Quantum corrections to the Classical Statistical Approximation within the Keldysh-Schwinger technique

Anna Radovskaya and Andrey Leonidov

Lebedev Physical Institute of RAS, Moscow, Russia

September 4, 2018

Motivation

- Evolution of nonequilibrium quantum field. Semiclassical systems: BEC in cold atoms, preheating processes in the early Universe, **the first stage of HIC** and ....
- Classical Statistical Approximation (CSA): averaging of system's classical trajectories with some weight function. Averaging $\to$ equilibrium.
- Quantum correction to the CSA: due to expansion the system become more quantum.
- CSA is LO term of semiclassical decomposition within the Keldysh-Schwinger technique. Quantum corrections to the CSA $\to$ NLO term.

outline

Keldysh contour

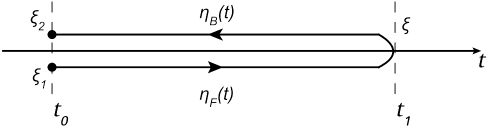
- Highly excited initial state (not vacuum) $\rightarrow$ density matrix

$$\hat{\rho}(t) = \sum_i P_i \, |\psi_i(t)\rangle\langle\psi_i(t)|$$

- Physical observable $\langle F(\hat{\varphi})\rangle = tr(F(\hat{\varphi})\hat{\rho})$

- Nonequilibrium evolution from t_0 to t_1:

$$\hat{\rho}(t) = \hat{U}(t, t_0)\hat{\rho}(t_0)\hat{U}^{\dagger}(t, t_0)$$



Path-integral representation

Observable

$$\langle F(\hat\varphi)\rangle_{t_1} = \int \mathfrak{D}\xi \int \mathfrak{D}\xi_1 \int \mathfrak{D}\xi_2 \; \langle \xi_1|\hat\rho(t_0)|\xi_2\rangle$$

$$\times\; F(\xi) \int_{\eta_F(t_0,\vec{x})=\xi_1(\vec{x})}^{\eta_F(t_1,\vec{x})=\xi(\vec{x})} \mathcal{D}\eta_F(t,\vec{x}) \int_{\eta_B(t_0,\vec{x})=\xi_2(\vec{x})}^{\eta_B(t_1,\vec{x})=\xi(\vec{x})} \mathcal{D}\eta_B(t,\vec{x})\; e^{iS_K[\eta_F,\eta_B]}.$$

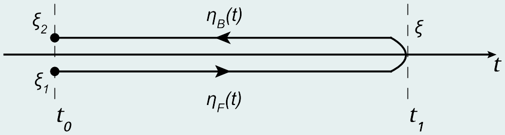
notations

$$\hat\varphi(\vec{x})|\xi\rangle = \xi(\vec{x})|\xi\rangle$$

$$\hat 1 = \int \mathfrak{D}\xi(\vec{x})\; |\xi\rangle\langle\xi|$$

$$S_K[\eta_F,\eta_B] = S[\eta_F] - S[\eta_B]$$

Keldysh contour



General formula

Classical and quantum components are:

$$\phi_c = \frac{\eta_F + \eta_B}{2}, \qquad \phi_q = \eta_F - \eta_B.$$

For almost any field theory

$$\langle F(\hat{\varphi})\rangle_{t_1} = \int \mathfrak{D}\chi_1 \int \mathfrak{D}\xi_1 \int \mathfrak{D}\xi_2 \; \langle \xi_1|\hat{\rho}(t_0)|\xi_2\rangle$$

$$\times \int_{\phi_c(t_0,\vec{x})=\frac{\xi_1(\vec{x})+\xi_2(\vec{x})}{2}}^{\phi_c(\infty,\vec{x})=\chi_1(\vec{x})} \mathcal{D}\phi_c \int_{\phi_q(t_0,\vec{x})=\xi_1(\vec{x})-\xi_2(\vec{x})}^{\phi_q(\infty,\vec{x})=0} \mathcal{D}\phi_q \; F(\phi_c(t_1))\; e^{iS_K[\phi_c,\phi_q]}$$

Scalar φ^4 theory

QCD $\to \varphi^4$ for simplicity

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{g^2}{4}\varphi^4 + J\varphi$$

J - auxiliary source

The Keldysh action

$$S_K[\phi_c,\phi_q] = \int d^3x\ \dot{\phi}_c(t_0,\vec{x})\phi_q(t_0,\vec{x}) - \int_{t_0}^{\infty} dt \int d^3x \left[\phi_q)\big(\underbrace{\partial_\mu\partial^\mu\phi_c + g^2\phi_c^3 - J}_{equation\ \ of\ \ motion}\big) - \frac{g^2}{4}\phi_c\phi_q^3\right]$$

- remember about Plank constant
- substitute $\phi_q \to \hbar\phi_q$
- use semiclassical decomposition

Scalar φ^4 theory

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J - auxiliary source

The Keldysh action

$$\frac{1}{\hbar}S_K[\phi_c,\phi_q] = \frac{1}{\hbar}\int d^3x\ \dot{\phi}_c(t_0,\vec{x})\phi_q(t_0,\vec{x})$$
$$-\frac{1}{\hbar}\int_{t_0}^{\infty} dt \int d^3x \left[\phi_q\left(\underbrace{\partial_\mu\partial^\mu\phi_c + g^2\phi_c^3 - J}_{equation\ \ of\ \ motion}\right) - \frac{g^2}{4}\phi_c\phi_q^3\right]$$

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$$-\int_{t_0}^{\infty} dt \int d^3x \left[\phi_q\left(\underbrace{\partial_\mu\partial^\mu\phi_c + g^2\phi_c^3 - J}_{equation\ \ of\ \ motion}\right) - \hbar^2\frac{g^2}{4}\phi_c\phi_q^3\right]$$

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- remember about Plank constant
- substitute $\phi_q \rightarrow \hbar\phi_q$
- use semiclassical decomposition

Semiclassical decomposition

$$e^{-i\frac{g^2\hbar^2}{4}\int\limits_{t_0}^{\infty} dt\int d^3x\ \phi_c\phi_q^3} = \underbrace{1}_{LO} - \underbrace{\frac{ig^2\hbar^2}{4}\int\limits_{t_0}^{\infty} dt\int d^3x\ \phi_c\phi_q^3}_{NLO} + \ldots$$

Leading Order

$$\langle F(\hat{\varphi})\rangle_{t_1}^{LO} = \int \mathfrak{D}\chi_{Init} \int\limits_{\phi_c(t_0,\vec{x})=\frac{\xi_1(\vec{x})+\xi_2(\vec{x})}{2}}^{\phi_c(\infty,\vec{x})=\chi_1(\vec{x})} \mathcal{D}\phi_c\ e^{i\int d^3x\ \dot{\phi}_c(t_0,\vec{x})(\xi_1(\vec{x})-\xi_2(\vec{x}))}$$

$$\times\ F(\phi_c(t_1)) \int\limits_{\phi_q(t_0,\vec{x})=\xi_1(\vec{x})-\xi_2(\vec{x})}^{\phi_q(\infty,\vec{x})=0} \mathcal{D}\phi_q e^{i\int dt\ d^3x\ \phi_q\left(\partial_\mu\partial^\mu\phi_c+g^2\phi_c^3-J\right)}$$

Notation $\int\mathfrak{D}\chi_{Init} = \int\mathfrak{D}\chi_1\int\mathfrak{D}\xi_1\int\mathfrak{D}\xi_2\ \langle\xi_1|\hat{\rho}(t_0)|\xi_2\rangle$

- integration over ϕ_q

Leading Order

calculation

$$\langle F(\hat{\varphi})\rangle^{LO}_{t_1} = \int \mathfrak{D}p(\vec{x}) \int \mathfrak{D}\chi_{Init} \int\limits_{\phi_c(t_0,\vec{x})=\frac{\xi_1(\vec{x})+\xi_2(\vec{x})}{2}}^{\phi_c(\infty,\vec{x})=\chi_1(\vec{x})} \mathcal{D}\phi_c \; e^{i\int d^3x\; \dot{\phi}_c(t_0,\vec{x})(\xi_1(\vec{x})-\xi_2(\vec{x}))}$$
$$\times \; F(\phi_c(t_1))\; \delta\left[\partial_\mu\partial^\mu\phi_c + g^2\phi_c^3 - J\right]\delta(p(\vec{x}) - \dot{\phi}_c(t_0,\vec{x}))$$

- integration over ϕ_q
- insert "initial velocity" unity $1 = \int \mathfrak{D}p(\vec{x})\delta(p(\vec{x}) - \dot{\phi}_c(t_0,\vec{x}))$
- perform integration over ϕ_c with help of EoM and initial conditions $\phi_c \to \phi_{cl}$
- change variables to $\alpha = \frac{\xi_1+\xi_2}{2}$, $\beta = \xi_1 - \xi_2$

Classical Statistical Approximation

- find solution of the classical EoM
- calculate observable on this solution

- average over all initial conditions with weight of the Wigner functional

$$\langle F(\hat{\varphi})\rangle_{t_1} = \int \mathfrak{D}\alpha(\vec{x})\mathfrak{D}p(\vec{x}) f_w[\alpha(\vec{x}), p(\vec{x})), t_0] F(\phi_{cl}(t_1))$$

$$f_w[\alpha(\vec{x}), p(\vec{x}), t_0] = \int \mathfrak{D}\beta(\vec{x})\langle \alpha + \frac{\beta}{2}|\hat{\rho}(t_0)|\alpha - \frac{\beta}{2}\rangle e^{i\int d^3x \; p(\vec{x})\beta(\vec{x})}$$

$$\partial_\mu\partial^\mu\phi_{cl} + g^2\phi_{cl}^3 = 0$$

$$\phi_{cl}(t_0, \vec{x}) = \alpha(\vec{x})$$

$$\dot{\phi}_{cl}(t_0, \vec{x}) = p(\vec{x})$$

-We should define the Wigner function at the initial time
-We do not need the small coupling constant for the CSA

-Where the CSA works?

Notation

We introduce notation for averaging over initial condition with the Wigner functional as

$$\langle \mathcal{O} \rangle_{i.c.} = \int \mathfrak{D}\alpha(\vec{x})\mathfrak{D}p(\vec{x}) f_w[\alpha(\vec{x}), p(\vec{x})), t_0]\ \mathcal{O}$$

So, the LO answer can be written simply as

$$\langle F(\hat{\varphi}) \rangle_{t_1}^{LO} = \langle\ F(\phi_{cl}(t_1))\ \rangle_{i.c.}$$

Auxiliary source at work

Semiclassical decomposition

$$e^{-i\frac{g^2}{4}\int_{t_0}^{\infty} dt \int d^3x\ \phi_c\phi_q^3} = \underbrace{1}_{LO} \underbrace{- \frac{ig^2}{4}\int_{t_0}^{\infty} dt \int d^3x\ \phi_c\phi_q^3}_{NLO} + \ldots$$

Keldysh action reminder

$$S_K = \int d^3x\, \dot{\phi}_c(t_0,\vec{x})\phi_q(t_0,\vec{x}) - \int_{t_0}^{\infty} dt \int d^3x \Big[\underbrace{\phi_q \left(\partial_\mu\partial^\mu\phi_c + g^2\phi_c^3 - J\right)}_{auxiliary\ \ source\ \ term} - \frac{g^2}{4}\phi_c\phi_q^3 \Big]$$

Functional derivative

$$\frac{\delta}{\delta J(t',\vec{x}')} e^{iS_K[\phi_c,\phi_q]} = i\phi_q(t',\vec{x}')e^{iS_K[\phi_c,\phi_q]}$$

Quantum corrections

LO+NLO terms

$$\langle F(\hat{\varphi})\rangle_{t_1}^{LO+NLO} = \left\langle \underbrace{F(\phi_{cl}(t_1,\vec{x}))}_{LO} + \underbrace{\frac{g^2}{4}\int_{t_0}^{t_1} dt' \int d^3x' \phi_{cl}(t',\vec{x}') \frac{\delta^3 F(\phi_{cl}(t_1,\vec{x}))}{\delta J^3(t',\vec{x}')}\bigg|_{J=0}}_{NLO} \right\rangle_{i.c.}$$

Coupling constant is not small

All terms

$$\langle F(\hat{\varphi})\rangle_{t_1} = \left\langle \bar{T} e^{\frac{g^2}{4}\int d\tau d\vec{y}\, \phi_{cl}(\tau,\vec{y}) \frac{\delta^3}{\delta J^3(\tau,\vec{y})}} F(\phi_{cl}(t_1,\vec{x})) \right\rangle_{i.c.}$$

Attention: Not for numerical use

For numerical use

Let us define k-th variation of the classical solution over source J as

$$\frac{\delta^k \phi_{cl}(t_1, \vec{x}_1)}{\delta J^k(t_2, \vec{x}_2)} = \Phi_k(t_1, \vec{x}_1; t_2, \vec{x}_2).$$

Then

$$\frac{\delta^3 F(\phi_{cl}(t_1, \vec{x}_1))}{\delta J^3(t_2, \vec{x}_2)} = \frac{\partial F}{\partial \phi_{cl}} \Phi_3 + 3 \frac{\partial^2 F}{\partial \phi_{cl}^2} \Phi_1 \Phi_2 + \frac{\partial^3 F}{\partial \phi_{cl}^3} \Phi_1^3.$$

$\Phi_k(t_1, \vec{x}_1; t_2, \vec{x}_2)$ can be found by variation of the classical EoM

$$\frac{\delta^3}{\delta J^3(t_2, \vec{x}_2)} \left(\partial_\mu \partial^\mu \phi_{cl}(t_1, \vec{x}_1) + g^2 \phi_{cl}^3(t_1, \vec{x}_1) = J(t_1, \vec{x}_1) \right),$$

Quantum corrections (numerical)

$$\langle F(\hat\varphi)\rangle_{t_1}^{LO+NLO} = \left\langle F(\phi_{cl}(t_1,\vec{x})) + \frac{g^2}{4}\int_{t_0}^{t_1} dt' \int d^3x' \phi_{cl}(t',\vec{x}')\frac{\delta^3 F(\phi_{cl}(t_1,\vec{x}))}{\delta J^3(t',\vec{x}')}\bigg|_{J=0}\right\rangle_{i.c.}$$

Variations

$$\frac{\delta^3 F(\phi_{cl}(t_1,\vec{x}_1))}{\delta J^3(t_2,\vec{x}_2)} = \frac{\partial F}{\partial \phi_{cl}}\Phi_3 + 3\frac{\partial^2 F}{\partial \phi_{cl}^2}\Phi_1\Phi_2 + \frac{\partial^3 F}{\partial \phi_{cl}^3}\Phi_1^3.$$

$$L_{t_1}\Phi_1(t_1,\vec{x}_1;t_2,\vec{x}_2) = \delta(t_1-t_2)\delta^{(3)}(\vec{x}_1-\vec{x}_2)$$

$$L_{t_1}\Phi_2(t_1,\vec{x}_1;t_2,\vec{x}_2) = -6g^2\phi_{cl}(t_1,\vec{x}_1)\Phi_1^2(t_1,\vec{x}_1;t_2,\vec{x}_2)$$

$$L_{t_1}\Phi_3(t_1,\vec{x}_1;t_2,\vec{x}_2) = -6g^2\Phi_1^3(t_1,\vec{x}_1;t_2,\vec{x}_2) - 18g^2\phi_{cl}(t_1,\vec{x}_1)\Phi_1(t_1,\vec{x}_1;t_2,\vec{x}_2)\Phi_2(t_1,\vec{x}_1;t_2,\vec{x}_2)$$

$$L_{t_1} = \partial_{t_1}^2 - \partial_{\vec{x}_1}^2 + 3g^2\phi_{cl}^2(t_1,\vec{x}_1)$$

Averaging over the initial conditions

$$f_w[\alpha(\vec{x}),p(\vec{x}),t_0] = \int \mathfrak{D}\beta(\vec{x})\langle \alpha + \frac{\beta}{2}|\hat\rho(t_0)|\alpha - \frac{\beta}{2}\rangle e^{i\int d^3x\, p(\vec{x})\beta(\vec{x})}$$

$$\partial_\mu\partial^\mu\phi_{cl} + g^2\phi_{cl}^3 = 0$$

$$\phi_{cl}(t_0,\vec{x}) = \alpha(\vec{x})$$

$$\dot\phi_{cl}(t_0,\vec{x}) = p(\vec{x})$$

Static box

Spatially homogeneous case $\partial_i \varphi(t, \mathbf{x}) = 0$

$$S = V \int dt \left(\frac{1}{2} \dot{\varphi}^2 - \frac{g^2}{4} \varphi^4 + J\varphi \right), \qquad V = \int d^3x$$

Equation of motion

$$\ddot{\varphi} + g^2 \varphi^3 = J.$$

Solution for J=0 is periodic Jacobi elliptic function with period T_{cl}:

$$\phi_{cl}(t) = \phi_m cn \left(\frac{1}{2}, g\phi_m t + C \right),$$

$$T_{cl} = \frac{4}{g\phi_m} K(1/2), \qquad K(1/2) \approx 1.85.$$

Initial conditions:

$$\phi_{cl}(t_0) = \alpha, \quad \dot{\phi}_{cl}(t_0) = p, \qquad \phi_m = \phi_m(\alpha, p), \quad C = C(\alpha, p)$$

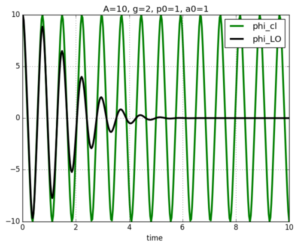
CSA at work

Wigner function

$$f_W(\alpha, p) = \frac{1}{\pi \alpha_0 p_0} e^{-\frac{(\alpha - A)^2}{\alpha_0^2}} e^{-\frac{p^2}{p_0^2}}$$

A - measure of the field excitation

Analytical answer



$$\langle \hat{\varphi} \rangle_{t_1}^{LO} \equiv \int d\alpha \int dp \; f_W(\alpha, p) \phi_{cl}(t_1) =$$

$$2A \sum_{k=0}^{\infty} u_k e^{-\frac{\pi^2 p_0^2}{g^2 A^4 T^2} k^2} e^{-\frac{\alpha_0^2 \pi^2 g^2}{T^2} k^2 t_1^2} cos\left(\frac{2Ag\pi k}{T} t_1\right)$$

$$u_m = \frac{1}{T} \int_0^T cn\left(\tfrac{1}{2}; t\right) e^{-imt\frac{2\pi}{T}} dt$$

Static T^μ_μ

Energy-momentum tensor

$$T^{\mu\nu} = \partial^\mu\varphi\partial^\nu\varphi - g^{\mu\nu}\left(\frac{1}{2}\partial_\lambda\varphi\partial^\lambda\varphi - \frac{g^2}{4}\varphi^4\right)$$

$$\varepsilon = T^{00} = \frac{1}{2}\dot{\varphi}^2 + \frac{g^2}{4}\varphi^4, \quad p = T^{ii} = \frac{1}{2}\dot{\varphi}^2 - \frac{g^2}{4}\varphi^4$$

Equilibrium stage = well defined equation of state (scale invariance)

$$\langle T^\mu_\mu\rangle = \varepsilon - 3p = 0$$

Classical level

$$T^\mu_\mu = \varepsilon - 3p = -\dot{\varphi}_{cl} + g^2\varphi^4_{cl} \neq 0$$

Quantum level (Keldysh-Schwinger technique)

$$\langle T^\mu_\mu\rangle_{t_1} = \int d\xi_{init} \int_{\varphi_c(t_0)=\frac{\xi_1+\xi_2}{2}}^{\varphi_c(\infty)=0} \mathcal{D}\varphi_c \int_{\varphi_q(t_0)=\xi_1-\xi_2}^{\varphi_q(\infty)=0} \mathcal{D}\varphi_q \; e^{iS_K[\varphi_c,\varphi_q]}\left(-\dot{\varphi}_c^2(t_1) + g^2\varphi_c^4(t_1)\right)$$

Static T^μ_μ

-Consider variation of the Keldysh action over the field φ_q

$$\frac{\delta S_K}{\delta \varphi_q}\Big|_{J=0} = -V_3\left(\ddot{\varphi}_c + g^2\varphi_c^3 + \frac{3}{4}g^2\varphi_c\varphi_q^2\right).$$

- Integrate by parts
- Neglect surface terms

without semiclassical decomposition

$$\langle T^\mu_\mu \rangle_{t_1} = -\frac{1}{2}\int d\xi_{init}\int \mathcal{D}\varphi_c\mathcal{D}\varphi_q\; e^{iS_K[\varphi_c,\varphi_q]}\;\partial^2_{t_1}\varphi_c^2(t_1)$$

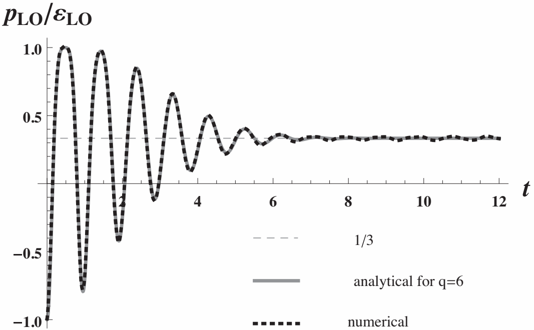
For large t_1:

- Field in the static box relaxes to some constant, means $\langle T^\mu_\mu\rangle_{t_1\to\infty} = 0$
- Scale invariance is restored (system forget about initial time)

Static T^{μ}_{μ} LO

$\langle T^{\mu}_{\mu}\rangle^{LO}_{t_1} \to 0$

$$\frac{p_{LO}(t_1\to\infty)}{\varepsilon_{LO}} = \left[\frac{1}{3} + 8\,\mathcal{I}(2)e^{-\frac{4\pi^2 p_0^2}{g^2 A^4 T^2}}e^{-\frac{4\alpha_0^2\pi^2 g^2}{T^2}t_1^2}\cos\left(\frac{4\pi A}{T}gt_1\right) + \dots\right]$$



Static T^μ_μ NLO

$$\langle T^\mu_\mu\rangle^{NLO}_{t_1} = -\frac{1}{2}\partial^2_{t_1}\left\langle \frac{1}{2V^2g^2\phi^4_m}\Big[\psi_0(g\phi_m t_1) + g\phi_m\psi_1(g\phi_m t_1)t_1 + g^2\phi^2_m\psi_2(g\phi_m t_1)t_1^2 + g^3\phi^3_m\psi_3(g\phi_m t_1)t_1^3\Big]\right\rangle_{i.c.} = 0$$

- $z = g\phi_m t$ -dimensionless variable

-$f_k(z_1, z)$ - variations of the classical solution

-$\psi(t) = \psi(t + T_{cl})$ - periodic functions. Numerical only.

$$t^n \int d\alpha dp\, f_W(\alpha, p) \sum_{k=-\infty}^{\infty} \psi_n^{(k)} e^{ikt\frac{2\pi}{T_{cl}}} = \sum_{k=-\infty}^{\infty} \psi_n^{(k)} A\, t^n e^{-Bk^2t^2} e^{iCkt}, \quad n = 0, 1, 2, 3$$

Finite time evolution!

$$\psi_n(t) = \sum_{k=-\infty}^{\infty} \psi_n^{(k)} e^{ikt\frac{2\pi}{T_{cl}}}$$

Longitudinally expanding box

$$\tau^2 = t^2 - z^2, \qquad \eta = \frac{1}{2}\ln\frac{t+z}{t-z},$$

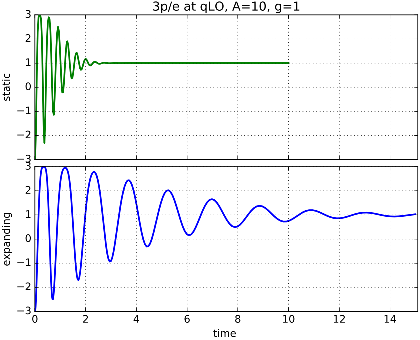
"Homogeneous" case $\partial_\eta \varphi = 0$ and $\partial_\perp \varphi = 0$

$$S = V_2 \int d\tau\ \tau \left(\frac{1}{2}\dot{\varphi}^2 - \frac{g^2}{4}\varphi^4 + J\varphi\right),$$
$$V_2 = \int d^2x_\perp d\eta$$

Equation of motion

$$\partial_\tau^2 \varphi + \frac{1}{\tau}\partial_\tau \varphi + g^2\varphi^3 = J$$

can not be calculated analytically



Expanding T^μ_μ

$$\langle T^\mu_\mu \rangle_{\tau_1} = -\frac{1}{2}\int d\xi_{init}\int \mathcal{D}\varphi_c \mathcal{D}\varphi_q \; e^{iS_K^{exp}[\varphi_c,\varphi_q]} \left(\partial^2_{\tau_1} + \frac{1}{\tau_1}\partial_{\tau_1}\right)\varphi_c^2(\tau_1)$$

The system is expanding:

- Asymptotically $\varepsilon = 0$ and $p = o$ due to expansion
- We are looking for intermediate quasi stationary state with definite EoS
- We need to take into account expansion, hence $T^\mu_\mu/\varepsilon_{LO}$, where $\varepsilon_{LO} \approx \tau^{-4/3}$

$$\frac{\langle T^\mu_\mu \rangle^{NLO}_{\tau_1}}{\varepsilon^e_{LO}} = -\frac{1}{2\varepsilon^e_{LO}(\tau_1)}\left(\partial^2_{\tau_1} + \frac{1}{\tau_1}\partial_{\tau_1}\right) \times \left\langle \frac{\tau_1^{-\frac{2}{3}}}{2V_2^2 g^2 \xi_m^4}\left(\int_{z_0^e}^{\tilde{z}} dz^e F_{exp}(z_1^e, z^e) + \int_{\tilde{z}}^{z_1^e} dz^e F^e_{static}(z_1^e, z^e)\right)\right\rangle^e_{i.c.}.$$

- $z^e = \frac{3}{2} g \xi_m \tau^{\frac{2}{3}}$
- $\xi_m \approx A^{\frac{3}{2}}$ - the first term is suppressed (A -measure of the field excitation)

Expanding T^μ_μ

$$\langle T^\mu_\mu \rangle_{\tau_1} = -\frac{1}{2}\int d\xi_{init}\int \mathcal{D}\varphi_c\mathcal{D}\varphi_q\ e^{iS_K^{exp}[\varphi_c,\varphi_q]}\left(\partial^2_{\tau_1}+\frac{1}{\tau_1}\partial_{\tau_1}\right)\varphi_c^2(\tau_1)$$

The system is expanding:

- Asymptotically $\varepsilon = 0$ and $p = o$ due to expansion
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$$\frac{\langle T^\mu_\mu\rangle^{NLO}_{\tau_1}}{\varepsilon^e_{LO}} = -\frac{1}{2\varepsilon^e_{LO}(\tau_1)}\left(\partial^2_{\tau_1}+\frac{1}{\tau_1}\partial_{\tau_1}\right) \times \left\langle \frac{\tau_1^{-\frac{2}{3}}}{2V_2^2 g^2\xi_m^4}\left(\int_{z_0^e}^{\tilde{z}} dz^e F_{exp}(z_1^e, z^e) + \cancel{\int_{\tilde{z}}^{z_1^e} dz^e F^e_{static}(z_1^e, z^e)}\right)\right\rangle^e_{i.c.}.$$

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- $\xi_m \approx A^{\frac{3}{2}}$ - the first term is suppressed (A -measure of the field excitation)

Quantum corrections: expanding box

$$\langle F(\hat{\varphi})\rangle_{\tau_1}^{LO+NLO} = \left\langle F(\phi_{cl}(\tau_1)) + \frac{g^2}{4V_2^2}\int_{\tau_0}^{\tau_1}\frac{d\tau_2}{\tau_2^2}\phi_{cl}(\tau_2)\frac{\delta^3 F(\phi_{cl}(\tau_1))}{\delta J^3(\tau_2)}\Big|_{J=0}\right\rangle_{i.c.}$$

Variations

$$\frac{\delta^3 F(\phi_{cl}(\tau_1))}{\delta J^3(\tau_2)} = \frac{\partial F}{\partial \phi_{cl}}\Phi_3(\tau_1,\tau_2) + 3\frac{\partial^2 F}{\partial \phi_{cl}^2}\Phi_1(\tau_1,\tau_2)\Phi_2(\tau_1,\tau_2) + \frac{\partial^3 F}{\partial \phi_{cl}^3}\Phi_1^3(\tau_1,\tau_2).$$

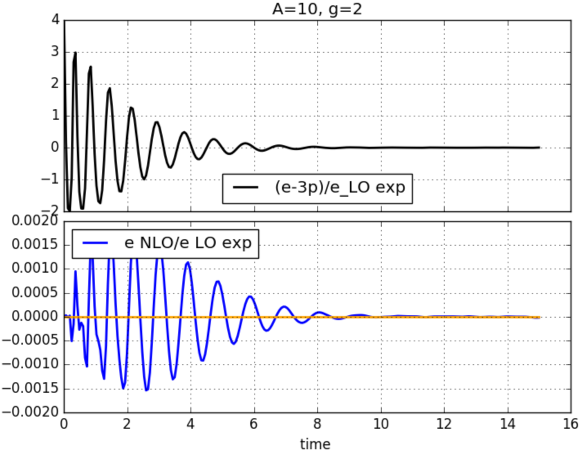
$$L_{t_1}\Phi_1(\tau_1,\tau_2) = \delta(\tau_1-\tau_2)$$
$$L_{t_1}\Phi_2(\tau_1,\tau_2) = -6g^2\phi_{cl}(\tau_1)\Phi_1^2(\tau_1,\tau_2)$$
$$L_{t_1}\Phi_3(\tau_1,\tau_2) = -6g^2\Phi_1^3(\tau_1,\tau_2) - 18g^2\phi_{cl}(\tau_1)\Phi_1(\tau_1,\tau_2)\Phi_2(\tau_1,\tau_2)$$
$$L_{\tau_1} = \partial_{\tau_1}^2 + \frac{1}{\tau_1}\partial_{\tau_1} + 3g^2\phi_{cl}^2(\tau_1)$$

Averaging over the initial conditions

$$f_w[\alpha,p,\tau_0] = \int d\beta\langle\alpha+\frac{\beta}{2}|\hat{\rho}(t_0)|\alpha-\frac{\beta}{2}\rangle e^{i\int V_2\tau_0 p\beta}$$
$$\partial_\tau^2\phi_{cl} + \frac{1}{\tau}\partial_\tau\phi_{cl} + g^2\phi_{cl}^3 = 0$$
$$\phi_{cl}(\tau_0) = \alpha$$
$$\partial_\tau\phi_{cl}(\tau_0) = p$$

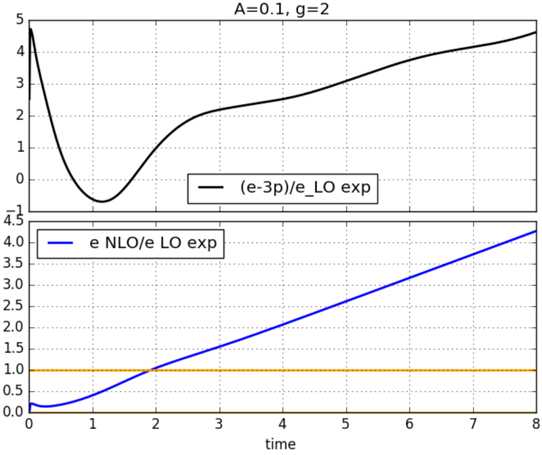
Numerical results

$A = 10$, $g = 2$. CSA works excellent. $\varepsilon - 3p = 0$



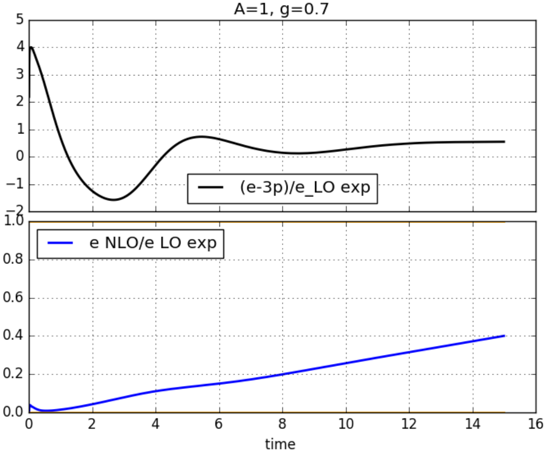
Numerical results

$A = 0.1$, $g = 2$. CSA does not work



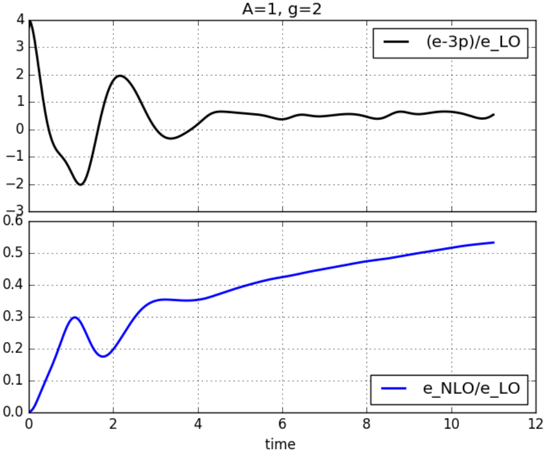
Numerical results

$A = 1$, $g = 0.7$. CSA works fine. $\varepsilon - 3p = const$



Numerical results

$A = 1$, $g = 2$. CSA works fine. $\varepsilon - 3p = const$



- Sangyong Jeon, "Color Glass Condensate in Schwinger-Keldysh QCD", Annals Phys. 340 (2014) 119-170

(small fluctuations around classical solution)

$$\mathcal{L} = -\frac{1}{2}F^a_{\mu\nu}F^{\mu\nu a} - J^a_\mu A^{\mu a}, \qquad \mathcal{L}_K = \mathcal{L}(A^F_\mu) - \mathcal{L}(A^B_\mu),$$

$$\mathcal{A}_\mu = \frac{1}{2}(A^F_\mu + A^B_\mu), \quad \eta_\mu = (A^F_\mu - A^B_\mu),$$

Keldysh action

$$\mathcal{L}_K = \eta^a_\nu([D_\mu, F^{\mu\nu}] - J^{\nu a}) - i\frac{g}{4}[D_\mu, \eta_\nu]^a[\eta^\mu, \eta^\nu]^a$$

$$D_\mu = \partial_\mu - ig\mathcal{A}_\mu, \qquad F_{\mu\nu} = \frac{i}{g}[D_\mu, D_\nu]$$

- Gauge invariance $\mathcal{A}_\mu \to U\mathcal{A}_\mu U^\dagger + \frac{i}{g}U\partial_\mu U^\dagger, \quad \eta_\mu \to U\eta_\mu U^\dagger$

CSA + quantum corrections

$$\langle F(A_\mu)\rangle_{t_1}^{LO+NLO} = \Big\langle \underbrace{F(\mathbf{A}_\mu(t_1,\vec{x}))}_{LO} +$$

$$\underbrace{\frac{ig}{4}\int_{t_0}^{t_1} dt' \int d^3x' \left[\partial_\mu - ig\mathbf{A}_\mu, \frac{\delta}{\delta J_\nu}\right]^a \left[\frac{\delta}{\delta J^\mu}, \frac{\delta}{\delta J^\nu}\right]^a F(\mathbf{A}_\mu(t_1,\vec{x}))\Big|_{J=0}}_{NLO} \Big\rangle_{i.c.}$$

- Classical solution $\mathbf{A}_\mu$: $[D_\mu, F^{\mu\nu}]^a = 0$
- Variations $\frac{\delta \mathbf{A}_\mu}{\delta J^\nu}$: $\quad \frac{\delta}{\delta J^\lambda}([D_\mu, F^{\mu\nu}]^a = J^{\nu a})$
- Gauge invariance $\to$ Unity decomposition $\to$ Physical variables $\to$ Coulomb gauge ???

- The systematic procedure for calculation of quantum corrections to the Classical Statistical Approximation is developed.
- Time evolution of the $\langle T^\mu_\mu \rangle$ is analyzed for homogeneous static and longitudinally expanding models.
- It i shown that quantum corrections can change the CSA predictions.

"Staticalization" of the solution

Change of variables → asymptotically periodic solution

$$y = \tau^{\frac{2}{3}}, \qquad \varphi(\tau) = \tau^{-\frac{1}{3}}\xi(\tau^{\frac{2}{3}}),$$

$$\ddot{\xi}(y) + \frac{1}{4y^2}\xi(y) + \frac{9}{4}g^2\xi(y)^3 = 0,$$

$$\xi^{eq}(y) = \xi_m cn(\bar{g}\xi_m y + C), \qquad \bar{g} = \frac{3}{2}g$$

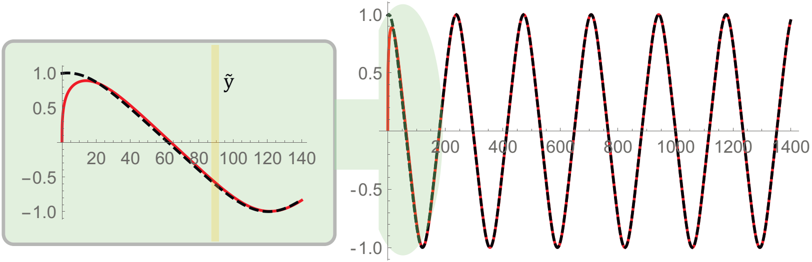


Figure: $\xi(y)$ - red line, $\xi^{eq}(y)$- black dashed line

Expanding case: one can't relate asymptotic periodic solution with the initial value