Stability Properties of Bright Solitons in Two-dimensional CFT

Yulia Galushkina[1] , **Eduard Kim**[1,2] , Emin Nugaev[1] , Yakov Shnir[3]

[1]Institute for Nuclear Research RAS

[2]Moscow Institute of Physics and Technology

[3]BLTP JINR

October 9, 2025

Models in Quantum Field Theory 2025

Conformal Field Theory

We recall free Schrödinger equation in arbitrary space dimensions

$$i \frac{\partial}{\partial t} \psi = - \frac{\nabla^2}{2m} \psi. \tag{1}$$

It is well known that this equation is invariant under space-time transformations of Schrödinger group.

Schrödinger group		
Subgroup	Transformations	Infinitesimal generators G
Time translation	$t^{'} = t + \beta$	$\frac{\partial}{\partial t}$
Space Translation	$x^{'} = x + a$	$\frac{\partial}{\partial x}$
Rotation	$x^{'} = x$	1
Galilean boost	$x^{'} = x + v \cdot t$	$t \frac{\partial}{\partial x} - imx$
Dilatation	$t^{'} = e^{2\sigma} t,\ x^{'} = e^{\sigma} x$	$2t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + \frac{1}{2}$
Special conformal symmetry	$t^{'} = \frac{t}{1+\eta t},\ x^{'} = \frac{x}{1+\eta t}$	$\frac{imx^2}{2} - \frac{t}{2} - xt \frac{\partial}{\partial x} - t^2 \frac{\partial}{\partial t}$

Unbroken dilatation and conformal symmetry are present in a model with potential term $|\psi|^{2n}$ that satisfies relation[1]

$$nd = d + 2, \tag{2}$$

where d is the number of space dimensions.

The simplest case possible is to consider a $(1+1)$-dimensional theory with $|\psi|^6$ self-interaction term which Lagrangian is written as

$$\mathcal{L}_{GP6} = i\psi^*\dot{\psi} - \frac{1}{2m}\nabla\psi^*\nabla\psi + \frac{\lambda}{24m^3}\left(\psi^*\psi\right)^3. \tag{3}$$

The corresponding equation of motion is a quintic Gross-Pitaevskii equation

$$i\frac{\partial}{\partial t}\psi = \left[-\frac{\nabla^2}{2m} - \frac{\lambda}{8m^3}\left(\psi^*\psi\right)^2\right]\psi, \tag{4}$$

where coupling $\lambda > 0$ (attractive potential).

[1] M. O. deKok and J.W. van Holten, Nucl. Phys. B 803 (2008), arXiv: 0712.3686 [hep-th].

Bright Soliton

Quintic Gross-Pitaevskii equation (4) supports bright soliton solutions

$$\psi(t,x) = e^{i\mu t}\left(\frac{24m^3\mu}{\lambda}\right)^{\frac{1}{4}}\sqrt{\mathrm{sech}\left(\sqrt{8m\mu}\cdot x\right)}. \tag{5}$$

It is worth studying the integral characteristics of these solutions, such as the $U(1)$ charge and the energy functional. Thus, straightforward calculations show that

$$\begin{aligned} N &= \int_{-\infty}^{\infty} dx|\psi(t,x)|^2 = \frac{\sqrt{3}\pi m}{\sqrt{\lambda}}, \\ H &= \int_{-\infty}^{\infty} dx\left[\frac{1}{2m}|\nabla\psi(t,x)|^2 - \frac{\lambda}{24m^3}|\psi(t,x)|^6\right] = 0. \end{aligned} \tag{6}$$

We support this result by considering scale invariance of theory (3) and the influence of unbroken conformal symmetry.

Dilatations: $e^{\sigma} = \sqrt{2m\mu}$, so that $t^{'} = 2m\mu t$ and $x^{'} = \sqrt{2m\mu} \cdot x$. The complex field $\psi^{'} = (2m\mu)^{-\frac{1}{4}}\psi$.

$$\nabla^2 \psi^{'} = \psi^{'} - \frac{\lambda}{4m^2} \left|\psi^{'}\right|^4 \psi^{'}. \qquad (7)$$

$$N = \frac{\sqrt{2m\mu}}{\sqrt{2m\mu}} \int_{-\infty}^{\infty} dx^{'} \left|\psi^{'}\left(t^{'}, x^{'}\right)\right|^2, \quad H = 0. \qquad (8)$$

The latter is a result of an unbroken scale invariance and conformal symmetry, the corresponding symmetry generators D and K

$$D = 2tH + \frac{i}{2} \int x \left(\psi^* \nabla \psi - \psi \nabla \psi^*\right) dx,$$
$$K = t^2 H - tD - \frac{m}{2} \int x^2 (\psi^* \psi) dx.$$

are conserved in accordance with equations [2]

$$\frac{dK}{dt} = -t \frac{dD}{dt}, \qquad \frac{dD}{dt} = 2H = 0. \qquad (9)$$

[2] M. O. deKok and J.W. van Holten, Nucl. Phys. B 803 (2008), arXiv: 0712.3686 [hep-th].

Linear Perturbations

Relations (6) impose a constraint on both the energy and the $U(1)$ charge of bright solitons (5).

$$\psi_p(t,x) = \psi(t,x) + \delta\psi(t,x) = e^{i\mu t} f(x) + \delta\psi(t,x) \tag{10}$$

one can derive linearized equation of motion

$$i\frac{\partial}{\partial t}\delta\psi(t,x) = -\frac{\nabla^2}{2m}\delta\psi(t,x) - \frac{\lambda}{8m^3}\left(3\cdot\delta\psi(t,x)|\psi(t,x)|^4 + \right. \\ \left. +2\cdot\delta\psi^*(t,x)\psi^3(t,x)\psi^*(t,x)\right). \tag{11}$$

Symmetry-related zero modes have a simple form

$$\delta\psi_0(t,x) = G\psi(t,x), \tag{12}$$

where G is an infinitesimal generator of Schrödinger group symmetry or a generator of $U(1)$ symmetry $G = i$.

The general ansatz for linear perturbations of the complex field ψ can be written as

$$\delta\psi(t,x) = e^{i\mu t}\left(e^{i\gamma t}\eta(t,x) + e^{-i\gamma^* t}\xi^*(t,x)\right). \tag{13}$$

By setting the parameter γ and the functions η, ξ to be real we study the vibrational modes of bright soliton.

$$\begin{aligned} \nabla^2\eta &= \left(1 + \frac{\gamma_{osc.}}{\mu}\right)\eta - \frac{1}{4m^2} f^4(3\eta + 2\xi), \\ \nabla^2\xi &= \left(1 - \frac{\gamma_{osc.}}{\mu}\right)\xi - \frac{1}{4m^2} f^4(3\xi + 2\eta). \end{aligned} \tag{14}$$

Considering decay modes requires redefinition $\gamma \to -i\gamma, \gamma \in \mathbb{R}$ and $\xi \equiv (\eta + \xi^*)$.

$$\begin{aligned} \nabla^2 \operatorname{Re}\xi &= \operatorname{Re}\xi + \frac{\gamma_{dec.}}{\mu}\operatorname{Im}\xi - \frac{5}{4m^2} f^4 \operatorname{Re}\xi, \\ \nabla^2 \operatorname{Im}\xi &= \operatorname{Im}\xi - \frac{\gamma_{dec.}}{\mu}\operatorname{Re}\xi - \frac{1}{4m^2} f^4 \operatorname{Im}\xi. \end{aligned} \tag{15}$$

An extensive numerical scanning[3] of normalizable modes which are localized in spatial dimension has failed to find any modes at any value of the parameter μ other than zero modes.
Vakhitov-Kolokolov criterion of stability

$$\frac{\mu}{N}\frac{d}{d\mu}N < 0 \tag{16}$$

and instability

$$\frac{\mu}{N}\frac{d}{d\mu}N > 0. \tag{17}$$

[3] Yulia Galushkina et al., Phys. Lett. B 865 (2025), arXiv: 2411.13514 [hep-ph].

Relativistic Generalization

In order to provide relativistic generalization of the model (3) we use a simple relation between the relativistic field ϕ and the non-relativistic field ψ that has the form

$$\phi(t,x) = \frac{1}{\sqrt{2m}} e^{-imt} \psi(t,x). \tag{18}$$

Thus, we are able to write down the following Lorentz-invariant Lagrangian

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi + \frac{\lambda}{3} \left(\phi^* \phi\right)^3 . \tag{19}$$

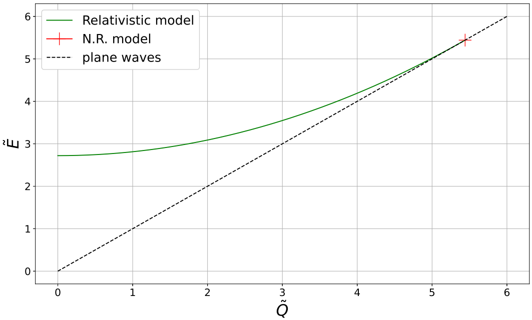
This theory also supports a soliton solution that can be written as

$$\phi(t,x) = e^{-i\omega t} g_\omega(x) = e^{-i\omega t} \left(\frac{3\left(m^2 - \omega^2\right)}{\lambda} \right)^{\frac{1}{4}} \sqrt{\operatorname{sech}\left(2\sqrt{m^2-\omega^2} \cdot x\right)}. \tag{20}$$

$$Q = 2\omega \int_{-\infty}^{\infty} dx\, |\phi(t,x)|^2 = \frac{\sqrt{3}\pi\omega}{\sqrt{\lambda}} \tag{21}$$

$$E = \int_{-\infty}^{\infty} dx \left[\left|\dot{\phi}\right|^2 + |\nabla\phi|^2 + m^2 |\phi|^2 - \frac{\lambda}{3} |\phi|^6 \right] = \frac{\sqrt{3}\pi(m^2+\omega^2)}{2\sqrt{\lambda}}. \tag{22}$$

It can be directly checked that the differential relation $\frac{dE}{dQ} = \omega$ is satisfied.



Decay Modes

Following scaling

$$\tilde{x} = x\sqrt{m^2 - \omega^2};$$
$$\tilde{g} = \frac{g_\omega \lambda^{\frac{1}{4}}}{(m^2 - \omega^2)^{\frac{1}{4}}}, \qquad (23)$$

allows us to write linearized equations of motion for decay modes $\delta\phi(t,x) = e^{-i\omega t} e^{\gamma_{dec.} t} \left(\operatorname{Re}\xi(x) + i \operatorname{Im}\xi(x)\right)$

$$\tilde{\nabla}^2 \operatorname{Re}\xi = \frac{\left(m^2 - \omega^2 + \gamma_{dec.}^2\right) \operatorname{Re}\xi + 2\omega\gamma_{dec.} \operatorname{Im}\xi}{m^2 - \omega^2} - 5\tilde{g}^4 \operatorname{Re}\xi,$$
$$\tilde{\nabla}^2 \operatorname{Im}\xi = \frac{\left(m^2 - \omega^2 + \gamma_{dec.}^2\right) \operatorname{Im}\xi - 2\omega\gamma_{dec.} \operatorname{Re}\xi}{m^2 - \omega^2} - \tilde{g}^4 \operatorname{Im}\xi. \qquad (24)$$

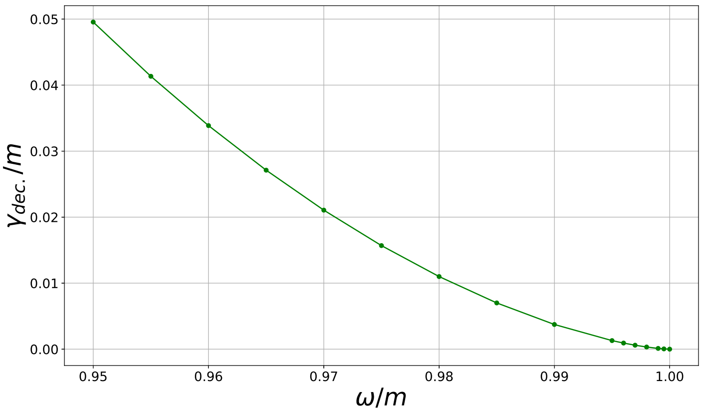


Figure 1: Spectrum of decay modes that are described by Eqs.(24). In the limit $\omega \to m$ parameter $\gamma_{dec.}$ tends to zero as $C \cdot (m - \omega)^{1.506}$.

It can be seen that in the limit $\omega \to m$ parameter $\gamma_{dec.}$ tends to zero. While $\frac{\gamma_{dec.}}{\omega} \ll 1$ decay modes might be generated by expanding a soliton solution in perturbation series as

$$i\phi_p(t,x) = ie^{-i\left(1+i\frac{\gamma}{\omega}\right)\omega t} g_{1+i\frac{\gamma}{\omega}}(x) \approx e^{-i\omega t}\left(1+\gamma t\right)\cdot\left(ig_\omega(x) - \gamma\frac{\partial g_\omega(x)}{\partial\omega}\right). \tag{25}$$

Comparison with the expansion of decay mode ansatz

$$\delta\phi(t,x) = e^{i\omega t}e^{\gamma t}\left(\mathrm{Re}\,\xi + i\,\mathrm{Im}\,\xi\right) \approx e^{i\omega t}(1+\gamma t)\left(\mathrm{Re}\,\xi + i\,\mathrm{Im}\,\xi\right) \tag{26}$$

helps to evaluate that $\mathrm{Re}\,\xi = -\gamma\frac{\partial g_\omega(x)}{\partial\omega}$ and $\mathrm{Im}\,\xi = g_\omega(x)$.

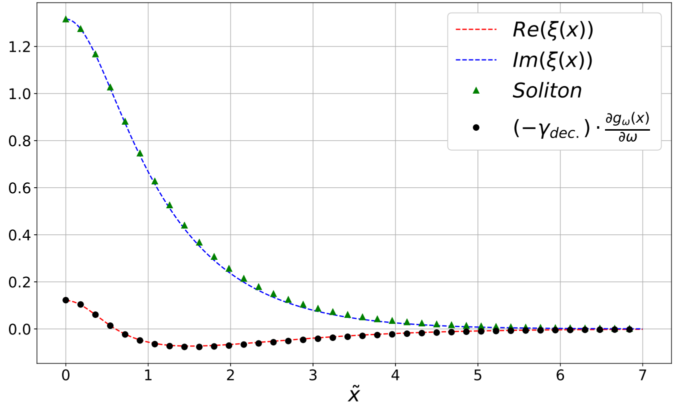


Figure 2: Decay mode profile at $\omega/m = 0.99$ and $\gamma_{dec.}/m = 3.74 \cdot 10^{-3}$. Scaled soliton profile and $(-\gamma_{dec.})\frac{\partial g_\omega}{\partial \omega}$ are added for comparison.

Summary

- We have found and examined bright solitons in two-dimensional conformal field theory.
- In a relativistic generalization of our theory, the restoration of conformal symmetry leads to enhanced stability of bright solitons.
- The presence of conformal symmetry allowed for the Vakhitov-Kolokolov series expansion.

Thank you for attention!

Numerical studies of relativistic decay modes were supported by the grant RSF 22-12-00215.