Frobenius-Perron Operator Approach to the Beam-Beam Interaction in Circular Colliders

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Beam-Beam Interaction. General Features

The now-common term ***beam-beam interaction*** is quite longstanding.

First machine to start operating in collider mode in 1970 is the **Intersecting Storage Rings (ISR)** at CERN.

At the present time, it can be without hesitation stated that beam-beam interaction represents one of the most complex problems in the physics of accelerators and charged particle beams.

Despite significant progress in understanding the relevant issues and underlying processes, there is still no comprehensive picture that encompasses all the features and physical details of beam-beam interaction.

It is fair to say that the progress in numerical simulation of beam-beam interaction is significantly greater than the achievements of the theoretical models proposed so far.

Beam-Beam Interaction. Weak-Strong Model

Historically, the first theoretical model of the beam-beam interaction is the so called **weak-strong model,** also known as the **incoherent beam-beam interaction.**

Special workshop has been dedicated → *M. Month and J.C. Herrera, editors*, ***Nonlinear Dynamics and the Beam-Beam Interaction***. **AIP Conference Proceedings, No. 57, (1980)**. **Enormous amount of articles hereafter**.

In the weak-strong model, it is assumed that one of the beams is **strong and rigid** and does not undergo significant changes (***practically unmodified***) in the collision process.

The role of the **strong beam** is to act on the other beam (considered **weak and mobile**), the latter playing ***the role of a dynamic probe and indicator of the interaction***.

The weak-strong beam-beam interaction **affects the single particle behaviour** and considers the beam-beam interaction as a ***static lens***. Obviously, highly simplified model, but in many cases provides good enough results.

Beam-Beam Interaction. Strong-Strong Model

The **realistic model reflecting the collective nature** of the interaction between the two counter-rotating beams is called ***the strong-strong or coherent beam-beam interaction model*** for short.

In this model, **the evolution** of the two beams occurs **synchronously**, as the electromagnetic field created by each beam is ***influencing and modifying the other one*** at the interaction point.

The coherent beam-beam interaction in one dimension **was first theoretically studied by Chao and Ruth by solving the linearized Vlasov-Poisson equations** → *A.W. Chao and R.D. Ruth*, "**Coherent beam-beam instability in colliding-beam storage rings**," *Particle Accelerators*, vol. 16, pp. 201–216, 1985.

Since the pioneering work of Chao and Ruth, numerous papers based on the self-consistent Vlasov technique have been published, **among which it is worth noting, at the first place, the article** by **Yu. Alexahin** → *Yu. Alexahin*, "**A study of the coherent beam–beam effect in the framework of the Vlasov perturbation theory**," *Nuclear Instruments and Methods in Physics Research Section A*, vol. 480, no. 2–3, pp. 253–288, 2002.

Beam-Beam Interaction. Strong-Strong Model

Based on the macroscopic hydrodynamic approach the results regarding the linear mode coupling, also known as the coherent beam-beam resonance have been generalized in → ***S.I. Tzenov and R.C. Davidson***, "**Macroscopic fluid approach to the coherent beam-beam interaction**," ***Proceedings of IEEE Particle Accelerator Conference (PAC 2001), 18-22 June 2001. Chicago, IL, United States***, vol. 0106181, pp. 2078–2080, 2001.

The standard technique for solving the Vlasov-Poisson system of equations mandatorily used is in terms of action-angle variables.

The approach used in → ***Stephan I. Tzenov and Ronald C. Davidson***, "**Hamiltonian formalism for solving the Vlasov-Poisson equations and its applications to periodic focusing systems and coherent beam-beam interaction**," ***Physical Review Special Topics - Accelerators and Beams***, vol. 5, p. 021001, 2002.

is implemented in a **"mixed" phase space (old coordinates and new canonical momenta)**.

In this way, the form of the Poisson equation for the beam-beam potential(s) in Cartesian coordinates is preserved, which is significantly simpler to handle analytically on one hand, and more computationally efficient on the other.

Beam-Beam Interaction Model Involving Symplectic Maps

Why use Symplectic Maps?

The **local nature of beam-beam interaction is an excellent testbed** for the application of the ***symplectic mappings approach***, which is unfortunately relatively **less popular as compared to the Vlasov-Poisson technique**.

In the weak-strong model the beam-beam potential is a ***fixed static electromagnetic element*** → maps can be defined in a straightforward manner. **Pioneering article** → *Alex J. Dragt*, "**Transfer map approach to the beam-beam interaction**," ***In Nonlinear Dynamics and the Beam-Beam Interaction, M. Month and J.C. Herrera, editors, AIP Conference Proceedings***, vol. 57, pp. 143–157, 1980.

A new approach, based on the symplectic twist map method with subsequent regularization of the one-turn beam-beam map, has been developed in → *Stephan I. Tzenov*, "**Renormalization Group Approach to the Beam-Beam Interaction in Circular Colliders**," ***Proceedings of EPAC 2002, Paris, France***, pp. 1422–1424, 2002.

A regularized symplectic beam-beam map has been proposed, which correctly describes the long-term asymptotic behavior of the original dynamical system. It has been shown that the regularized map possesses an integral of motion that can be calculated in any desired order. The invariant density in phase space (stationary distribution function) has been constructed as a generic function of the integral of motion and a coupled system of nonlinear functional equations has been obtained for the distributions of the two colliding beams.

Hamiltonian Description of Beam-Beam Interaction

Two-dimensional model of coherent beam-beam interaction in a plane transversal to the individual particle orbits in each beam is described by the Hamiltonian

$$\mathcal{H}_k = \frac{R}{2}\left(p_x^2 + p_y^2\right) + \frac{1}{2R}\left(G_x^{(k)} x^2 + G_y^{(k)} y^2\right) + \delta_p(\theta)\frac{R q_k}{E_{sk}\beta_{sk}^2}\left(\varphi_{3-k} - c\beta_{sk} A_{s(3-k)}\right)$$

(x, p_x, y, p_y) → canonical conjugate pair of transverse variables, R → mean machine radius

$G_{x,y}^{(k)}$ → linear machine focusing strengths for each beam in the transverse directions

q_k → corresponding particle charges,

E_{sk} and β_{sk} → energy and the relative velocity of the synchronous particle, respectively

$\delta_p(\theta)$ → periodic delta-function

φ_{3-k} and $A_{s(3-k)}$ → scalar and the longitudinal component of the vector potential, respectively.

Hamiltonian Description of Beam-Beam… Continued

In the ultra-relativistic limit

$$\nabla_\perp^2 \varphi_{3-k} = -\frac{q_{3-k} N_{3-k} \varrho_{3-k}}{\epsilon_0}, \qquad \nabla_\perp^2 A_{s(3-k)} = -\mu_0 q_{3-k} N_{3-k} J_{s(3-k)}, \qquad \nabla_\perp^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

Since $J_{s(3-k)} = -c\beta_{s(3-k)} \varrho_{3-k}$ → $A_{s(3-k)} = -\frac{\beta_{s(3-k)}}{c} \varrho_{3-k}$

Appropriate scaling of the beam-beam potential and normalized canonical variables

$$\varphi_k = \frac{q_k N_k \beta_{1k}^*}{4\pi\epsilon_0} V_k, \qquad x = q_1 \sqrt{\beta_{1k}}, \qquad p_x = \frac{1}{\sqrt{\beta_{1k}}} (p_1 - \alpha_{1k} q_1)$$

α-s and β-s → Twiss parameters, β^* → corresponding Twiss parameter at the interaction point.

$$\mathcal{H}_k = \frac{\dot{\chi}_{1k}}{2} \left(p_1^2 + q_1^2\right) + \frac{\dot{\chi}_{2k}}{2} \left(p_2^2 + q_2^2\right) + \delta_p(\theta) \lambda_k V_{3-k}$$

$$\lambda_k = \frac{R r_p Z_k Z_{3-k} N_{3-k} \beta_{1(3-k)}^*}{A_k \gamma_{sk}} \frac{1 + \beta_{sk} \beta_{s(3-k)}}{\beta_{sk}^2}$$

Hamiltonian Description of Beam-Beam… Continued

The normalized beam-beam potential satisfies the Poisson equation

$$\left(\frac{\partial^2}{\partial q_1^2} + \kappa_{3-k}\frac{\partial^2}{\partial q_2^2}\right)V_{3-k} = -4\pi\varphi_{3-k}, \qquad \kappa_{3-k} = \frac{\beta^*_{1(3-k)}}{\beta^*_{2(3-k)}}$$

For the sake of simplicity and clarity, consider the **one-dimensional case in one of the transversal degrees of freedom**.

The particle **distribution function $f_k(q,p;\theta)$** of each beam is a solution to the ***Vlasov equation***

$$\frac{\partial f_k}{\partial \theta} + \dot{\chi}_k p\frac{\partial f_k}{\partial q} - \frac{\partial \mathcal{H}_k}{\partial q}\frac{\partial f_k}{\partial p} = 0$$

the normalized beam density is expressed as

$$\varrho_k(q;\theta) = \int_{-\infty}^{\infty} dp\, f_k(q,p;\theta)$$

The Frobenius-Perron Operator for the Beam-Beam Map

The locality of the beam-beam interaction **suggests a substantial simplification** of the problem.

What is the **Frobenius-Perron Operator**?

Consider a continuous multidimensional finite degree-of-freedom dynamical system (not necessarily Hamiltonian) defined by a state vector ***x(t)***. The evolution is described by the set of equations

$$\frac{dx}{dt} = \mathrm{F}(x, \lambda; t) \quad \rightarrow \quad \textbf{Liouville equation} \quad \frac{\partial f(x;t)}{\partial t} + \nabla \cdot [\mathrm{F}(x,\lambda;t) f(x;t)] = 0$$

Formal solution

$$x(t) = X(x_0, \lambda; t) \rightarrow f(x;t) = \int dz\, \delta[x - X(z,\lambda;t)] f_0(z), \qquad f_0(z) \ - \textbf{initial distribution}$$

For one-dimensional maps of the form $x_{n+1} = F(x_n, \lambda)$

$$f_{n+1}(x) = \Im f_n(x) = \int dz\, \delta[x - F(z,\lambda)] f_n(z)$$

$\Im$ is the **Frobenius-Perron operator**.

The Frobenius-Perron Operator for… Continued

The Frobenius-Perron operator can be written in a **more explicit form** as

$$\widehat{\mathfrak{I}} f_n(x) = \sum_b \frac{f_n\left[F_b^{-1}(x,\lambda)\right]}{\left|F'\left[F_b^{-1}(x,\lambda)\right]\right|}$$

Index b runs over all branches of the inverse map F^{-1} and F' → differentiation with respect to x

The beam-beam map is derived by formally solving the Hamilton's equations of motion

$$\dot{q} = \dot{\chi}_k p, \qquad \dot{p} = -\dot{\chi}_k q - \lambda_k \delta_p(\theta) V'_{3-k}(q;\theta)$$

The result is

$$q_{n+1} = q_n \cos\omega_k + \left[p_n - \lambda_k V'_{3-k}(q_n)\right] \sin\omega_k$$

$$p_{n+1} = -q_n \sin\omega_k + \left[p_n - \lambda_k V'_{3-k}(q_n)\right] \cos\omega_k$$

$\omega_k = 2\pi\nu_k$ ν_k → ***betatron tune related to the k-th beam***.

The Frobenius-Perron Operator for… Continued

The Frobenius-Perron operator can be written as

$$f_k^{(n+1)}(q,p) = \int d\xi d\eta\, \delta\{q - \xi c_k - [\eta - \lambda_k V'_{3-k}(\xi)]s_k\}\delta\{p + \xi s_k - [\eta - \lambda_k V'_{3-k}(\xi)]c_k\} f_k^{(n)}(\xi,\eta)$$

$$c_k = \cos\omega_k \qquad\qquad s_k = \sin\omega_k$$

Manipulate the arguments → $qc_k - ps_k - \xi = 0 \qquad qs_k + pc_k - \eta + \lambda_k V'_{3-k}(\xi) = 0$

The integral becomes trivial, and the final form of the Frobenius-Perron operator is

$$f_k^{(n+1)}(q,p) = f_k^{(n)}[Q, P + \lambda_k V'_{3-k}(Q)]$$

$$\binom{Q}{P} = \mathcal{R}_k^T \binom{q}{p}, \qquad \mathcal{R}_k = \begin{pmatrix} \cos\omega_k & \sin\omega_k \\ -\sin\omega_k & \cos\omega_k \end{pmatrix}$$

The Frobenius-Perron Operator for… Continued

Formal small parameter ϵ and the action-angle variables

$$q = \sqrt{2J}\cos a\,, \qquad p = -\sqrt{2J}\sin a$$

$$J = \frac{1}{2}\left(q^2 + p^2\right), \qquad a = -\arctan\left(\frac{p}{q}\right)$$

The Frobenius-Perron operator becomes

$$f_k^{(n+1)}(a + \omega_k, J) = f_k^{(n)}\left[q, p + \epsilon\lambda_k V'_{3-k}(q)\right]$$

Exactly the same considerations are valid for the counter-circulating beam, for which a similar Frobenius-Perron operator can be derived.

Renormalization Group Reduction of the Frobenius-Perron Operator

Exponentiation of the Frobenius-Perron Operator

$$f_k^{(n+1)}(a+\omega_k, J) = f_k^{(n)}[q, p + \epsilon\lambda_k V'_{3-k}(q)]$$

$$\downarrow$$

$$f_k^{(n+1)}(a+\omega_k, J) = \exp\left[\epsilon\lambda_k \left(\partial_q V_{3-k}\right)\partial_p\right] f_k^{(n)}(a, J)$$

Since the beam-beam potential V_{3-k} **does not depend on the momentum variable** p, we can write

$$\hat{\mathfrak{L}}_{3-k} = \left(\partial_q V_{3-k}\right)\partial_p - \left(\partial_p V_{3-k}\right)\partial_q = \left(\partial_q V_{3-k}\right)\partial_p$$

$\hat{\mathfrak{L}}_{3-k}$ → ***Liouvillian operator associated with*** V_{3-k}. In ***action-angle variables***

$$\hat{\mathfrak{L}}_{3-k} = \left(\partial_a V_{3-k}\right)\partial_J - \left(\partial_J V_{3-k}\right)\partial_a$$

$$f_k^{(n+1)}(a+\omega_k, J) = \exp\left[\epsilon\lambda_k \hat{\mathfrak{L}}_{3-k}\right] f_k^{(n)}(a, J)$$

Renormalization Group Reduction of the Frobenius-… Continued

Premise for the time being that the beam-beam potential $V_k(q)$ is a **known function** of q.

The **Fourier image** of the beam-beam potential $\widetilde{V}_k(\lambda)$, defined as

$$V_k(q) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda \, \widetilde{V}_k(\lambda) e^{i\lambda q}, \qquad \widetilde{V}_k(\lambda) = \int_{-\infty}^{\infty} dq \, V_k(q) e^{-i\lambda q}$$

possesses the following **symmetry property** → $\widetilde{V}_k^*(\lambda) = \widetilde{V}_k(-\lambda)$

The Fourier image can be written as

$$\widetilde{V}_k(\lambda) = \frac{4\pi}{\lambda^2} \int_{-\infty}^{\infty} dq dp \, f_k(q,p) e^{-i\lambda q} = \frac{4\pi}{\lambda^2} \int_0^{\infty} dJ \int_0^{2\pi} da \, f_k(a,J) e^{-i\lambda\sqrt{2J}\cos a}$$

Using the Jacobi-Anger expansion

$$e^{iz\cos\varphi} = \sum_{m=-\infty}^{\infty} i^m \mathcal{J}_m(z) e^{im\varphi}, \qquad \mathcal{J}_m(z) \quad \rightarrow \quad \textbf{Bessel function of first kind}$$

Renormalization Group Reduction of the Frobenius-… Continued

Represent the beam-beam potential in a Fourier series in the angle variable as follows

$$V_k(a,J) = V_k^{(0)}(J) + V_k^a(a,J) = V_k^{(0)}(J) + 2\sum_{m=1}^{\infty} V_k^{(m)}(J)\cos a$$

where

$$V_k^{(0)}(J) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\,\widetilde{V}_k(\lambda)\mathcal{J}_0\left(\lambda\sqrt{2J}\right), \qquad V_k^{(m)}(J) = \frac{i^m}{2\pi}\int_{-\infty}^{\infty} d\lambda\,\widetilde{V}_k(\lambda)\mathcal{J}_m\left(\lambda\sqrt{2J}\right)$$

If **both rotation frequencies ω_k are far from nonlinear resonances** excited by the beam-beam potentials V_k the Frobenius-Perron operator can be ***renormalized***. This can be done also when **one or both ω_k are relatively close to certain structural resonance(s)** driven by the beam-beam potentials.

Details → ***Stephan I. Tzenov***, "**Stochastic Properties of the Frobenius- Perron Operator**," ***arXiv:nlin/0606003, p. 14 pages, 2006***.

Renormalization Group Reduction of the Frobenius-… Continued

Consider the non-resonant case for the renormalized amplitude of the distribution function in the continuous limit

$$\frac{\partial \widetilde{F}_k}{\partial n} = -\omega_k \partial_a \widetilde{F}_k + \left[\lambda_k \widehat{\mathfrak{L}}^{(0)}_{3-k} + \lambda_k^2 \left(\frac{1}{2} \widehat{\mathfrak{L}}^{(0)2}_{3-k} + \Omega_{3-k} \partial_a\right)\right] \widetilde{F}_k$$

$\widehat{\mathfrak{L}}^{(0)}_{3-k} = -\omega^{(u)}_{3-k}(J)\partial_a$, $\quad \omega^{(u)}_{3-k}(J) = \partial_a V^{(0)}_{3-k}(J)$ → ***nonlinear first-order incoherent tune-shift***

$$\Omega_{3-k}(\omega_k, J) = \sum_{m=1}^{\infty} m \cot\left(\frac{m\omega_k}{2}\right) \partial_J \left(V^{(m)}_{3-k} \partial_J V^{(m)}_{3-k}\right) \quad \rightarrow \quad \textbf{second–order incoherent tune–shift}$$

$$f^{(n)}_k(a, J) \sim \widetilde{F}_k(a - n\omega_k, J; n)$$

The diffusion equation exhibits a very important and far-reaching property - there exists an equilibrium solution for the renormalized distribution function $\widetilde{F}^{(0)}_k(J)$, which depends only on the action variables. Moreover, there exist a damping mechanism acting on the fluctuation harmonics with respect to the angle variables, such that the general solution of the Fokker-Planck equation rapidly relaxes towards the invariant density distribution.

Renormalization Group Reduction of the Frobenius-… Continued

Relaxation rate to the invariant distribution depends on the first-order incoherent tune-shift

$$\omega_{3-k}^{(u)}(J) = -\frac{4}{\sqrt{2J}} \int dq dp\, f_{3-k}(q,p) \int_0^\infty \frac{d\lambda}{\lambda} \mathcal{J}_1\left(\lambda\sqrt{2J}\right) \cos(q\lambda)$$

Second integral (with respect to λ) is tabular

$$\int_0^\infty \frac{dx}{x} \mathcal{J}_n(cx) \begin{Bmatrix} \sin bx \\ \cos bx \end{Bmatrix} = \frac{1}{n} \begin{Bmatrix} \sin[n\arcsin(b/c)] \\ \cos[n\arcsin(b/c)] \end{Bmatrix}, \qquad for \qquad 0 < b \leq c$$

$$\omega_{3-k}^{(u)}(J) = -\frac{4}{\sqrt{2J}} \int_{-\infty}^{\infty} dp \int_{-\sqrt{2J}}^{\sqrt{2J}} dq\, f_{3-k}(q,p) \cos\left[\arcsin\left(\frac{q}{\sqrt{2J}}\right)\right]$$

Renormalization Group Reduction of the Frobenius-… Continued

Taking into account the equilibrium distribution function

$$f_{3-k}^{(0)}(q,p) = \frac{1}{2\pi\sigma_{3-k}^2} exp\left(-\frac{p^2+q^2}{2\sigma_{3-k}^2}\right)$$

Representation of the modified Bessel function

$$\mathfrak{T}_n(z) = \frac{1}{\pi}\int_0^{\pi} d\tau\, e^{z\cos\tau}\cos(n\tau)$$

Finally

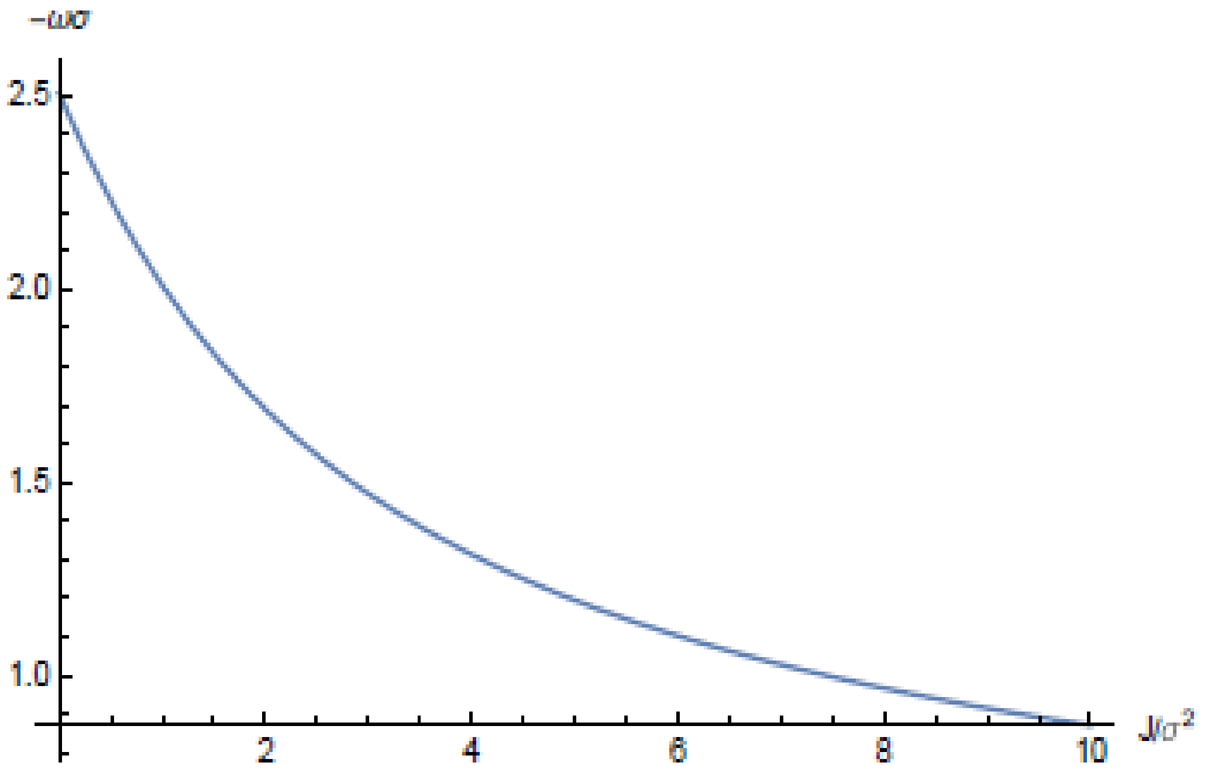
$$\omega_{3-k}^{(u)}(J) = -\frac{\sqrt{2\pi}}{\sigma_{3-k}}\left[\mathfrak{T}_0\left(\frac{J}{2\sigma_{3-k}^2}\right)+\mathfrak{T}_1\left(\frac{J}{2\sigma_{3-k}^2}\right)\right] exp\left(-\frac{J}{2\sigma_{3-k}^2}\right)$$

It is sometimes useful in practice to evaluate the averaged incoherent tune shift

$$\langle\omega_{3-k}\rangle = (2\pi)\int_0^{\infty} dJ\, f_{3-k}^{(0)}(J)\omega_{3-k}^{(u)}(J) = -\frac{(2\pi)}{2\sigma_{3-k}\sqrt{\pi}}\frac{4+2\sqrt{2}}{3+2\sqrt{2}}$$

Renormalization Group Reduction of the Frobenius-… Continued

Dependence of the **first-order incoherent tune shift $-\omega_k^{(u)}\sigma_k$** as a function of the **action variable J/σ_k^2**.



For typical characteristic parameters of the magnetic structure for the NICA collider and the number of particles in each of the beams $N_k \sim 4 \times 10^9$, the Incoherent tune shift is of the order of $\lambda_k \langle \omega_{3-k} \rangle \sim 0.016$.

Linearized Frobenius-Perron Operator and Stability of Coherent Beam-Beam Resonances

Temporal evolution of the dynamic motions of the beam distributions around the equilibrium distributions $G_k(J)$

$$f_k^{(n)}(a, J) = \mathcal{F}_k^{(n)}(a, J) + G_k(J)$$

Substituting the above ansatz into the Frobenius-Perron operator and retaining only the first order terms in $\mathcal{F}_k^{(n)}$, we obtain the linearized Frobenius-Perron operator

$$\mathcal{F}_k^{(n+1)}(a + \omega_k, J) = \mathcal{F}_k^{(n)}\left(a - \lambda_k \omega_{3-k}^{(u)}, J\right) + \lambda_k \left[\partial_a \mathcal{V}_{3-k}^{(n)}\left(a - \lambda_k \omega_{3-k}^{(u)}, J\right)\right] G_k'(J)$$

Here

$$\mathcal{V}_k^{(n)}(a, J) = 2 \int_{-\infty}^{\infty} \frac{d\lambda}{\lambda^2} e^{i\lambda\sqrt{2J}\cos a} \int da' dJ' \, \mathcal{F}_k^{(n)}(a', J') e^{-i\lambda\sqrt{2J'}\cos a'}$$

This is a ***recurrence Fredholm integral equation of second type***.

Linearized Frobenius-Perron Operator and Stability of Coherent… Continued

To solve the linear recurrence equation introduce the Fourier transform

$$\mathcal{F}_k^{(n)}(a,J) = G_k(J) \sum_{l=-\infty}^{\infty} g_k^{(l)}(J,n) e^{ila}$$

Assuming the equilibrium distribution function $G_k(J)$ to be of the form

$$G_k(J) = \frac{1}{2\pi\sigma_k^2} exp\left(-\frac{J}{\sigma_k^2}\right)$$

for small beam sizes σ_k, use the following formal trick

$$G_k(J)G_{3-k}(J') = \mathfrak{C}_k exp\left(-\frac{J}{\sigma_k^2} - \frac{J'}{\sigma_{3-k}^2}\right) = \mathfrak{C}_k exp\left(-\frac{J'}{\sigma_{3-k}^2} + \frac{J'}{\sigma_k^2} - \frac{2\sqrt{JJ'}}{\sigma_k^2}\right)$$

$$\times exp\left[-\frac{\left(\sqrt{J} - \sqrt{J'}\right)^2}{\sigma_k^2}\right] \sim \sigma_k \sqrt{\pi} G_k(J) G_{3-k}(J') \delta\left(\sqrt{J} - \sqrt{J'}\right)$$

The above expression can be symmetrized with respect to the sizes of both beams

Linearized Frobenius-Perron Operator and Stability of Coherent… Continued

$$G_k(J)G_{3-k}(J') = \bar{\sigma}\sqrt{\pi}G_k(J)G_{3-k}(J')\delta\left(\sqrt{J} - \sqrt{J'}\right), \qquad \bar{\sigma} = \frac{1}{2}(\sigma_1 + \sigma_2)$$

Equating similar harmonics with respect to the angle variable in the linearized Frobenius-Perron operator, we obtain

$$G_k(J)g_k^{(l)}(n+1) = e^{-il\left(\omega_k + \lambda_k \omega_{3-k}^{(u)}\right)} G_k(J)\left[g_k^{(l)}(n) + 2J\lambda_k \frac{\bar{\sigma}}{\sigma_k^2}\sqrt{2\pi}G_{3-k}(J)\sum_{m=-\infty}^{\infty}\mathfrak{M}_{lm}\, g_{3-k}^{(m)}(n)\right] \quad (*)$$

The infinite matrix $\mathfrak{M}_{lm}$ can be expressed as

$$\mathfrak{M}_{lm} = \frac{32il}{[(l+m)^2 - 1][(l-m)^2 - 1]}, \qquad l+m = \text{even} \quad \rightarrow \quad \mathfrak{M}_{lm} = 0, \qquad l+m = \text{odd}$$

If $g_k^{(l)}(n)$ does **not depend on the action variable**, equation (*) can be simplified by integrating away the action variable. This approximation however, is valid if and only if the perturbed betatron tunes $\omega_{3-k}^{(u)}$ **do not depend on the action** J, which obviously is not the case. The dependence on the action variable ***leads to an effect similar to Landau damping, well-known in plasma physics***, which we shall neglect.

Linearized Frobenius-Perron Operator and Stability of Coherent… Continued

Another **justification for the validity of such an approximation** is the **rapid decrease of the incoherent tune shift as a functional dependence on the action variable *J*** clearly visible in Figure. Thus, the first-order incoherent tune shift can be approximately replaced by its average value

$$g_k^{(l)}(n+1) = e^{-il(\omega_k + \lambda_k \langle \omega_{3-k} \rangle)} \left[g_k^{(l)}(n) + \tilde{\lambda}_k \sum_{m=-\infty}^{\infty} \mathfrak{M}_{lm}\, g_{3-k}^{(m)}(n) \right]$$

$$\tilde{\lambda}_k = \sqrt{\frac{2}{\pi}} \lambda_k \frac{\overline{\sigma} \sigma_{3-k}^2}{\Sigma^4}, \qquad \Sigma^2 = \sigma_1^2 + \sigma_2^2$$

Consider now an isolated coherent beam-beam resonance of the form

$$n_1 \widetilde{\omega}_1 + n_2 \widetilde{\omega}_2 = 2\pi s + \Delta, \qquad \widetilde{\omega}_k = \omega_k + \lambda_k \langle \omega_{3-k} \rangle$$

To study the stability of the **isolated coherent beam-beam resonance**, we retain only the $\pm n_1$ and the $\pm n_2$ elements in the infinite matrix $\mathfrak{M}_{lm}$

Linearized Frobenius-Perron Operator and Stability of Coherent… Continued

The transformation matrix of the coupled map equations can be expressed as

$$\begin{pmatrix} exp(-i\psi_1) & 0 & \alpha_1 exp(-i\psi_1) & \alpha_1 exp(-i\psi_1) \\ 0 & exp(i\psi_1) & -\alpha_1 exp(i\psi_1) & -\alpha_1 exp(i\psi_1) \\ \alpha_2 exp(-i\psi_2) & \alpha_2 exp(-i\psi_2) & exp(-i\psi_2) & 0 \\ -\alpha_2 exp(i\psi_2) & -\alpha_2 exp(i\psi_2) & 0 & exp(i\psi_2) \end{pmatrix}$$

$$\psi_k = n_k \tilde{\omega}_k, \qquad \alpha_1 = \tilde{\lambda}_1 \mathfrak{M}_{n_1 n_2}, \qquad \alpha_2 = \tilde{\lambda}_2 \frac{n_2}{n_1} \mathfrak{M}_{n_1 n_2}$$

The eigenvalues of the transition matrix are the roots of the secular equation

$$(\mu^2 - 2c_1\mu + 1)(\mu^2 - 2c_2\mu + 1) = 0,$$

roots $\Rightarrow \quad c_{1,2} = \frac{1}{2}(\cos\psi_1 + \cos\psi_2) \pm \frac{1}{2}\sqrt{(\cos\psi_1 - \cos\psi_2)^2 - 4A \sin\psi_1 \sin\psi_2},$

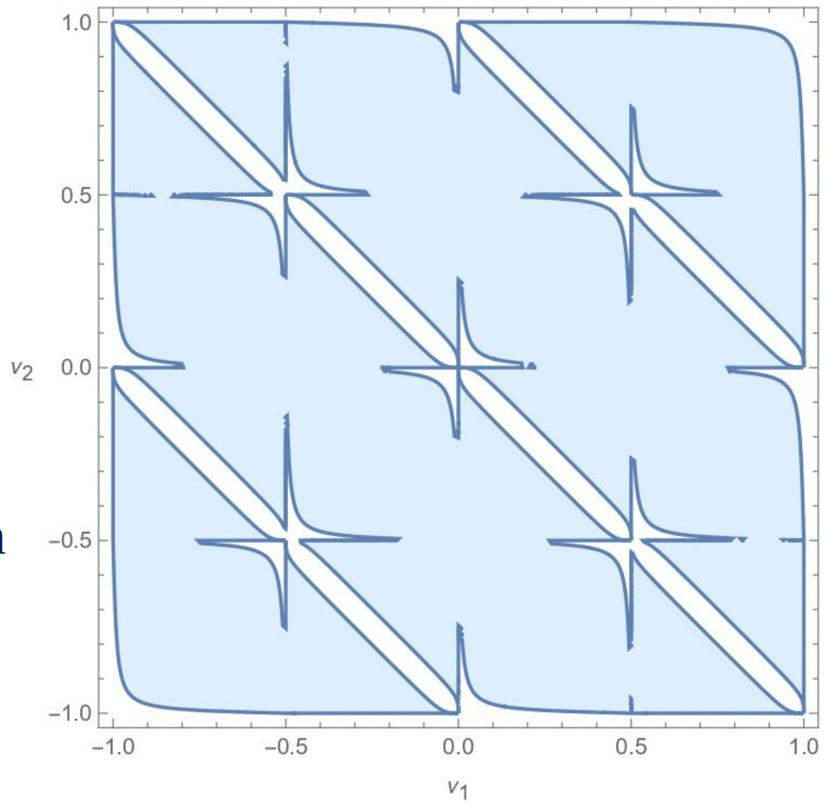
where $A = \tilde{\lambda}_1 \tilde{\lambda}_2 \frac{n_2}{n_1} \mathfrak{M}^2_{n_1 n_2}$

Linearized Frobenius-Perron Operator and Stability of Coherent… Continued

The motion is stable if the coefficients $c_{1,2}$ simultaneously satisfy the conditions

$$-1 \leq c_{1,2} \leq 1$$

Linear beam-beam coupling resonance $\tilde{\omega}_1 + \tilde{\omega}_2 = 2\pi s + \Delta$ in the space of the fractional part of the shifted betatron tunes. For a better clarity of the structure and shape of the islands of instability, an increased value of the **beam-beam parameter** λ_k corresponding to $N_k \sim 4 \times 10^{10}$ number of particles in each beam has been taken.

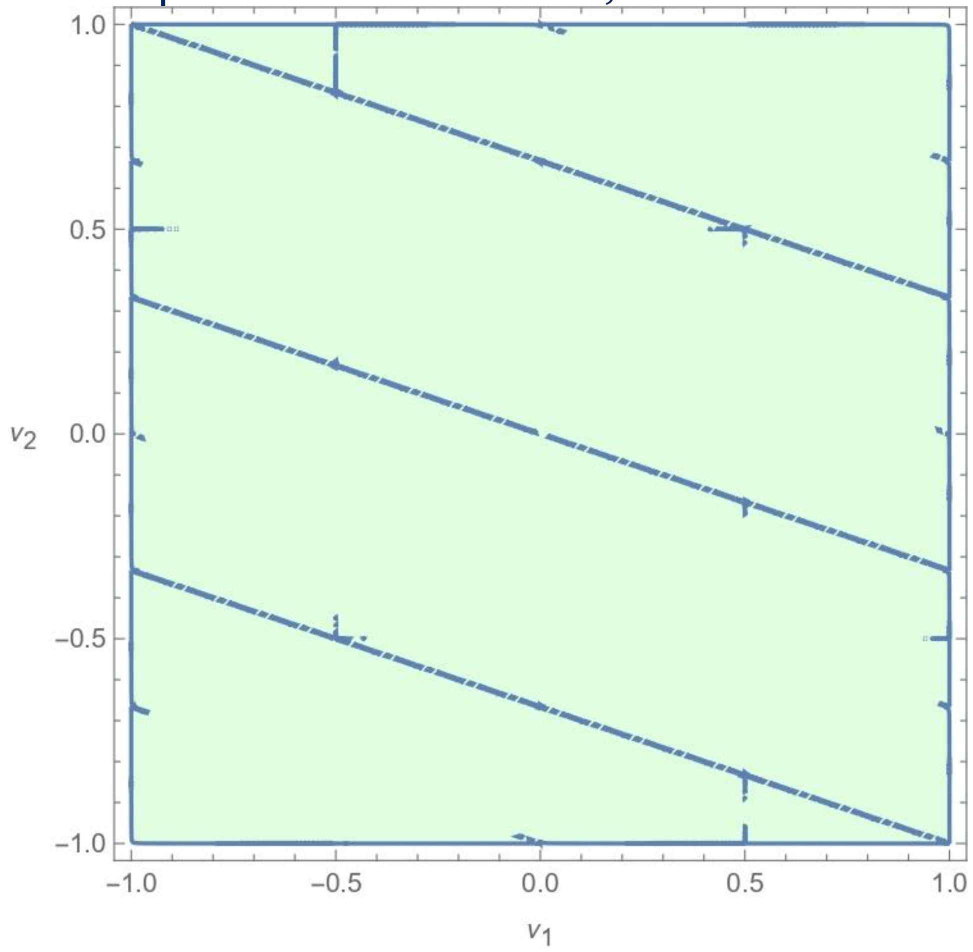


Linearized Frobenius-Perron Operator and Stability of Coherent… Continued

Note that only nonlinear beam-beam resonances of even order are possible. Furthermore, the elements of the infinite matrix $\mathfrak{M}_{lm}$ decrease quite rapidly with the resonance order, which leads to a drastic reduction of the resonant driving term.

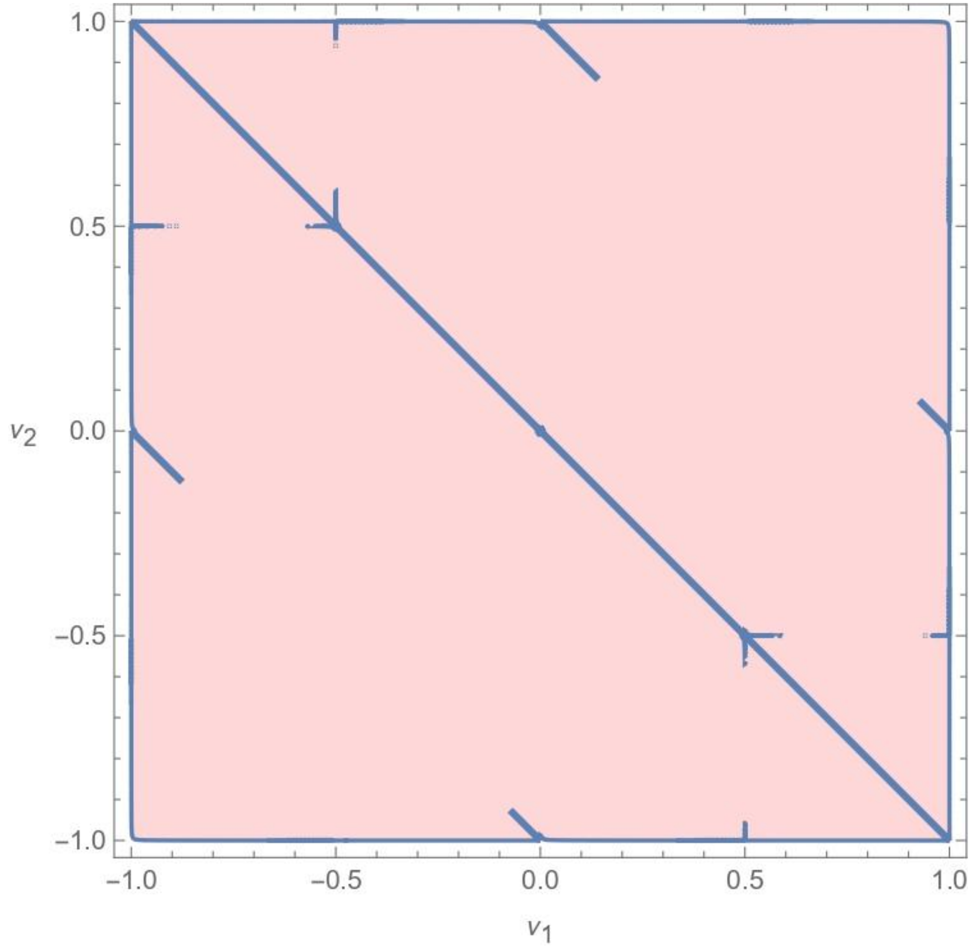
The figure shows the stability diagram in the case of fourth-order coherent nonlinear beam-beam resonance $\tilde{\omega}_1 + 3\tilde{\omega}_2 = 2\pi s + \Delta$. The instability region consists of narrow resonance stopbands together with islands of instability scattered around them. There is sufficiently wide band of stability, which greatly facilitates the felicitous selection of the operating betatron tunes.

In this sense, nonlinear coherent beam-beam resonances are significantly less dangerous than the linear coupling resonance.



Linearized Frobenius-Perron Operator and Stability of Coherent… Continued

The Figure presents the realistic situation showing the stability diagram of the linear coherent beam-beam resonance at a value of the beam-beam parameter $\boldsymbol{\lambda_k \sim 4.7712 \times 10^{-6}}$ corresponding to $\boldsymbol{N_k \sim 4 \times 10^{9}}$ number of particles in each beam. A central narrow resonance stopband and scattered satellite narrow stopbands and small islands of instability are clearly visible.



What has been achieved and what more can be done

- **An innovative, unconventional approach** to the problem of beam-beam interaction.
- Detailed analysis of the **establishment of an equilibrium density distribution** in phase space and the relaxation towards the latter has been studied analytically.
- The **behavior of the perturbed from equilibrium distribution function** with respect to the coherent stability of the colliding beams, is carried out in linear approximation.
- The **Renormalization Group (RG) method has been applied to study the stochastic properties of the Frobenius-Perron operator** for symplectic twist maps of the most general type and in particular for the beam-beam twist map.
- It has been shown that up to second order in the beam-beam perturbation kick, the **renormalized map propagator (equivalently, the renormalized Frobenius-Perron operator) with nonlinear stabilization describes a random walk of the angle variable**.
- The **incoherent beam-beam tune shift as a function of the action variable** has been calculated explicitly.

What has been achieved and what more can be done

- **The linearized Frobenius-Perron operator for each of the two beams actually implies a discrete form of the linearized Vlasov equations**.

- This essentially is equivalent to and signifies a **new method for calculating coherent beam-beam instabilities using a matrix mapping technique**. **In the special case of an isolated coherent beam-beam resonance, a stability criterion for coherent beam-beam resonances has been found in closed form**.

What has been achieved and what more can be done

What next:

- The Frobenius-Perron operator approach ***can be generalized without much difficulty to systems with more than one degree of freedom, so as to cover both transverse directions and, if necessary, the longitudinal degree of freedom as well***.
- Combined with an **adequate Poisson solver, the Frobenius-Perron operator, especially in its Cartesian coordinate and momentum representation**, can represent a **tool of particular value for the numerical simulation of the beam-beam interaction**.
- Its **numerical implementation** may provide a wonderful opportunity not only to track the orbits of individual particles, but also to **follow and describe the dynamic evolution of an entire statistical distribution of an ensemble of particles**.


Thank you for your
attention!