
IBR-2 reactor

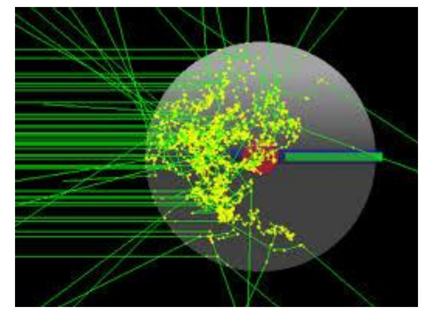
U-400M: heavy ions 50 MeV/u

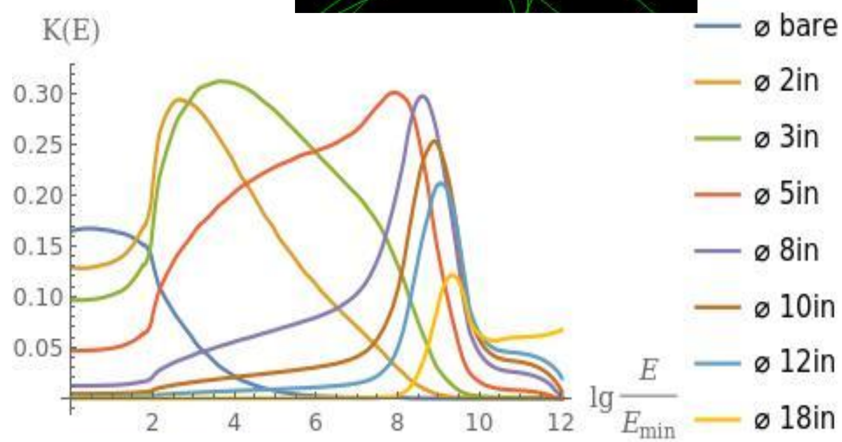


Measurements using Bonner spectrometers

Bonner spectrometer consists of a thermal neutron detector placed in spherical polyethylene moderators of various diameters, the number and size of which are specified by dynamic spectrum range



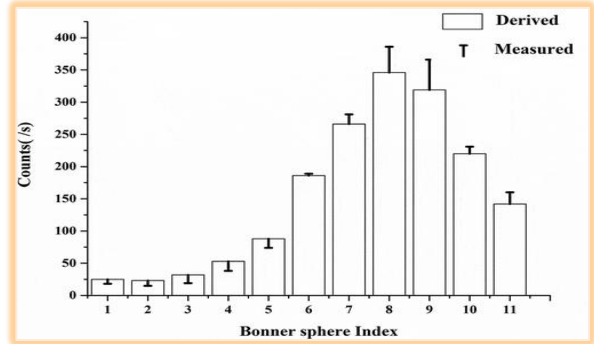




Sensitivity (response) functions for ^{6}Li detector surrounded by 7 moderators (Rad. Protect. Dosimetry, ***10*** *(1985) 89)*

Measurements using Bonner spectrometers

In order to reconstruct the full spectrum of neutrons $\varphi(E)$ from the results of measurements, it is necessary to solve the system of M Fredholm integral equations of the 1st kind



$$\begin{cases} \int_{E_{\min}}^{E_{\max}} K_1(E)\varphi(E)dE = Q_1, \\ \vdots \\ \int_{E_{\min}}^{E_{\max}} K_M(E)\varphi(E)dE = Q_M, \end{cases}$$

where Q_j is the Bonner spectrometer reading for the j-th sphere, and M is the number of spheres used to measure the spectrum, the integration limits $E_{\min}$ and $E_{\max}$ are determined by the spectrum definition area and the set of detectors used for measurements.

In terms of the new variable lethargy $u(E) = \lg(E/E_{\min})$ the integral equations take the form

$$\ln 10 \times \int_0^{l_E} K_j(u) \cdot \varphi(u)E(u)du = Q_j, \qquad j = 1, \dots, M,$$

where $l_E \equiv \lg(E_{\max}/E_{\min})$.

Therefore, instead of the spectrum $\varphi(E)$, it is more rational to find its product $\varphi(u)E(u)$.

Legendre polynomial expansion method

Our approach to finding the product $\varphi(u)E(u)$ is to expand it into *N* shifted Legendre polynomials
(*Chizhov K., Beskrovnaya L., Chizhov A*. Physics of Particles and Nuclei, **55** (2024) 532)

$$\Phi(u) \equiv \varphi(u)E(u) = \sum_{j=1}^{N} C_j \cdot P_j\left(\frac{2u}{l_E} - 1\right).$$

As a result, to find the unknown expansion coefficients C_j we obtain a system of *N* linear algebraic equations

$$\boldsymbol{A}^T \mathbf{AC} = \boldsymbol{A}^T \mathbf{Q},$$

where elements of matrix $\mathbf{A}_{\boldsymbol{M}\times\boldsymbol{N}}$

$$A_{ij} = \ln 10 \times \int_0^{l_E} K_i(u) \cdot P_j\left(\frac{2u}{l_E} - 1\right) du\,,$$

and *N*-dimensional vector $\mathbf{C}^T = (C_1, \ldots, C_M)$, *M*-dimensional vector $\mathbf{Q}^T = (Q_1, \ldots, Q_M)$.

However, a direct solution of such a system of equations by inverting matrix $\boldsymbol{A}^T\mathbf{A}$ does not give a physically justified solution to the spectrum due to the fact that *this matrix is ill-conditioned.*

Truncated SVD method as regularization

Truncated singular value decomposition is a method of *dimensionality reduction* that is able to preserve most of the original information.

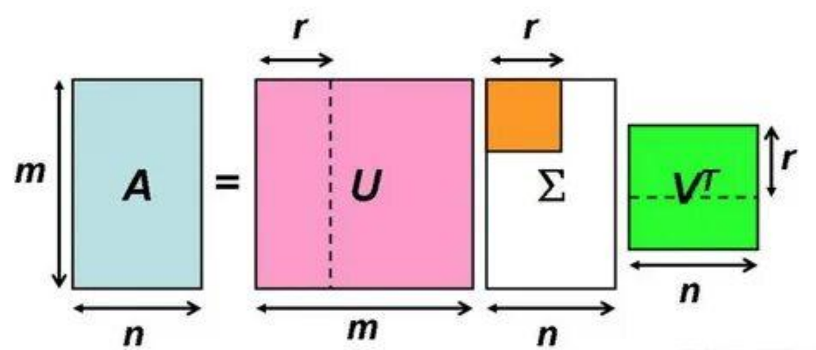
The essence of the method: matrix $\mathbf{A}_{M\times N}$, represented in the **singular value decomposition**

$$\mathbf{A}=\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T,$$

where unitary matrices $\boldsymbol{U}_{M\times M}$, $\boldsymbol{V}_{N\times N}$ are *left* and *right singular vectors* and rectangular diagonal matrix $\boldsymbol{\Sigma}_{M\times N} = \text{diag}\{\sigma_1, \sigma_2, \dots, \sigma_M, 0, \dots, 0\}$ with non-negative numbers on the diagonal $\sigma_1 > \sigma_2 > \ldots > \sigma_M > 0$ (σ_i being the singular values of $\mathbf{A}$), is approximated by retaining only the *r* upper singular values and their corresponding singular vectors:

$$\mathbf{A}_r=\mathbf{U}_r\boldsymbol{\Sigma}_r\mathbf{V}_r^T,$$

where $m \times r$ matrix $\mathbf{U}_r$, $k \times r$ matrix $\mathbf{V}_r^T$ and $r \times r$ diagonal matrix $\boldsymbol{\Sigma}_r$.



By discarding *small singular values*, the truncated matrix $\mathbf{A}_r$ becomes *more conditioned*:

$$cond(\mathbf{A}_r) = \sigma_1/\sigma_r < \sigma_1/\sigma_M = cond(\mathbf{A}).$$

Therefore, the truncated singular value decomposition can be used as a regularization method for solving a system of equations for the coefficients of the neutron spectrum decomposition.

Iterative TSVD method

Thus, to find the expansion coefficients, we solve the following system of N linear algebraic equations with the truncated matrix $\mathbf{A}_r$:

$$\mathbf{A}_r^T \mathbf{A}_r \mathbf{C}_r = \boldsymbol{A}^T \mathbf{Q}.$$

To find a numerical solution to such a system of equations, it is convenient to use *an iterative method*. As such a method we use the one of *Landweber* (Amer. J. Math. **73** (1951), 615):

$$\mathbf{C}_r^{(k+1)} = (\mathbf{I} - \tau \mathbf{A}_r^T \mathbf{A}_r)\mathbf{C}_r^{(k)} + \tau \boldsymbol{A}^T \mathbf{Q},$$

where *the relaxation factor* $\tau = 1/\|\mathbf{A}_r\|^2$ and

$$\mathbf{C}_r^{(0)} = (\mathbf{A}_r^T \mathbf{A}_r)^{-1} \boldsymbol{A}^T \mathbf{Q} = \mathbf{V}_r (\boldsymbol{\Sigma}_r^{-1})^2 \mathbf{U}_r^T \boldsymbol{A}^T \mathbf{Q}.$$

So, the neutron spectrum as a result after K iterations has the form

$$\Phi_r^K(u) \equiv \varphi_r^K(u) E(u) = \sum_{j=1}^{N} C_{r,j}^{(K)} \cdot P_j\left(\frac{2u}{l_E} - 1\right).$$

As *the criterion* for choosing the numbers of truncation r and iterations K is *the condition* of ensuring *the non-negativity of* the neutron spectrum $\varphi_r^K(E)$.

Then *the radiation dose rate* can be calculated as

$$\dot{H}_r^K = \int_{E_{\min}}^{E_{\max}} h(E) \cdot \varphi_r^K(E) dE = \ln 10 \times \int_0^{l_E} h(u) \cdot \Phi_r^K(u) du\,,$$

where $h(E)$ are the external radiation dose *conversion coefficients*.

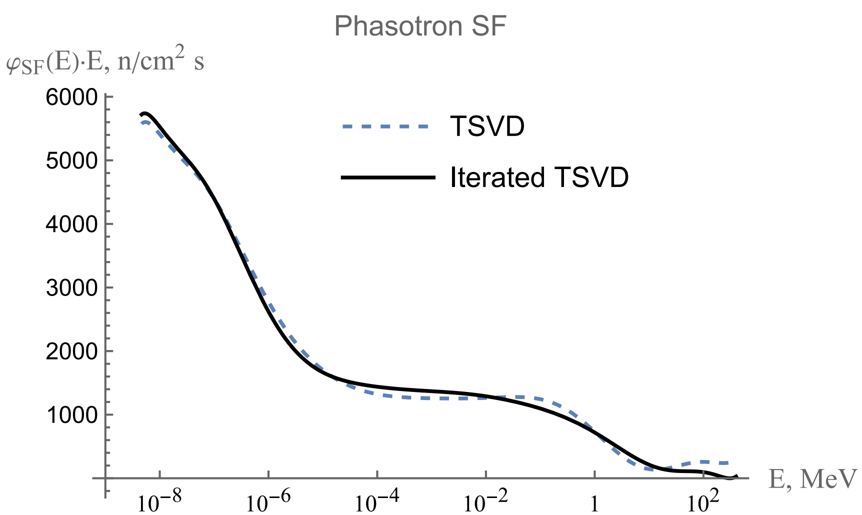
Neutron spectrum unfolding at Phasotron: Soft field

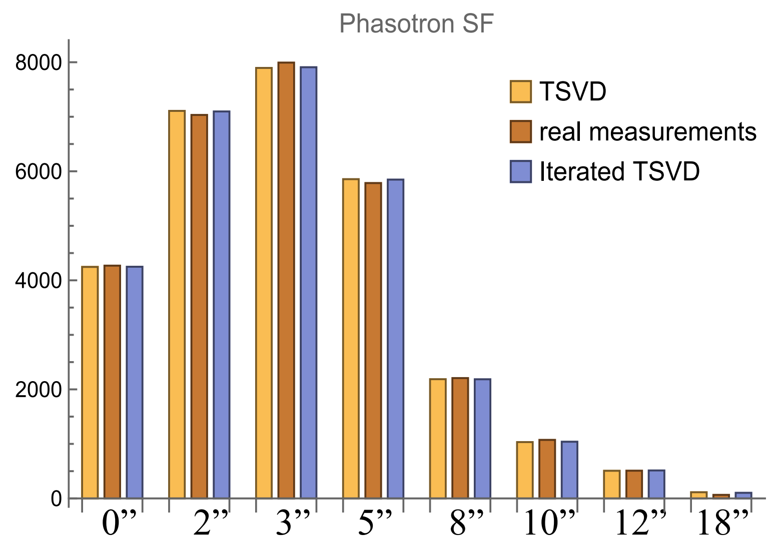
Diameters of $M = 8$ polyethylene spheres = {0”, 2”, 3”, 5”, 8”, 10”, 12”, 18”} surrounded 4 mm x 4 mm ^{6}Li detector (*Awschalom M., Sanna R.S.* Rad. Protect. Dosimetry, **10** (1985) 89).

Spectrometer counts: $Q_{\mathrm{SF}} = \{4269, 7033, 7993, 5784, 2207, 1073, 508.4, 64.5\}$ (*Aleinikov V. et al.* Rad. Protect. Dosimetry, **54** (1994) 57).

$E_{\min} = 5 \times 10^{-9}$ MeV, $E_{\max} = 400$ MeV; $N = 15$ shifted Legendre polynomials: $\{P_0, \ldots, P_{14}\}$

$$\Sigma_r = \mathrm{diag}\{\underbrace{3.17551,\ 0.841275,\ 0.379885,\ 0.183081,\ 0.0773665}_{r\,=\,5},\ \underbrace{\cancel{0.02807,\ 0.00775113,\ 0.000745156}}_{0}\}$$

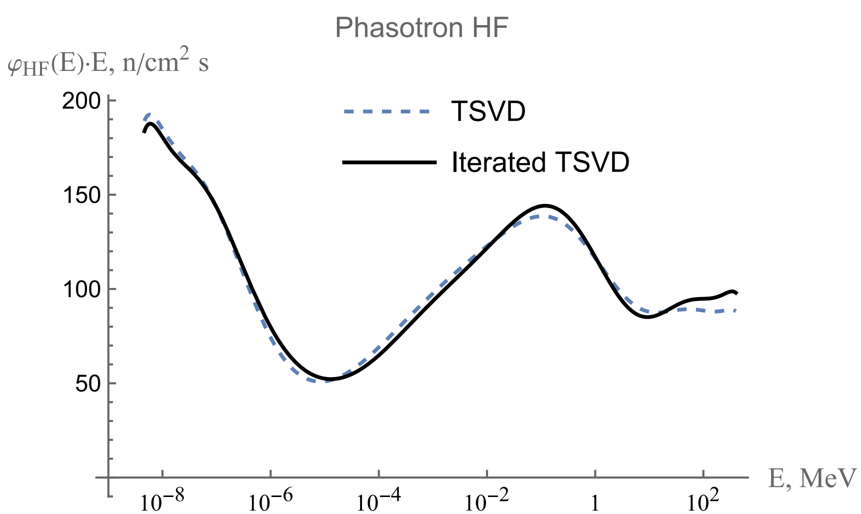


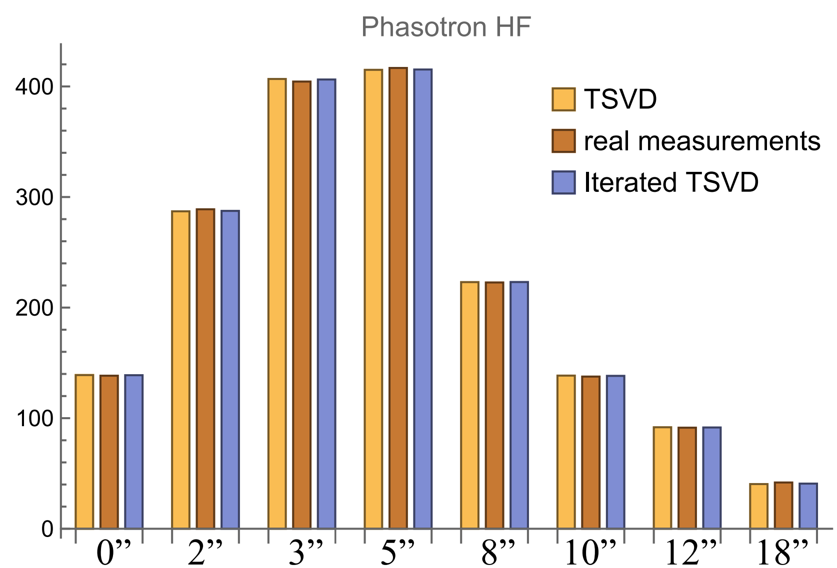


Unfolded soft neutron spectra by TSVD ($r = 5$) *and iterated TSVD* ($K = 2054$) *(left) and effective detector readings (right) with relative accuracies* $\delta_{SF}^{TSVD} = 0.012$ (1.2%) *and* $\delta_{SF}^{ITSVD} = 0.010$ (1.0%).

Neutron spectrum unfolding at Phasotron: Hard field

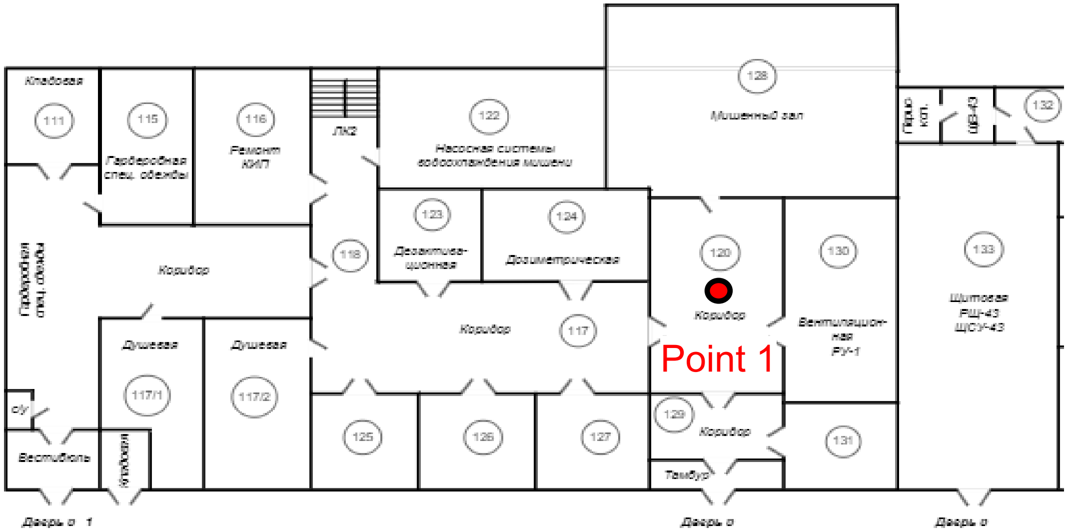
Spectrometer counts: $Q_{\mathrm{HF}} = \{138.4, 288.9, 404.4, 416.7, 222.7, 137.6, 91.39, 41.82\}$
(*Aleinikov V. et al.* Rad. Protect. Dosimetry, **54** (1994) 57).

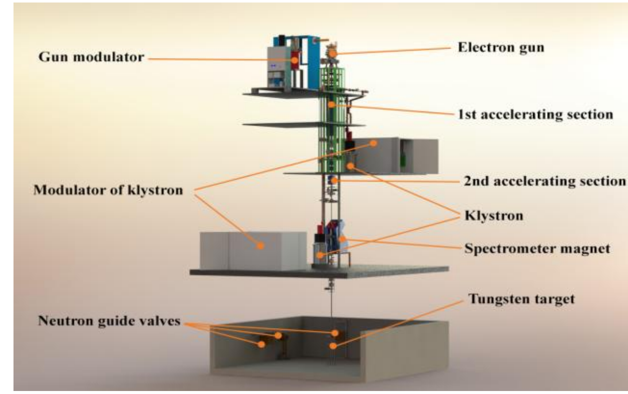


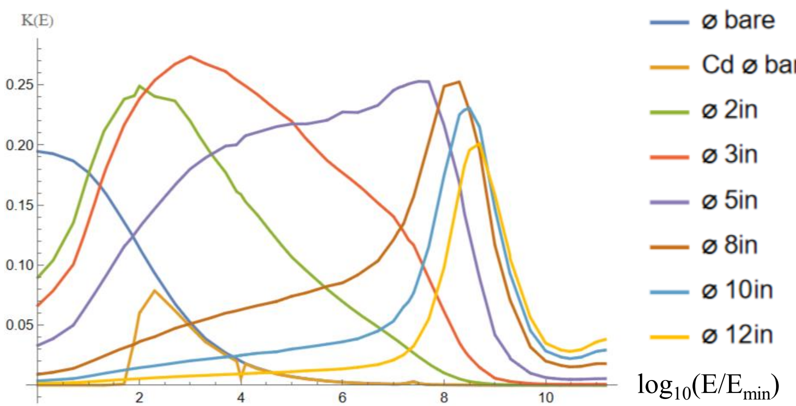


Unfolded hard neutron spectra by TSVD ($r = 5$) *and iterated TSVD* ($K = 3000$) (*left*) *and effective detector readings* (*right*) *with relative accuracies* $\delta_{HF}^{TSVD} = 0.0054$ (0.54%) *and* $\delta_{HF}^{ITSVD} = 0.0043$ (0.43%).

Neutron spectrum unfolding at IREN







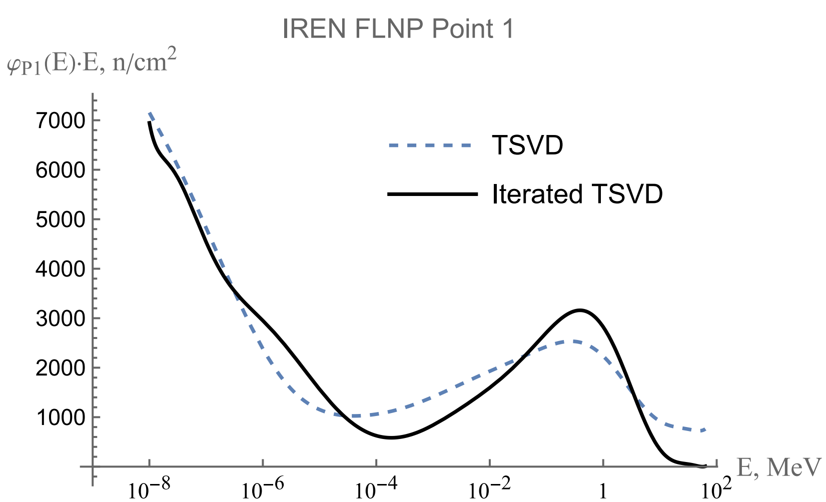
Response functions for LiI detector surrounded by 7 moderators (*Martinkovic J., Timoshenko G.N. JINR preprint P16-2005-105, 2005*)

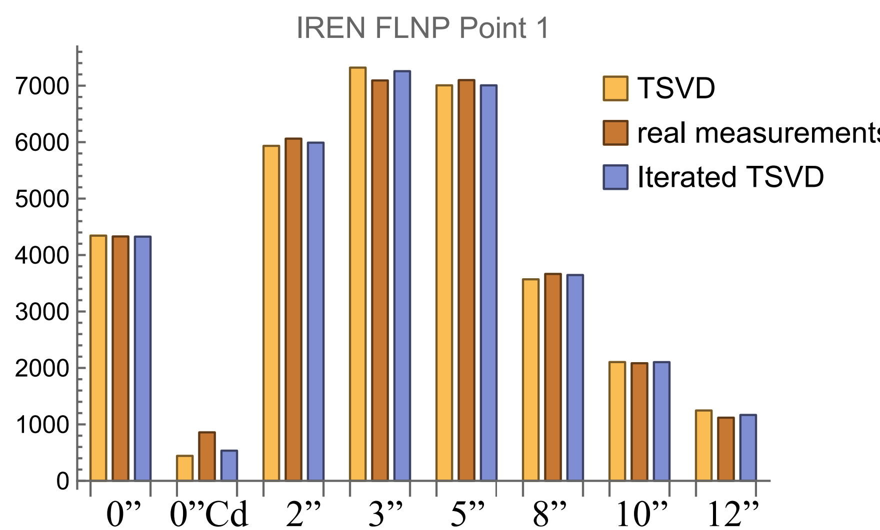
Neutron spectrum unfolding at IREN: Point 1

Spectrometer counts: $Q_{P1} = \{4329, 858.6, 6061.7, 7092.7, 7098.7, 3664.8, 2083.6, 1118.5\}$
(*Krylov A.R., Timoshenko G.N., Aleinikov V.E.* JINR preprint P16-91-177, 1991).

$E_{\min} = 10^{-8}$ MeV, $E_{\max} = 65$ MeV; $N = 15$ shifted Legendre polynomials: $\{P_0, \dots, P_{14}\}$

$$\Sigma_r = \mathrm{diag}\{\underbrace{2.72091,\ 0.737033,\ 0.333551,\ 0.148978}_{r\,=4},\ \underbrace{\cancel{0.0501416,\ 0.0236747,\ 0.0137851,\ 0.00241383}}_{0}\}$$

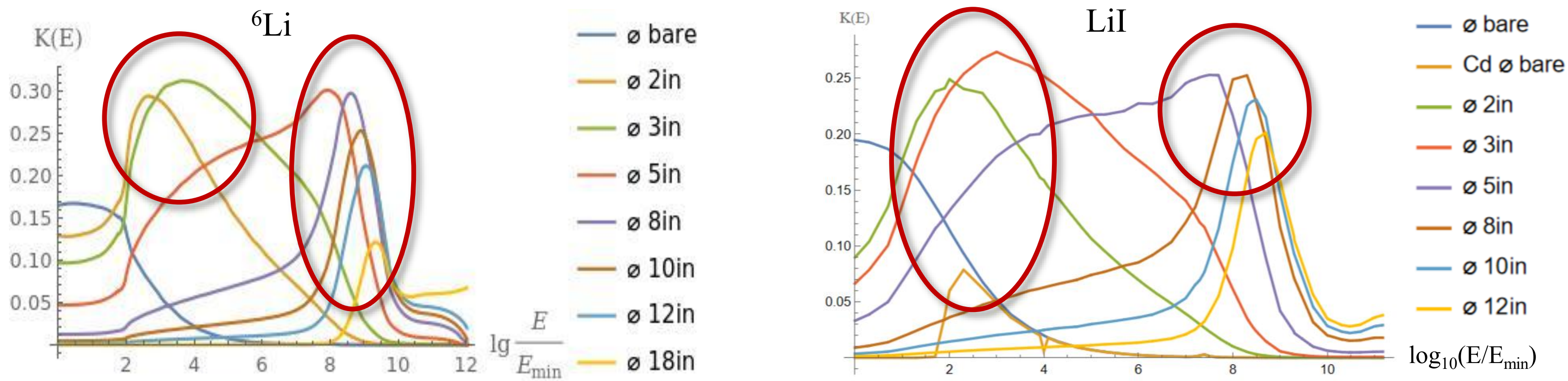




Unfolded neutron spectra by TSVD ($r = 4$) *and iterated TSVD* ($K = 1150$) (*left*) *and effective detector readings* (*right*) *with relative accuracies* $\delta_{P1}^{TSVD} = 0.040$ (4%) *and* $\delta_{P1}^{ITSVD} = 0.029$ (2.9%).

Truncation vs Sets of moderation spheres

Since the spectrometer sensitivity functions are *not linearly independent*,



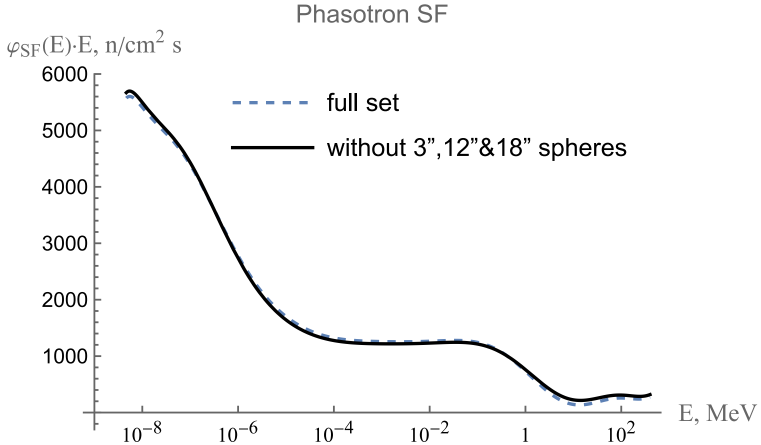
the question of choosing the optimal set of spectrometer moderation spheres is rather important
(*Chizhov A., Chizhov K*. Math. Modeling, **8** (2024) 89).

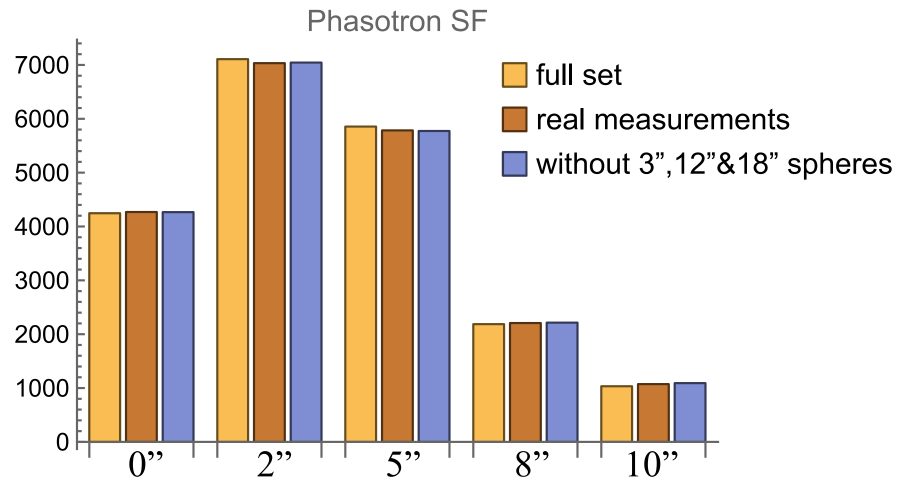
In this regard, a properly chosen truncation parameter r which denotes the actual rank of the matrix in the system of equations can indicate an effective set of moderation spheres M_{eff} for performing measurements of neutron spectra without qualitatively loss in the radiation dose assessment:

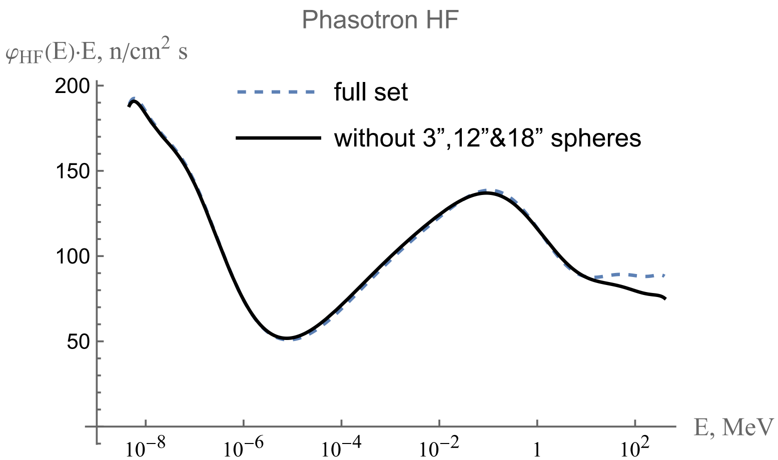
$$r \sim M_{eff}.$$

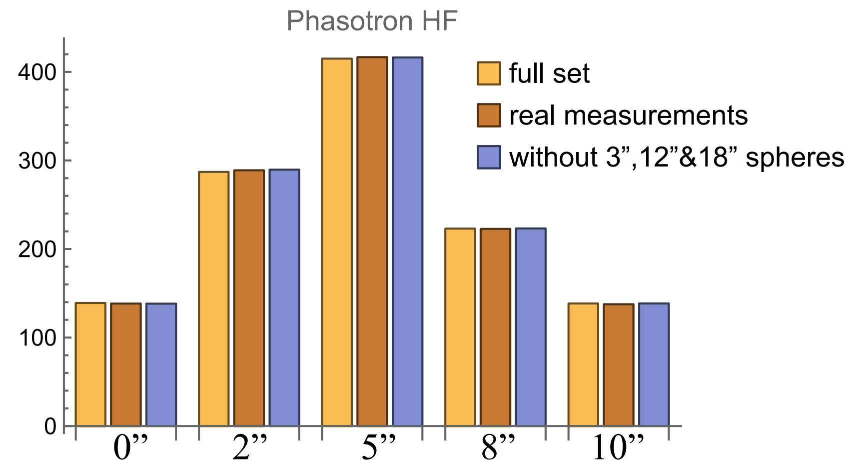
Neutron spectrum unfolding at Phasotron with reduced set of spheres

Comparison of the results of unfolding neutron field spectra using the TSVD method based on readings from the full set of Bonner detectors and SVD method with the reduced one (without 3”, 12”, and 18” moderator spheres)





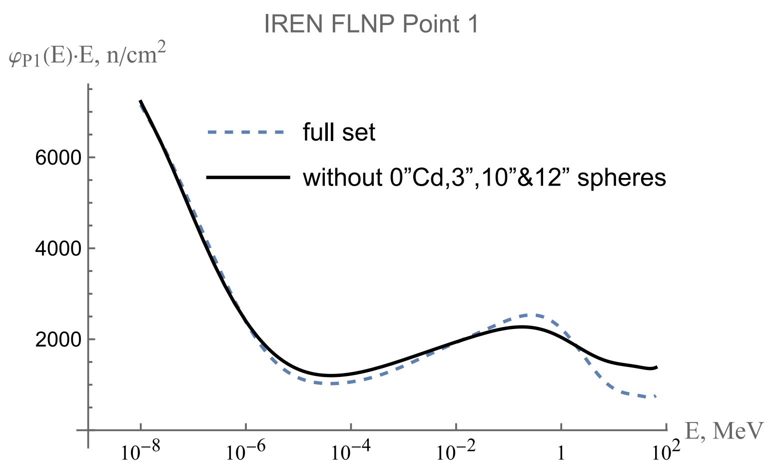


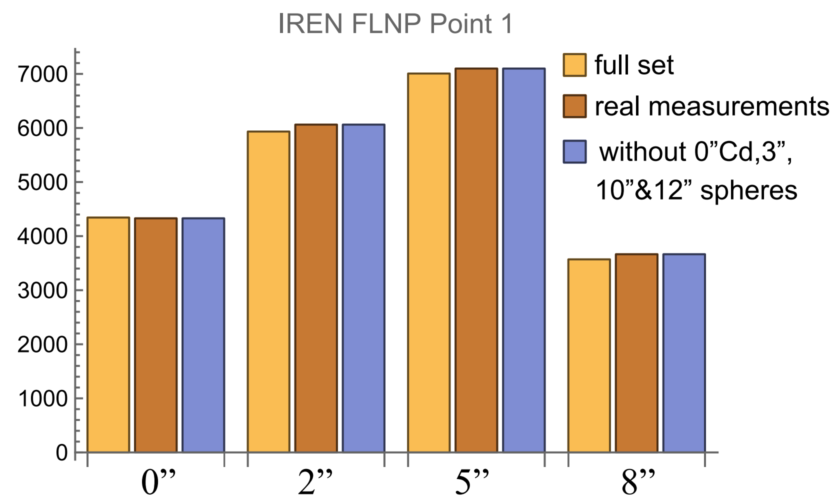


Unfolded neutron spectra (left) and detector readings (right) with the reduced set of spheres with relative accuracies $\delta_D^{SF} = 0.018$ (1.8%) *and* $\delta_D^{HF} = 0.009$ (0.9%).

Neutron spectrum unfolding at IREN with reduced set of spheres

Comparison of the results of unfolding neutron field spectra using the TSVD method based on readings from the full set of Bonner detectors and SVD method with the reduced one (without 0”Cd, 3”, 10”, and 12” moderator spheres)





Unfolded neutron spectrum (left) and detector readings (right) with the reduced set of spheres with relative accuracy $\delta_D^{P1} = 0.049$ (4.9%)

Conclusions

The truncated singular value decomposition method for unfolding neutron spectra from measurements of a Bonner multi-sphere detector can be used to:

- *regularize* an ill-defined system of equations
- determine *the optimal set of moderator spheres* (their sizes and number) for effective practical measurements

Thank you for your attention!