RG analysis of random walk on a KPZ fluctuating rough surface

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Physics of Elementary Particles and Statistical Physics

In this talk we deal with two areas of physic: statistical physics and high energy physics:

- Fokker-Planck equation for a particle in a gravitational field;
- Kardar-Parizi-Zhang equation describing the growth of the surface;
- stochastic description of the system;
- functional integration and calculation of Feynman graphs;
- renormalization group (RG).

The problem under consideration is walking on the surface under the fluctuation of surface.

Plan of the talk

The main steps (general scheme) are following:

- Stochastic formulation of the model;
- quantum field action and Feynman diagrams;
- divergences of the diagrams;
- renormalization, RG, RG flow and fixed points.

Fokker-Planck equation

In the macroscopic description of the model, the particles' density $\theta(t, \mathbf{x})$ satisfies the diffusion-type equation

$$\{\partial_t + \partial_i(F_i - \kappa_0 \partial_i)\} \; \theta(t, \mathbf{x}) = 0.$$

The drift F in a gravitational field obeys some symmetries and in the simplest linear approximation reads

$$F_i = -\alpha_0 \partial_i h,$$

where $h(t, \mathbf{x})$ is the height of the surface.

Thus, particles' density θ obeys equation

$$\partial_t \theta = \varkappa_0 \, \partial^2 \theta + \alpha_0 \, \partial_i (\theta \, \partial_i h),$$

with height of the surface in r.h.s.

Description of h

There are different possible ways to describe dependence of height h on time t and point $\mathbf{x}$.

For example, linear Edwards–Wilkinson model:

$$\{\partial_t - \kappa_0 \partial^2\} \, h(t, \mathbf{x}) = f(t, \mathbf{x}),$$

where f is a Gaussian random noise with zero mean and a given pair correlation function.

This model was studied *Universe 9, 139 (2023)*, non-trivial fixed point (all charges are non-zero) was found.

Description of h: Kardar-Parisi-Zhang equation

Now we use famous (simplest non-linear) Kardar-Parisi-Zhang equation:

$$\partial_t h = \nu_0 \partial^2 h + \frac{\lambda_0}{2} \left(\partial_i h\right) \left(\partial_i h\right) + f,$$

where $f = f(x)$ is a random Gaussian noise with zero mean $\langle f \rangle = 0$ and the pair correlator

$$\langle f(x)\, f(x') \rangle = D_0\, \delta\left(t - t'\right)\, \delta^{(d)}(\mathbf{x} - \mathbf{x}').$$

Action functional: General rules

Theorem: any stochastic equation of the type

$$\partial_t \phi(x) = U(x, \phi) + f(x), \quad \left\langle f(x) f(x') \right\rangle = D(x, x'),$$

where $\phi(x) = \phi(t, \mathbf{x})$ is a random field, $U(x, \phi)$ is a t-local functional depending on the fields and their derivatives, $f(x)$ is a random force, **is equivalent to quantum field model** of the double set of fields $\widetilde{\phi} = \{\phi, \phi'\}$ and action functional

$$S[\varphi] = \underbrace{\frac{1}{2}\varphi' D \varphi'}_{\text{noise term}} + \varphi' \underbrace{[-\partial_t \varphi + U]}_{\text{dynamics}},$$

integration over t and $\mathbf{x}$ implied.

Action functional: General rules

What does it mean:

- statistical average is equivalent to functional integration with weight $\exp S[\phi]$;
- classical random field $\to$ quantum field;
- we may use all techniques from quantum field theory: Feynman graphs, renormalization group, operator product expansion, *etc.*

Actions functional

Quantum field action:

$$S(\Phi) = \theta' \left[-\partial_t \theta + \varkappa_0 \partial^2 \theta + \alpha_0 \partial_i (\theta \partial_i h) \right] + S_h(h, h'),$$

where

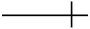
$$S_h(h, h') = \frac{1}{2} h' h' + h' \left[-\partial_t h + \nu_0 \partial^2 h + \frac{\lambda_0}{2} (\partial_i h)(\partial_i h) \right].$$

All integrations are implied:

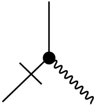
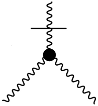
$$h' h' \equiv \int dt \int d^d x \, h'(t, x) h'(t, x).$$

Actions functional

According to general rules both models contain three propagators

$\langle \theta\theta' \rangle_0 =$ , $\langle hh' \rangle_0 =$ , $\langle hh \rangle_0 =$ , (1)

and two vertices

$\theta\theta' h =$ , $h' hh =$ . (2)

The propagators are

$$\langle hh \rangle_0 = \frac{1}{\omega^2 + \nu_0^2 k^4}, \quad \langle hh' \rangle_0 = \frac{1}{-i\omega + \nu_0 k^2}, \quad \langle \theta\theta' \rangle_0 = \frac{1}{-i\omega + \varkappa_0 k^2},$$

Actions functional

Logarithmic dimension is $d = 2$ and there are four divergent Green functions: $\langle hh' \rangle, \langle \theta\theta' \rangle, \langle h'h' \rangle$ and $\langle \theta\theta' h \rangle$.

Renormalized action has form:

$$S_R(\Phi) = \theta' \left[-\partial_t \theta + \varkappa Z_3 \partial^2 \theta + \alpha Z_4 \partial_i (\theta \partial_i h) \right] + S_{hR}(h, h'),$$

$$S_{hR}(h, h') = \frac{1}{2} Z_1 h' h' + h' \left[-\partial_t h + Z_2 \nu \partial^2 h + \frac{\lambda}{2} (\partial_i h)(\partial_i h) \right].$$

Three charges are

$$g_0 = \lambda_0 \, \nu_0^{-3/2}, \qquad w_0 = \alpha_0 \, \nu_0^{-3/2}, \qquad u_0 = \frac{\varkappa_0}{\nu_0}.$$

Renormalization constants

One-loop answers for constants Z are

$$Z_1 = 1 - \frac{1}{16\pi} \frac{g^2}{\varepsilon}, \quad Z_2 = 1,$$

$$Z_3 = 1 + \frac{1}{8\pi} \frac{w^2}{\varepsilon} \frac{(u-1)}{u(u+1)^2}, \quad Z_4 = 1 + \frac{1}{8\pi} \frac{w}{\varepsilon} \frac{(w-g)}{(u+1)^2}.$$

Since the field θ is passive the constants Z_1 and Z_2 coincide with their analogs in pure KPZ model:

$$Z_1^{-1} = 1 + \frac{1}{16\pi} \frac{g^2}{\varepsilon}, \qquad Z_2 = 1.$$

RG functions

RG functions (*beta*-functions) are

$$\beta_g = -g\left(\frac{\varepsilon}{2} + \frac{1}{2}\frac{g^2}{16\pi}\right),$$

$$\beta_w = -w\left(\frac{\varepsilon}{2} + \frac{1}{2}\,\frac{g^2}{16\pi} - \frac{w(w-g)}{8\pi}\,\frac{1}{(u+1)^2}\right),$$

$$\beta_u = \frac{w^2}{8\pi}\,\frac{(u-1)}{(u+1)^2}.$$

Fixed points are governed by the rule

$$\beta(g_*) = \beta(w_*) = \beta(u_*) = 0.$$

Fixed point is IR-attractive if real parts of all eigenvalues of the matrix $\Omega_{ik} \equiv \partial_i \beta_k(g_*, w_*, u_*)$ are positive.

Fixed points

1. **IR attractive** if $\varepsilon < 0$
$$g_* = 0, \quad w_* = 0 \qquad \text{for all } u_*;$$
2. saddle point, eigenvalues of Ω_{ik} are $\{-\frac{\varepsilon}{2}, \varepsilon, \frac{\varepsilon}{2}\}$
$$g_* = 0, \quad w_*^2 = 16\,\pi\varepsilon, \quad u_* = 1;$$
3. **IR attractive** if $\varepsilon > 0$
$$g_*^2 = -16\pi\varepsilon, \quad w_* = 0 \qquad \text{for all } u_*;$$
4. saddle point, eigenvalues of Ω_{ik} are $\{\varepsilon, -\frac{\varepsilon}{2}, -\frac{\varepsilon}{2}\}$
$$g_*^2 = -16\,\pi\varepsilon, \quad w_* = g_*, \quad u_* = 1;$$
5. point in the system g_*, w_*, $y_* = \frac{1}{u_*}$
$$y_* = 0, \quad g_*^2 = -16\pi\varepsilon \qquad \text{for all } w_*;$$
6. point in the system $g_*, \widetilde{w}_* = w_*/u_*^{3/2}, u_*$
$$u_* = 0, \quad g_*^2 = -16\,\pi\varepsilon \qquad \text{for all } \widetilde{w}_*.$$

Kardar-Parizi-Zhang model itself

Kardar-Parizi-Zhang model corresponds to action

$$S_h(h,h') = \frac{1}{2}h'h' + h'\left[-\partial_t h + \nu_0\partial^2 h + \frac{\lambda_0}{2}(\partial_i h)(\partial_i h)\right]$$

with the only *beta*-function (the same as in the full model!)

$$\beta_g = -g\left(\frac{\varepsilon}{2} + \frac{1}{2}\frac{g^2}{16\pi}\right).$$

The only IR attractive nontrivial fixed point reads

$$g_*^2 = -16\pi\varepsilon.$$

Kardar-Parizi-Zhang model itself

However, various non-perturbative considerations imply existence of a strong-coupling scaling regime for all $d \geq 1$.

Within the RG framework, it is natural to associate this strong-coupling (non-trivial) regime with a certain non-perturbative IR attractive fixed point, not "visible" in the perturbative expression.

This point governs the IR behaviour for $\varepsilon > 0$ and $g_* > 0$ for it.

Full model

Since $\beta_g = 0$ for some unknown point g_* we are left with two *beta*-functions:

$$\beta_w = -w \left(\frac{\varepsilon}{2} + \frac{1}{2} \frac{g^2}{16\pi} - \frac{w(w-g)}{8\pi} \frac{1}{(u+1)^2} \right),$$

$$\beta_u = \frac{w^2}{8\pi} \frac{(u-1)}{(u+1)^2}.$$

which have solutions $w_* = g_*$ (unknown) and $u_* = 1$.

Full model: stability

Fixed point is IR-attractive if real parts of all eigenvalues of the matrix $\Omega_{ik} \equiv \partial_i \beta_k(g_*, w_*, u_*)$ are positive.

What we have: $\Omega_{gg} > 0$, $\Omega_{ug} = \Omega_{wg} = 0$.

$\Omega_{wu} \sim (u-1) = 0$ at $u_* = 1$.

Thus, the matrix Ω_{ik} is block triangular and its eigenvalues coincide with the diagonal elements which reads

$$\partial_w \beta_w = \partial_u \beta_u = w_*^2/(32\pi) > 0.$$

Thus, this hybrid point is IR attractive, lies in the physical range of parameters and, therefore, governs the IR asymptotic behaviour of the Green's functions of our model.

Conclusion

We applied methods of **quantum field theory** to the model of moving of the particle on the surface described by KPZ equation.

- The key point is the possibility to reformulate initial stochastic problem into some quantum field theory.
- Feynman graphs are divergent. Renormalization group allows us to work with these objects and, moreover, provides critical dimensions of measurable quantities.
- KPZ model itself has well-known problem of negative coordinate of fixed point; as a consequence our model has the same result.
- If we use well-known non-perturbative result for KPZ model, two others *beta*-functions connecting with walking of the particled will produce good (positive) IR attractive non-trivial fixed point.

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Thank you for your attention!