The Quantum Primordial Black Holes in Early Universe, Inflationary Cosmology and Relic Gravitational Waves

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Sourse : ***The quantum primordial black holes, dimensionless small parameter, inflationary cosmology and non-gaussianity,***

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1. Primordial Black Holes

The most general scenario is a collapse of high matter densities.

The mass **of pbh** $M(t_M)$, formed during the time t after the big bang

$$M(t_M) \approx \frac{c^3 t_M}{G} \approx 10^{15} \left(\frac{t}{10^{-23}} \sec\right) g. \qquad (1)$$

From (1) **pbh** have a wide range of masses, in particular for the Planck time $t_M \approx 10^{-43}$s, **pbh** has a Planck mass $M(t_M) \approx 10^{-5} g$, quantum-gravitational effects will be significant.

Quantum black holes **qbh** are currently understood as Schwarzschild black holes, $r = r_{qbh} \propto l_p$and mass $m = m_{qbh} \propto M_p$.

However, due to formula (1), **qbh** can be formed in the Early Universe as **pbh** in the Planck time $\approx 10^{-43}$s. Regardless of the way **qbh is formed** , quantum gravitational effects are essential for them.

In all cases, one can find quantum-gravitational corrections for the main characteristics of black holes, which will be significant for **qbh .**

In particular, if the Generalized Uncertainty Principle (GUP) is valid in the transition to high (Planck) energies **(*Nouicer, PLB,2007*)**:

$$(\boldsymbol{\delta X})(\boldsymbol{\delta P}) \geq \frac{\hbar}{2}\langle \mathbf{exp}\,\frac{\alpha^2 l_p^2}{\hbar^2}\boldsymbol{P}^2\rangle$$

In first order $(\boldsymbol{\delta X})(\boldsymbol{\delta P}) \geq \frac{\hbar}{2}\left(\mathbf{1}+\frac{\alpha^2 l_p^2}{\hbar^2}(\boldsymbol{\delta P})^2\right)$ (2*)

then there can exist a Planck black hole (which will be called “minimal” below) with a minimum mass M_0 and radius of the event horizon r_{min}:

$$\boldsymbol{M_0 \propto m_p, r_{min} = r_{M_0} = l_{min} \propto l_p;\ \hbar = \mathbf{c} = \mathbf{k_B} = 1},\text{ then } \boldsymbol{l_p^2 = G,\ m_p{}^2 = 1/G.}$$

Semiclassical approximation , Hawking temperature, $\boldsymbol{T_{\mathrm{H}} = \frac{1}{8\pi GM}}$

Within the framework of GUP, quantum-gravitational corrections (qgc):

$$T_{\mathrm{H},q} = \frac{1}{8\pi M l_p^2}\exp\left(-\frac{1}{2}W\left(-\frac{1}{e}\left(\frac{M_0}{M}\right)^2\right)\right),$$

$W\left(-\frac{1}{e}\left(\frac{M_0}{M}\right)\right)^2$ − −value at the corresponding point of the Lambert function $W(u)$, $\boldsymbol{W(u)e^{W(u)} = u}$; $\boldsymbol{W(u)}-$ **Lambert W-function.**

$W(u)$ is the multifunction for complex variable $u = x + y\imath$. However for

example for real $u = x, -\frac{1}{e} \leq u \leq 0,$ *,* $W(u)$ *is the single-valued continuous function with two branches* $W_{0,}W_{-1}$ *is present picture .*

Johann Heinrich Lambert (1728--1777)

Corless, R. M.; Gonnet, G. H.; Hare, D. E. G.; Jeffrey, D. J.; Knuth, D. E. (1996). "On the Lambert *W* function" (PDF). *Advances in Computational Mathematics*. **5**: 329–359.

$\boldsymbol{T_{\mathrm{H},q}}$ can be expanded into a series in terms of a small parameter $(\boldsymbol{M_0/M})^2$ with the the leading first term:

$$\boldsymbol{T_{\mathrm{H},q} \simeq \frac{1}{8\pi M l_p^2}\left(1+\frac{1}{2e}\left(\frac{M_0}{M}\right)^2+\frac{5}{8e^2}\left(\frac{M_0}{M}\right)^4+\frac{49}{48e^3}\left(\frac{M_0}{M}\right)^6+\cdots\right)},$$

Then, within the framework of GUP, $M \to M_q$:

$$\boldsymbol{T_{\mathrm{H},q} \simeq \frac{1}{8\pi M_q l_p^2}; M_q \doteq M\exp\left(\frac{1}{2}W\left(-\frac{1}{e}\left(\frac{M_0}{M}\right)\right)^2\right)}$$

For Schwarzschild black holes,and $c=\hbar=1; r_M = 2MG \to r_{M_q} = 2M_qG$.

$$\boldsymbol{T_{H,q} \simeq \frac{1}{8\pi M l_p^2}\left(1+\frac{1}{2e}\left(\frac{r_{min}}{r_M}\right)^2+\frac{5}{8e^2}\left(\frac{r_{min}}{r_M}\right)^4+\frac{49}{48e^3}\left(\frac{r_{min}}{r_M}\right)^6+\cdots\right)},$$

$$\boldsymbol{\left(\frac{M_0}{M}\right)^2 = \frac{r_{min}^2}{{r_M}^2} \doteq \alpha_{r_M}},$$

$$\boldsymbol{exp\left(\pm\frac{1}{2}W\left(-\frac{1}{e}\left(\frac{M_0}{M}\right)^2\right)\right) = exp\left(\pm\frac{1}{2}W\left(-\frac{1}{e}\alpha_{r_M}.\right)\right)}$$

3 Primordial Black Holes and "Quantum Shifts" (QS) for Cosmological Parameters in Inflation Models

The metric of a Schwarzschild black hole

$$ds^2 = -\left(1 - \frac{2MG}{r}\right)dt^2 + \left(1 - \frac{2MG}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \quad (*)$$

When studying the Early Universe for **PBHs** , $(*)$ is replaced by the Schwarzschild-de Sitter (SdS) metric,

$$\boldsymbol{ds^2 = -f(\tilde{r})dt^2 + \frac{d\tilde{r}^2}{f(\tilde{r})} + \tilde{r}^2 d\Omega^2} \quad (**)$$

where $\boldsymbol{f(\tilde{r}) = 1 - \frac{2GM}{\tilde{r}} - \frac{\Lambda \tilde{r}^2}{3} = f(\tilde{r}) = 1 - \frac{2GM}{\tilde{r}} - \frac{\tilde{r}^2}{L^2}}$,

$\boldsymbol{L = \sqrt{3/\Lambda}}$, $\boldsymbol{M}$ is the black hole mass , $\tilde{r}$ is a small quantity, and Λ is the cosmological term.In this case for small radii $\boldsymbol{r_{\mathrm{SdS,M}} \approx 2GM = r_{\mathrm{M}}}$.

In the general case, in particular in inflationary cosmology,last metric is written in terms of conformal time η:

$$\boldsymbol{ds^2 = a^2(\eta)\{-d\eta^2 + \left(1 + \frac{\mu^3\eta^3}{r^3}\right)^{4/3}\left[\left(\frac{1-\mu^3\eta^3/r^3}{1+\mu^3\eta^3/r^3}\right)^2 dr^2 + r^2 d\Omega^2\right]\}},$$

where $\boldsymbol{\mu} = (\boldsymbol{GMH_0/2})^{1/3}$**,** where $\boldsymbol{H_0}$ is the deSitter Hubble parameter

$$\boldsymbol{a = a(\eta) = -\frac{1}{H_0\eta}, \eta < 0;} \qquad \boldsymbol{d\eta = \frac{dt}{a(t)}, \eta = \int_0^t \frac{dt'}{a(t')}}.$$

Here $\boldsymbol{r}$ it satisfies the condition $\boldsymbol{r_0 < r < \infty}$ and the value $\boldsymbol{r_0 = -\mu\eta}$ corresp. to the black hole singularity.$\boldsymbol{\mu = (r_M H_0/4)^{1/3}}$, with high accuracy.

$\boldsymbol{\mu = const}$**, (*Prokopec, Reska,JCAP 2011*).** Then if in formula "shifts" $r_M: r_M \mapsto \widetilde{r_M}$, then "shifts" accordingly, and $H_0: H_0 \mapsto \widetilde{H}_0$ so that $\boldsymbol{\mu =} \boldsymbol{(r_M H_0/4)^{1/3} = (\widetilde{r_M}\widetilde{H}_0/4)^{1/3}, \widetilde{H}_0 = \frac{r_M}{\widetilde{r_M}} H_0}$ and **all others cosm.param.** *All "shifts" relatively* ***qgc***

3.1 Initially, a primordial black hole is considered in the absence of absorption and emission processes. Since $\mu = const$, the replacement $r_M \mapsto r_{M_q}$ leads to $H_0 \to H_{0,q}$ that satisfies

$$\boldsymbol{\mu = (r_M H_0/4)^{\frac{1}{3}} = (r_{M_q} H_{0,q}/4)^{\frac{1}{3}} \ , H_{0,q} = H_0 \exp\left(-\frac{1}{2} W\left(-\frac{1}{e}\alpha_{r_M}\right)\right)} \ \textbf{(!)}$$

And **all cosm.param. (q-deformation)**

3.2 *Case of "minimal" absorption of particles by a black hole .*

Let M the initial mass of the black hole with the area of the event horizon be A. **Bekenstein 1973**: the minimum increment of the area of the black hole event horizon absorbing the particle of energy and size R was estimated E: $(\Delta A)_0 \simeq 4l_p^2(\ln 2)ER$. In quantum consideration $R \sim 2\delta X$ and $E \sim \delta P$. Semiclassical Picture.

low energies $\boldsymbol{E \ll E_p}$, that is, **HUP,** $(\delta X)(\delta P) \geq \frac{\hbar}{2}$, $(\Delta A)_0 \simeq 4l_p^2(\ln 2)$.

For all energies: $\boldsymbol{E \leq E_p}$**, GUP,** $\left(\Delta\hat{A}\right)_{0,q} \approx 4l_p^2 \ln 2 \exp\left(-\frac{1}{2}W\left(-\frac{1}{e}\boldsymbol{\alpha_{r_M}}\right)\right)$.

3.3 *Black Hole Evaporation and qgc*

Similarly can be obtained analog **(!)** for ***Black Hole Evaporation***

$$\boldsymbol{H_0 \mapsto \mathcal{H}_{0,q} = \frac{R(M_{Evap})}{R(M_{q,Evap})} H_0},$$

where $\boldsymbol{R(M_{Evap})}$ – radius **qbh** after evaporation in semiclassical approximation and $\boldsymbol{R(M_{q,Evap})}$ – **in quant-grav. picture (qgc)**

for $\boldsymbol{t_{Evap} = t_{Infl} - t_M}$.

4 Black Holes Formation Probability Corrections

It is assumed that non-relativistic particles with a mass $m < m_p$ dominate in the pre-inflationary period and, for convenience, denote the Schwarzschild radius as R_S. $\boldsymbol{N(R,t)}$ the number of particles in ball with physical radius $R = R(t)$ and volume $V_R(t)$.

Due to above formulae, it is necessary to replace the Schwarzschild radius $\boldsymbol{r_M}$ with

$$\boldsymbol{r_{M_q} = r_M \exp\left(\frac{1}{2} W\left(-\frac{1}{e}\alpha_{r_M}\right)\right); \; M_q = M \exp\left(\frac{1}{2} W\left(-\frac{1}{e}\alpha_{r_M}\right)\right)}$$

Thus, $\boldsymbol{r_{M_q} < r_M; M_q < M. \Rightarrow \mathcal{P}_{form}(M_q) > \mathcal{P}_{form}(M)}$

5. *q-deformation of Friedmann Equations*

Well-known Friedman Equation without term with curvature

$$\boldsymbol{\frac{a\prime^2}{a^4} = \frac{8\pi}{3} G\rho},$$

$a(\eta) \rightarrow a(\eta)_q$; Quantum Deformation (**QD**)

$$\frac{{a_q}'^2}{a_q^4} = \frac{a'^2}{\exp\left(W\left(-\frac{1}{e}\alpha_{r_M}\right)\right) a^4} = \frac{8\pi}{3} G\rho$$

or

$$\frac{a'^2}{a^4} = \frac{8\pi}{3} G\rho \exp\left(W\left(-\frac{1}{e}\alpha_{r_M}\right)\right) \doteq \frac{8\pi}{3} G\rho_q,$$

$$\rho_q \doteq \rho \exp\left(W\left(-\frac{1}{e}\alpha_{r_M}\right)\right) < \rho.$$

Similarly procedure for other Friedman equations, in part

$$\boldsymbol{2\frac{a''}{a^3} - \frac{a\prime^2}{a^4} = -\frac{8\pi}{3} Gp}$$

In result $\boldsymbol{p_q \doteq p\exp\left(W\left(-\frac{1}{e}\alpha_{r_M}\right)\right) < p}$.

And the equation of the covariant energy conservation remains unaltered with replacement of $\rho \mapsto \rho_q, p \mapsto p_q$

$$\left[\boldsymbol{\rho' = -3\frac{a'}{a}(\rho + p)}\right] \to \left[\boldsymbol{\rho'_q = -3\frac{a'}{a}\left(\rho_q + \rho_q\right)}\right]$$

6. Enhancement of Non-Gaussianity by the qgcs-corrections

Deviation from the Gaussian distribution for the random field $g(\mathbf{x})$ is a nonzero value of the three-point correlator

$$\boldsymbol{\langle g_{k_1} g_{k_2} g_{k_3}\rangle = (2\pi)^3 \delta^3_{k_1+k_2+k_3} B_g(k_1, k_2, k_3) \neq 0},$$

where $g_{\mathbf{k_i}}, i = 1,2,3$ represent the Fourier transform $g(\mathbf{x})$ in the momentum k_i and the quantity B_g, named the *bispectrum*, is a measure of non-Gaussinity for the random field $g(\mathbf{x})$.

We can show that, with due regard for the foregoing **qgcs**, the absolute value of the *bispectrum* B_g in the inflation pattern for different random fields is growing.

6.1 Non-gaussianity corrections from the inflaton self-interaction

$\delta\phi$ – perturbation of inflaton ϕ.

We suppose that the metric perturbations $\boldsymbol{\delta g \approx 0}$.

Then *bispectrum* $\mathbf{B}_{g}$ in this case

$$\mathbf{B}_{\delta\phi}^{self} \sim H_*^2 V_*''' , \ \mathbf{B}_{\delta\phi}^{self} \equiv \mathbf{B}_{\delta\phi,M}^{self}$$

where the asterisk denotes a value of the corresponding quantity during inflation which is taken to be constant

$$\mathbf{B}_{\delta\phi,q}^{self} \equiv \mathbf{B}_{\delta\phi,M_q}^{self} \sim H_{q*}^2 V_{q*}''' = \mathbf{e}xp\left(-2W\left(-\frac{1}{e}\alpha_{r_M}\right)\right) H_*^2 V_*''' ;$$

$$\mathbf{B}^{self}_{\delta\phi,M_q} = \ \mathbf{exp}\left(-2W\left(-\frac{1}{e}\alpha_{r_M}\right)\right)\mathbf{B}^{self}_{\delta\phi}, \ \left|\mathbf{B}^{self}_{\delta\phi,M_q}\right| > \left| \ \ \mathbf{B}^{self}_{\delta\phi}\right|.$$

6.2 The Non-Gaussianity Correction of Field Perturbations from Gravitational interactions

In the above analysis of non-Gaussianity there was no metric perturbation because the gravitational interaction was excluded. When it is taken into consideration, the non-Gaussianity in this case arises from the metric perturbation and the *bispectrum* takes the following form:

$$\mathbf{B}^{grav}_{\delta\phi,M} \equiv \mathbf{B}^{grav}_{\delta\phi} \sim -\frac{1}{8}H_*^4\left(\frac{V'}{V}\right).$$

Then similarly **6.1**

$$\mathbf{B}^{grav}_{\delta\phi,M_q} = \ \mathbf{exp}\left(-2W\left(-\frac{1}{e}\alpha_{r_M}\right)\right)\mathbf{B}^{grav}_{\delta\phi}, \ \left|\mathbf{B}^{grav}_{\delta\phi,M_q}\right| > \left| \ \ \mathbf{B}^{grav}_{\delta\phi}\right|$$

6.3 Non-Gaussianity Correction for the Tensor Primordial Perturbations

$$\boldsymbol{B_M^{tens} \equiv B^{tens} \sim \frac{H_*^4}{m_p^4}}.$$

$$\mathbf{B}_{M_q}^{tens} = \mathbf{e}xp\left(-2W\left(-\frac{1}{e}\alpha_{r_M}\right)\right)B^{tens}; \quad \left|\mathbf{B}_{M_q}^{tens}\right| > |\mathbf{B}^{tens}|$$

In all cases the same (universal) Non-Gaussianity Enhancement coefficient

$$\mathbf{e}xp\left(-2W\left(-\frac{1}{e}\alpha_{r_M}\right)\right) > 1.$$

7. The Quantum PBH in Pre-inflation Era and RGW. Beginning, *(E. Lifhits,1946; L.P. Grishchuk,1974,1975,1976,… 1993,…)*

$$\overline{g}_{ij} = g_{ij} + h_{ij}(x_i) \qquad (i, j = 0, 1, 2, 3),$$

$$|h_{ij}| \ll |g_{ij}|$$

$h_{00} = h_{0\alpha} = 0, (\alpha = 1, 2, 3)$; $h^{ij}{}_{;j} = 0$ **– gauge conditions (; covariant derivative with respect** ds^2)

$$ds^2 = a(\eta)^2[-d\eta^2 + dr^2 + r^2 d\Omega^2], \quad {ds_q}^2 = {a(\eta)_q}^2[-d\eta^2 + dr^2 + r^2 d\Omega^2]$$

Next, the ansatz $h_{\alpha\beta}(\eta, x) = h(\eta)G_{\alpha\beta}(k, x)$; $G_{\alpha\beta} -$ *combination of plane-waves solutions* **with the two polarizations** $\exp(\pm \mathrm{i}\boldsymbol{kx})$.

The evolution equation for the temporal part of the wave

$$h''(k,\eta) + 2\frac{a'(\eta)}{a(\eta)}h'(k,\eta) + k^2 h(k,\eta) = 0,$$

What is the definition $h(k,\eta) \equiv \frac{\mu(k,\eta)}{a(\eta)}$

$$\mu''(\eta) + \left(k^2 - \frac{a''(\eta)}{a(\eta)}\right)\mu(\eta) = 0, \rightarrow \mu(\eta)_q{}'' + \left({k_q}^2 - \frac{a''(\eta)}{a(\eta)}\right)\mu(\eta)_q = 0 \quad \textbf{(7.1)}$$

Here $k = |\mathrm{k}|$ **is the wave-number, this is the same as the wave number** $k = a\omega$**.**

$$ds^2 \rightarrow ds_q^2, k \rightarrow k_q = \frac{2\pi a_q}{\lambda} = a_q\omega; h(k,\eta) \rightarrow h(k,\eta)_q; \mu(\eta) \rightarrow \mu(\eta)_q = \mu(\eta, k_q).$$

7.a $\left(k^2 \gg \frac{a''}{a}\right), {k_q}^2 \gg \frac{{a_q}''}{a_q} = \frac{a''}{a}$ **; i. e.** ***high frequency waves***

Eq. (7.1) for 7.a -- ***harmonic oscillator with the varying frequency where solution is a free wave*** $\mu(\eta)_q = \mu(\eta, k_q) = e^{\pm i k_q \eta}$, **and then corr.** ***amplitude***

$\mathrm{h}(\eta)_q = \mathrm{a}(\eta)_q{}^{-1} sin(k_q\eta + \phi)$ ***decreases adiabatically in an expanding universe***

$a_q^{-1} = a^{-1} exp\left(-\frac{1}{2}W\left(-\frac{1}{e}\alpha_{r_M}\right)\right)$ ***and thus sharper then*** a^{-1}

7.b $(k^2 \ll \frac{a''}{a})$; $k_q{}^2 \ll \frac{a_q{}''}{a_q} = \frac{a''}{a}$; **general solution (7.1) is a linear combination of pair**

$$\mu_{1,q} = a(\eta)_q = \mu_1 \exp\left(\frac{1}{2} W\left(-\frac{1}{e}\alpha_{r_M}\right)\right),$$

$$\mu_{2,q} = a(\eta)_q \int d\eta\, a(\eta)_q{}^{-2} = \mu_2 \exp\left(-\frac{1}{2} W\left(-\frac{1}{e}\alpha_{r_M}\right)\right).$$

In expanding Universe $\boldsymbol{\mu_1}$ *will grow faster than* $\boldsymbol{\mu_2}$ *and similarly for* $\boldsymbol{\mu_{1,q}}$ *and* $\boldsymbol{\mu_{2,q}}$.

However for ***q-deformation this growth is dampened*** by $\mathbf{exp}\left(\pm\frac{1}{2} W\left(-\frac{1}{e}\alpha_{r_M}\right)\right)$.

Nevertheless general solution **(7.1)** μ_q for **7.b:** $\mu_q \approx \mu_{1,q}$ **and** ***L.P. Grishchuk 1993*** *showed that the amplitude of* $\boldsymbol{h_{\alpha\beta}}$ *(and in present case* $(\boldsymbol{h_{\alpha\beta}(\eta, x)_q})$) *is the constant if 7.b.* The *amplitude of* $\boldsymbol{h_{\alpha\beta}}$ *will grow if* **7.b** *is not satisfied for* $\boldsymbol{\eta \to 0}$.

This phenomenon is "superadiabatic" or "parametric" amplification of gravitational waves.

Other scenarios of pbhs presence: ***quantum tunneling, finite temperature***

D. J. Gross, M.J. Perry, and M.J. Yaffe, ***Phys. Rev. D 25, 330 (1982);***

radiation-dominated stage of universe $p = \frac{1}{3}\rho; \implies$ ***the Quantum PBH***

Other scenarios of RGWs appearing in pbhs presence**:**

7.2 *Alexander D. Dolgov, Damian Ejlli,*

Relic gravitational waves from light primordial black holes,

Phys.Rev.D 84 (2011) 024028*;*

7.3 *Martti Raidal, Ville Vaskonen, Hardi Veermäe,*

Gravitational Waves from Primordial Black Hole Mergers,

***JCAP* 09 (2017) 037**

8. Conclusion and Further Steps

In this way it has been demonstrated that, within the scope of natural assumptions, the ***qgcs*** calculated for pbhs arising in the pre-inflationary epoch contribute significantly to the inflation parameters, enhancing non-Gaussianity in the case of cosmological perturbations. Besides, withy due regard for these ***qgcs***, the probability of arising pbhs is higher. Based on these results, the following steps may be planned to study the corrections of cosmological parameters and cosmological perturbations due to ***qgcs*** for ***pbhs*** in the pre-inflationary era:

8.1. Comparison of these results obtained in Section 6 with the experimental data accumulated by space observatories: (***Planck Collaboration***), (***WMAP Collaboration***),(***Hubble***),),(***James Webb***),... and solution of direct and ***inverse*** problems;

8.2. Elucidation of the fact, how closely the author's results are related to general approaches to inclusion of the ***quantum-gravitational effects*** in studies of inflationary perturbations (for example, ***Claus Kiefer and other***);

8.3. These studies are part of the following program:

Pbhs formed in pre-inflation era → ***pbhs*** formed in the inflation result →

→ ***Astroparticle physics*** in the presence of **these** ***pbhs*** **(**or same ***Cosmomicrophysics,***

Maxim Khlopov and other **).**