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Nonrelativistic approximation in the Pauli – Fierz theory for a spin 3/2 particle in presence of external fields

It is known that for relativistic equations for higher spin particles, some anomalous solutions are found which may be associated with particles moving with velocity greater than the light one. There arises the question how behave these anomalous solutions in the non-relativistic limit.

Also it is evident that non-relativistic equations are solved easier than relativistic ones. In the present paper we derive the non-relativistic equation for spin 3/2 particle in presence of external electromagnetic fields.

We start with the relativistic system of equations for 16-component wave functions with transformation properties of vector-bispinor under the lorentz group. When performing the non-relativistic approximation, for separating in the complete wave function big and small components we apply the method of projective operators.

Correspondingly, the complete wave function is presented as a sum of three parts: the big Ψ_+, depending on 6 variables, and the small Ψ_0 and Ψ_-, depending on 14 variables. There are found 2 linear constraints on big components, and 2 constraints on the small ones.

The system of equations is presented in explicit form with the use of 20 new variables. After performing the procedure of non-relativistic approximation we derive 6 equations with the needed non-relativistic structure, in which enter only 4 main primary big components. It is proved that only 4 equations are independent, so we arrive at the generalized Pauli -like equation for 4-component wave function.

The structure of the derived equation definitely indicates that any anomalous solutions cannon appear, for instance in presence of external uniform electric or magnetic fields.

1 Initial covariant equation

Let us start with tetrad based form of the master equation for spin 3/2 particle

$$\gamma^5 \epsilon_k{}^{can} \gamma_c \left[i(D_a)_n{}^l - \frac{1}{2} M \gamma_a \delta_n{}^l \right] \Psi_l = 0, \tag{1}$$

where $M = mc/\hbar$ is a mass parameter, the presence of the multiplier is meaningful; the generalized derivatives are

$$D_a = e^{\alpha}_{(a)} (\partial_\alpha + ieA_\alpha) + \frac{1}{2} (\sigma^{ps} \otimes I + I \otimes j^{ps})\, \gamma_{[ps]a}. \tag{2}$$

With the use of six matrices $\epsilon_k{}^{can} = (\mu^{[ca]})_k{}^n$, eq. (1) may be presented as follows

$$\gamma^5 (\mu^{[ca]})_k{}^n \gamma_c \left[i(D_a)_n{}^l - M \gamma_a \delta_n{}^l \right] \Psi_l = 0, \tag{3}$$

whence we derive the detailed form of eq. (1):

$$\left(\gamma^1 \otimes \mu^{[01]} + \gamma^2 \otimes \mu^{[02]} + \gamma^3 \otimes \mu^{[03]}\right) D_0\Psi +$$

$$+ \left(\gamma^0 \otimes \mu^{[01]} + \gamma^2 \otimes \mu^{[12]} - \gamma^3 \otimes \mu^{[31]}\right) D_1\Psi$$

$$+ \left(\gamma^0 \otimes \mu^{[02]} + \gamma^3 \otimes \mu^{[23]} - \gamma^1 \otimes \mu^{[12]}\right) D_2\Psi +$$

$$+ \left(\gamma^0 \otimes \mu^{[03]} + \gamma^1 \otimes \mu^{[31]} - \gamma^2 \otimes \mu^{[23]}\right) D_3\Psi$$

$$+ iM\frac{1}{2}\Big\{ s_{01} \otimes \mu^{[01]} + s_{02} \otimes \mu^{[02]} + s_{03} \otimes \mu^{[03]} + s_{23} \otimes \mu^{[23]} +$$

$$+ s_{31} \otimes \mu^{[31]} + s_{12} \otimes \mu^{[12]} \Big\}\Psi = 0\,, \quad s_{ab} = \gamma_a\gamma_b - \gamma_b\gamma_a$$

The above equation may be presented shortly as follows

$$\left(\Gamma^0 D_0 + \Gamma^0 D_0 + \Gamma^0 D_0 + \Gamma^0 D_0 + iM\Gamma\right)\Psi = 0. \tag{4}$$

It is convenient to multiply eq. (4) by the matrix Γ^{-1}, so we get

$$\left(Y^0 D_0 + Y^1 D_1 + Y^2 D_2 + Y^3 D_3 + iM\right)\Psi = 0. \tag{5}$$

2 Nonrelativistic approximation

We restrict ourselves to Minkowski space-time model and Cartesian coordinate. The wave function may be presented in the matrix form (the first index is bispinor one, the second is vector one)

$$\Psi_{A(n)} = \left| \begin{array}{cccc} f_0 & f_1 & f_2 & f_3 \\ g_0 & g_1 & g_2 & g_3 \\ h_0 & h_1 & h_2 & h_3 \\ d_0 & d_1 & d_2 & d_3 \end{array} \right| . \qquad (6)$$

We calculate the term

$$\Gamma^0 \Psi = \gamma^1 \Psi \widetilde{\mu}^{[01]} + \gamma^2 \Psi \widetilde{\mu}^{[02]} + \gamma^3 \Psi \widetilde{\mu}^{[03]} =$$

$$= \left| \begin{array}{cccc} 0 & i d_3 + h_2 & d_3 - h_1 & -i d_1 - d_2 \\ 0 & -d_2 - i h_3 & d_1 + h_3 & i h_1 - h_2 \\ 0 & -f_2 - i g_3 & f_1 - g_3 & i g_1 + g_2 \\ 0 & i f_3 + g_2 & -f_3 - g_1 & f_2 - i f_1 \end{array} \right| ,$$

whence we derive its 16-dimensional representation (its explicit form is omitted)

$$\Psi = \text{the column}\{f_0, g_0, h_0, d_0;\ f_1, g_1, h_1, d_1;\ f_2, g_2, h_2, d_2;\ f_3, g_3, h_3, d_3\};$$

the same is done for all other involved matrices.

We can readily prove that the minimal equation for the matrix для $Y^0 = Y_0$ is

$$Y_0^2(Y_0^2 - 1) = 0.$$

Therefore, we can define three projective operators

$$P_0 = 1 - Y_0^2,\ P_1 = P_+ = +\frac{1}{2}Y_0^2(Y+1),\ P_2 = P_- = -\frac{1}{2}Y_0^2(Y-1).$$

They are found in explicit form (their expressions are omitted).

Further we get presentation for three projective constituents

$$\Psi_0 = P_0\Psi\,, \quad \Psi_+ = P_+\Psi, \quad \Psi_- = P_-\Psi.$$

$$\Psi_0 = \left| \begin{array}{c} f_0 \\ g_0 \\ h_0 \\ d_0 \\ \frac{1}{3}\left(f_1 + if_2 - g_3\right) \\ \frac{1}{3}\left(f_3 + g_1 - ig_2\right) \\ \frac{1}{3}\left(-d_3 + h_1 + ih_2\right) \\ \frac{1}{3}\left(d_1 - id_2 + h_3\right) \\ \frac{1}{3}\left(-if_1 + f_2 + ig_3\right) \\ \frac{1}{3}i\left(f_3 + g_1 - ig_2\right) \\ \frac{1}{3}\left(i\left(d_3 - h_1\right) + h_2\right) \\ \frac{1}{3}i\left(d_1 - id_2 + h_3\right) \\ \frac{1}{3}\left(f_3 + g_1 - ig_2\right) \\ \frac{1}{3}\left(-f_1 - if_2 + g_3\right) \\ \frac{1}{3}\left(d_1 - id_2 + h_3\right) \\ \frac{1}{3}\left(d_3 - h_1 - ih_2\right) \end{array} \right| = \left| \begin{array}{r} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \\ S_7 \\ S_8 \\ -iS_5 \\ iS_6 \\ -iS_7 \\ iS_8 \\ S_6 \\ -S_5 \\ S_8 \\ -S_7 \end{array} \right|,$$

$$\Psi_+ = \Psi_1 = \left| \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{6}\left(d_3 + 2f_1 - if_2 + g_3 + 2h_1 - ih_2\right) \\ \frac{1}{6}\left(2d_1 + id_2 - f_3 + 2g_1 + ig_2 - h_3\right) \\ \frac{1}{6}\left(d_3 + 2f_1 - if_2 + g_3 + 2h_1 - ih_2\right) \\ \frac{1}{6}\left(2d_1 + id_2 - f_3 + 2g_1 + ig_2 - h_3\right) \\ -\frac{1}{6}i\left(d_3 - f_1 + 2if_2 + g_3 - h_1 + 2ih_2\right) \\ -\frac{1}{6}i\left(d_1 + 2id_2 + f_3 + g_1 + 2ig_2 + h_3\right) \\ -\frac{1}{6}i\left(d_3 - f_1 + 2if_2 + g_3 - h_1 + 2ih_2\right) \\ -\frac{1}{6}i\left(d_1 + 2id_2 + f_3 + g_1 + 2ig_2 + h_3\right) \\ \frac{1}{6}\left(-d_1 + id_2 + 2f_3 - g_1 + ig_2 + 2h_3\right) \\ \frac{1}{6}\left(2d_3 + f_1 + if_2 + 2g_3 + h_1 + ih_2\right) \\ \frac{1}{6}\left(-d_1 + id_2 + 2f_3 - g_1 + ig_2 + 2h_3\right) \\ \frac{1}{6}\left(2d_3 + f_1 + if_2 + 2g_3 + h_1 + ih_2\right) \end{array} \right| = \left| \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ L_1 \\ L_2 \\ L_1 \\ L_2 \\ L_3 \\ L_4 \\ L_3 \\ L_4 \\ L_5 \\ L_6 \\ L_5 \\ L_6 \end{array} \right|,$$

$$
\Psi_- = \Psi_2 = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{6}\left(-d_3 + 2f_1 - if_2 + g_3 - 2h_1 + ih_2\right) \\ \frac{1}{6}\left(-2d_1 - id_2 - f_3 + 2g_1 + ig_2 + h_3\right) \\ \frac{1}{6}\left(d_3 - 2f_1 + if_2 - g_3 + 2h_1 - ih_2\right) \\ \frac{1}{6}\left(2d_1 + id_2 + f_3 - 2g_1 - ig_2 - h_3\right) \\ \frac{1}{6}i\left(d_3 + f_1 - 2if_2 - g_3 - h_1 + 2ih_2\right) \\ \frac{1}{6}i\left(d_1 + 2id_2 - f_3 - g_1 - 2ig_2 + h_3\right) \\ -\frac{1}{6}i\left(d_3 + f_1 - 2if_2 - g_3 - h_1 + 2ih_2\right) \\ -\frac{1}{6}i\left(d_1 + 2id_2 - f_3 - g_1 - 2ig_2 + h_3\right) \\ \frac{1}{6}\left(d_1 - id_2 + 2f_3 - g_1 + ig_2 - 2h_3\right) \\ \frac{1}{6}\left(-2d_3 + f_1 + if_2 + 2g_3 - h_1 - ih_2\right) \\ \frac{1}{6}\left(-d_1 + id_2 - 2f_3 + g_1 - ig_2 + 2h_3\right) \\ \frac{1}{6}\left(2d_3 - f_1 - if_2 - 2g_3 + h_1 + ih_2\right) \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ P_1 \\ P_2 \\ -P_1 \\ -P_2 \\ P_3 \\ P_4 \\ -P_3 \\ -P_4 \\ P_5 \\ P_6 \\ -P_5 \\ -P_6 \end{vmatrix} .
$$

When performing the non-relativistic approximation, we should consider the Ψ_+ as large one, whereas Ψ_- and Ψ_0 should be considered as small:

$$P_i << L_i, \quad S_i << L_i;$$

projective constituents consist of the following variables

$$\Psi_+, \ \{L_1, ..., L_6\}; \quad \Psi_0, \ \{S_1, ..., S_8\}; \quad \Psi_-, \ \{P_1, ..., P_6\}. \quad (7)$$

3 Constraints in big and small components

Let us consider relations which define the large variable $L_1, ..., L_6$:

$$L_1 = \frac{1}{6} \left(d_3 + 2f_1 - if_2 + g_3 + 2h_1 - ih_2 \right),$$

$$L_3 = -\frac{1}{6} i \left(d_3 - f_1 + 2if_2 + g_3 - h_1 + 2ih_2 \right),$$

$$L_6 = \frac{1}{6}\left(2d_3 + f_1 + if_2 + 2g_3 + h_1 + ih_2\right),$$

whence it follows the constraint

$$L_1 + iL_3 - L_6 = 0;$$

and

$$L_2 = \frac{1}{6}\left(2d_1 + id_2 - f_3 + 2g_1 + ig_2 - h_3\right),$$

$$L_4 = -\frac{1}{6}i\left(d_1 + 2id_2 + f_3 + g_1 + 2ig_2 + h_3\right),$$

$$L_5 = \frac{1}{6}\left(-d_1 + id_2 + 2f_3 - g_1 + ig_2 + 2h_3\right),$$

whence it follows

$$L_2 - iL_4 + L_5 = 0.$$

Therefore, there exist only 4 independent large functions, for definiteness we will eliminate L_3, L_4:

$$iL_3 = L_6 - L_1, \quad iL_4 = L_5 + L_2. \tag{8}$$

Now let us consider relations which determine the constituent Ψ_- . ... Combing the relevant rows, we derive two identities

$$(A) \quad P_1 + iP_3 - P_6 = 0, \qquad (B) \quad P_2 - iP_4 + P_5 = 0; \qquad (9)$$

they provide us with two constraints which will be use below.

Now, let us consider relations which determine the sum od two small constituent

$$\Psi_0 + \Psi_- = \left| \begin{array}{c} S_5 + P_1 \\ S_6 + P_2 \\ S_7 - P_1 \\ S_8 - P_2 \\ \\ iS_5 + P_3 \\ iS_6 + P_4 \\ iS_7 - P_3 \\ iS_8 - P_4 \\ \\ S_6 + P_5 \\ -S_5 + P_6 \\ S_8 - P_5 \\ -S_7 - P_6 \end{array} \right| = \left| \begin{array}{c} +y_1 \\ +y_2 \\ +y_3 \\ +y_4 \\ \\ +y_5 \\ +y_6 \\ +y_7 \\ +y_8 \\ \\ +y_9 \\ +y_{10} \\ +y_{11} \\ +y_{12} \end{array} \right| , \qquad (10)$$

From these relations we can derive

$$y_1 + y_3 = S_5 + S_7, y_2 + y_4 = S_6 + S_8, y_5 + y_7 = i(S_5 + S_7),$$

$$y_6 + y_8 = i(S_6 + S_8), y_9 + y_{11} = S_6 + S_8, y_{10} + y_{12} = -(S_5 + S_7);$$

and

$$(y_1 + y_3) + (y_{10} + y_{12}) = 0, \quad (y_1 + y_3) + i(y_5 + y_7) = 0,$$
$$(y_2 + y_4) - (y_9 + y_{11}) = 0, \quad (y_2 + y_4) + i(y_6 + y_8) = 0.$$

and

$$S_5 - S_7 = (y_1 - y_3) + i(y_5 - y_7) - (y_{10} - y_{12}),$$
$$3(S_6 - S_8) = (y_2 - y_4) - i(y_6 - y_8) + (y_9 - y_{11}).$$

4 The study of the main system

Let us find 16 equations (5), using the presence of large and small variables, also taking into account the constraints (8); we omit their explicit form.

Further we perform several steps in calculations

1, Divide equations into 8 pairs

2. Sum and subtract equations within each pair

3. When performing the non-relativistic approximation, we should take into account the separation of rest energy by formal change

$$D_0 \Rightarrow (-iM + D_0) \tag{11}$$

4. We should tale into account the presence of small variables of different orders:

$$S_i \sim x, \quad y_s \sim x,, \quad \frac{D_0}{M} \sim x^2, \quad \frac{D_j}{M} \sim x. \tag{12}$$

5. We transform all equations to the new variables

$$y_1 + y_3 = \frac{1}{2} Z_1, \quad y_1 - y_3 = \frac{1}{2} Z_2,$$
$$y_2 + y_4 = \frac{1}{2} Z_3, \quad y_2 - y_4 = \frac{1}{2} Z_4,$$

$$y_5 + y_7 = \frac{1}{2} Z_5, \quad y_5 - y_7 = \frac{1}{2} Z_6,$$
$$y_6 + y_8 = \frac{1}{2} Z_7, \quad y_6 - y_8 = \frac{1}{2} Z_8,$$
$$y_9 + y_{11} = \frac{1}{2} Z_9, \quad y_9 - y_{11} = \frac{1}{2} Z_{10},$$
$$y_{10} + y_{12} = \frac{1}{2} Z_{11}, \quad y_{10} - y_{12} = \frac{1}{2} Z_{12},$$

We derive the six constraints (only 4 are independent):

$$Z_{11} = -Z_1, \quad iZ_5 = -Z_1, \quad (Z_{11} = iZ_5);$$
$$Z_9 = Z_3, \quad Z_7 = iZ_3, \quad (Z_9 = -iZ_7). \tag{13}$$

After that we can express all independent small components through the large ones

$$\left| \begin{array}{c} X \\ Y \\ Z_2 \\ Z_4 \\ Z_6 \\ Z_8 \\ Z_{10} \\ Z_{12} \end{array} \right| = \frac{1}{M} \left| \begin{array}{c} 2iD_1L_1 + 2iD_3L_5 + 2D_2\left(L_6 - L_1\right) \\ 2iD_1L_2 + 2D_2\left(L_2 + L_5\right) + 2iD_3L_6 \\ -2iD_3L_1 - 2iD_1L_2 - 2D_2L_2 \\ -2iD_1L_1 + 2D_2L_1 + 2iD_3L_2 \\ -2D_1\left(L_2 + L_5\right) + 2iD_2\left(L_2 + L_5\right) + 2D_3\left(L_1 - L_6\right) \\ 2D_3\left(L_2 + L_5\right) + 2D_1\left(L_1 - L_6\right) + 2iD_2\left(L_1 - L_6\right) \\ -2iD_3L_5 - 2iD_1L_6 - 2D_2L_6 \\ -2iD_1L_5 + 2D_2L_5 + 2iD_3L_6 \end{array} \right| . \quad (14)$$

and then substitute these relations into the reaming equations, in this way we arrive at 6 equations with non-relativistic structure, and they contain only the 4 large components L_1, L_2, L_5, L_6. After that we prove that only 4 equations from six ones are independent.

These 4 independent equations are transformed to the new variables:

$$\Psi = \begin{vmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 2 & 0 & 0 & -1 \\ 0 & 2 & 1 & 0 \end{vmatrix} \begin{vmatrix} L_1 \\ L_2 \\ L_5 \\ L_6 \end{vmatrix}; \qquad (15)$$

the final 4-component equation is presented in the form

$$iD_0\Psi = -\frac{1}{2M}\Delta\Psi + \frac{e}{3M}\Big(S_1F_{23} + S_2F_{31} + S_3F_{12}\Big)\Psi; \qquad (16)$$

$$D_0 = \partial_0 + ieA_0, \quad \Delta = (\partial_1 + ieA_1)^2 + (\partial_2 + ieA_2)^2 + (\partial_3 + ieA_3)^2.$$

Three matrices S_i obey the $su(2)$ algebra, they may be considered as referring to spin matrices:

$$\bar{S}_1 = \begin{vmatrix} 0 & -\frac{1}{2} & 0 & 0 \\ -\frac{3}{2} & 0 & -1 & 0 \\ 0 & -1 & 0 & -\frac{3}{2} \\ 0 & 0 & -\frac{1}{2} & 0 \end{vmatrix}, \bar{S}_2 = \begin{vmatrix} 0 & -\frac{i}{2} & 0 & 0 \\ \frac{3i}{2} & 0 & -i & 0 \\ 0 & i & 0 & -\frac{3i}{2} \\ 0 & 0 & \frac{i}{2} & 0 \end{vmatrix},$$

$$\bar{S}_3 = \begin{vmatrix} -3/2 & 0 & 0 & 0 \\ 0 & -1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 3/2 \end{vmatrix}.$$

5 Conclusion

We have derived the following Pauli-like equation for spin 3/2 particle. The structure of the derived equation definitely indicates that any anomalous solutions cannon appear, for instance in presence of external uniform electric or magnetic fields.

The next interesting task is to derive a non-relativistic equation starting with the tetrad based covariant equation (1). At this we should allow for that the non-relativistic approximation is possible (irrespective of the value of spin of a particle) only for space-time metrics with the following structure:

$$dS^2 = (dx^0)^2 + g_{ij}(x)dx^i dx^j. \tag{17}$$

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Spin 3/2 particle in the Coulomb field, the non-relativistic approximation

In the present paper, we have studied the nonrelativistic problem for a spin 3/2 particle in presence in the external Coulomb field. The known general procedure for performing the nonrelativistic approximation is based on the method of projective operators. This approach is applied directly in the relativistic system of radial equation derived previously for a free spin 3/2 particle for states with the spherical symmetry within the covariant tetrad formalism. The system of two 2-nd order differential equations describing the nonrelativistic particle has been derived. Solutions of radial equations have been constructed in terms of confluent hypergeometric functions, the corresponding energy spectra are found.

1 Separation of the variables

The basic equation for a spin 3/2 particle used in [5], has the form (it is equivalent to the known Rarita - Schwinger equation [3])

$$\gamma^5 \left(\gamma^1 \otimes \mu^{[01]} + \gamma^2 \otimes \mu^{[02]} + \gamma^3 \otimes \mu^{[03]}\right) iD_0 \Psi$$

$$+\gamma^5 \left(\gamma^0 \otimes \mu^{[01]} + \gamma^2 \otimes \mu^{[12]} - \gamma^3 \otimes \mu^{[31]}\right) iD_1 \Psi +$$

$$+\gamma^5 \left(\gamma^0 \otimes \mu^{[02]} + \gamma^3 \otimes \mu^{[23]} - \gamma^1 \otimes \mu^{[12]}\right) iD_2 \Psi$$

$$+\gamma^5 \left(\gamma^0 \otimes \mu^{[03]} + \gamma^1 \otimes \mu^{[31]} - \gamma^2 \otimes \mu^{[23]}\right) iD_3 \Psi -$$

$$-\gamma^5 M \Big\{ s_{01} \otimes \mu^{[01]} + s_{02} \otimes \mu^{[02]} + s_{03} \otimes \mu^{[03]}$$

$$+ s_{23} \otimes \mu^{[23]} + s_{31} \otimes \mu^{[31]} + s_{12} \otimes \mu^{[12]} \Big\} \Psi = 0 \, ,$$

where γ^a, γ^5 denote the Dirac matrices, $s_{ab} = \gamma_a \gamma_b - \gamma_b \gamma_a$, square brackets stand for the antisymmetry, symbol $\otimes$ designates the direct product of the matrices; the wave function is presented as a matrix with two indices, the first index is bispinor, and the second is vector one. The system of equations refers to the cyclic basis (in which the 16-dimensional generator J^{12} is diagonal, expressions for operators D_a will be given below. Six μ-matrices are defined by relations

$$\mu_{sp}^{[01]} = \frac{1}{\sqrt{2}} \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & -i \\ 0 & 0 & -i & 0 \end{vmatrix}, \mu_{sp}^{[02]} = \frac{1}{\sqrt{2}} \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{vmatrix},$$

$$\mu_{sp}^{[03]} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \end{vmatrix}, \mu_{sp}^{[23]} = \frac{1}{\sqrt{2}} \begin{vmatrix} 0 & -1 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix},$$

$$\mu_{sp}^{[31]} = \frac{1}{\sqrt{2}} \begin{vmatrix} 0 & -i & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{vmatrix}, \mu_{sp}^{[12]} = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}.$$

This Equation was considered [5] in spherical coordinates $x^{\alpha} = (t, r, \theta, \phi)$ and corresponding tetrad [6]. For components of the operator $D_c = e^{\alpha}_{(c)} \partial_{\alpha} + \frac{1}{2}(\sigma^{nm} \otimes I + I \otimes j^{nm})\, \gamma_{[nm]c}$ explicitly read

$$D_0 = \partial_t \, , \quad D_3 = \partial_r \, , \quad D_1 = \frac{1}{r}\partial_{\theta} + \frac{1}{r}(\sigma^{31} \otimes I + I \otimes j^{31}) \, ,$$

$$D_2 = \frac{1}{r}(\sigma^{32} \otimes I + I \otimes j^{32}) + \frac{1}{r}\frac{\partial_{\phi} + \cos\theta(\sigma^{12} \otimes I + I \otimes j^{12})}{\sin\theta} \, ,$$

where σ^{ab}, j^{ab} define generators for the bispinor and vector representations respectively.

The substitution for the wave function has the form [7, 8]

$$\Psi = \Psi_{A(l)} = e^{-i\epsilon t} \left| \begin{array}{cccc} f_0 D_{-1/2} & f_1 D_{-3/2} & f_2 D_{-1/2} & f_3 D_{+1/2} \\ g_0 D_{+1/2} & g_1 D_{-1/2} & g_2 D_{+1/2} & g_3 D_{+3/2} \\ h_0 D_{-1/2} & h_1 D_{-3/2} & h_2 D_{-1/2} & h_3 D_{+1/2} \\ d_0 D_{+1/2} & d_1 D_{-1/2} & d_2 D_{+1/2} & d_3 D_{+3/2} \end{array} \right| , \quad (1)$$

here we use the Wigner -functions, $D_{\sigma} = D^{j}_{-m,\sigma}(\phi, \theta, 0)\,;\; j = 1/2, 3/2, 5/2, \ldots$. This substitution contains 16 radial functions

f_a, g_a, h_a, d_a. At $j = 1/2$ it simplifies, $f_1 = 0, g_3 = 0, h_1 = 0, d_3 = 0$. From requirement of diagonalization of the space reflection operator [8], we derive the following restrictions (where $\delta = \pm 1$):

$$d_0 = \delta f_0, d_1 = \delta f_3, d_2 = \delta f_2, d_3 = \delta f_1,$$

$$h_0 = \delta g_0, h_1 = \delta g_3, h_2 = \delta g_2, h_3 = \delta g_1;$$

so only only 8 independent functions are preserved:

$$\Psi = \left| \begin{array}{cccc} f_0 D_{-1/2} & f_1 D_{-3/2} & f_2 D_{-1/2} & f_3 D_{+1/2} \\ g_0 D_{+1/2} & g_1 D_{-1/2} & g_2 D_{+1/2} & g_3 D_{+3/2} \\ \delta g_0 D_{-1/2} & \delta g_3 D_{-3/2} & \delta g_2 D_{-1/2} & \delta g_1 D_{+1/2} \\ \delta f_0 D_{+1/2} & \delta f_3 D_{-1/2} & \delta f_2 D_{+1/2} & \delta f_1 D_{+3/2} \end{array} \right| . \qquad (2)$$

Using the known relations for the Wigner functions [7],

$$\partial_\theta \; D_{+1/2} = \frac{1}{2}(a \; D_{-1/2} - b \; D_{+3/2}), \partial_\theta \; D_{-1/2} = \frac{1}{2}(b \; D_{-3/2} - a \; D_{+1/2}),$$

$$\partial_\theta \; D_{+3/2} = \frac{1}{2}(b \; D_{+1/2} - c \; D_{+5/2}), \quad \partial_\theta \; D_{-3/2} = \frac{1}{2}(c \; D_{-5/2} - b \; D_{-1/2}),$$

$$\frac{1}{\sin\theta}(-m - \frac{1}{2}\cos\theta)D_{+1/2} = \frac{1}{2}(-a\ D_{-1/2} - b\ D_{+3/2}),$$

$$\frac{1}{\sin\theta}(-m + \frac{1}{2}\cos\theta)D_{-1/2} = \frac{1}{2}(-b\ D_{-3/2} - a\ D_{+1/2}),$$

$$\frac{1}{\sin\theta}(-m - \frac{3}{2}\cos\theta)D_{+3/2} = \frac{1}{2}(-b\ D_{+1/2} - c\ D_{+5/2}),$$

$$\frac{1}{\sin\theta}(-m + \frac{3}{2}\cos\theta)D_{-3/2} = \frac{1}{2}(-c\ D_{-5/2} - b\ D_{-1/2}),$$

$$a = j+1/2\,, \quad b = \sqrt{(j-1/2)(j+3/2)}\,, \quad c = \sqrt{(j-3/2)(j+5/2)}\,,$$

after separating the variables, in [5] where found 8 equations (the equations for states with opposite parities differ only in sign at the mass parameter, so we follow only the case $\delta = +1$; radial equations for states with opposite parities differ only in sign at the mass parameter M)

$$\sqrt{2}\frac{d}{dr}g_1 + \frac{1}{r}(f_2 + \frac{3}{\sqrt{2}}g_1) + \frac{1}{\sqrt{2}r}(bf_1 - af_3 + a\sqrt{2}g_2) + iM(g_2 - \sqrt{2}f_3) = 0,$$

$$\sqrt{2}\frac{d}{dr}f_3+\frac{1}{r}(g_2+\frac{3}{\sqrt{2}}f_3)+\frac{1}{\sqrt{2}r}(-ag_1+bg_3+a\sqrt{2}f_2)+iM(\sqrt{2}g_1-f_2)=0,$$

$$-i\epsilon f_1+\frac{d}{dr}f_1+\frac{1}{r}f_1+\frac{1}{\sqrt{2}r}(bf_2+bf_0)+iMg_3=0,$$

$$-i\epsilon(\sqrt{2}f_2-g_1)+(-\sqrt{2}\frac{d}{dr}f_0+\frac{}{dr}g_1)-\frac{1}{r}(\frac{1}{\sqrt{2}}(f_0-f_2)-g_1)$$

$$+\frac{1}{\sqrt{2}r}(ag_2-ag_0)-iM(f_3+\sqrt{2}(g_0-g_2))=0,$$

$$-i\epsilon\sqrt{2}g_1+\frac{1}{r}(f_0-\frac{1}{\sqrt{2}}g_1)+\frac{1}{\sqrt{2}r}(-bf_1+af_3+a\sqrt{2}g_0)+iM(\sqrt{2}f_3+g_0)=0,$$

$$-i\epsilon\sqrt{2}f_3+\frac{1}{r}(g_0+\frac{1}{\sqrt{2}}f_3)+\frac{1}{\sqrt{2}r}(-ag_1+bg_3+a\sqrt{2}f_0)+iM(\sqrt{2}g_1-f_0)=0,$$

$$-i\epsilon(\sqrt{2}g_2-f_3)+(-\sqrt{2}\frac{d}{dr}g_0-\frac{d}{dr}f_3)$$

$$-\frac{1}{r}(\frac{1}{\sqrt{2}}(g_0+g_2)+f_3)+\frac{1}{\sqrt{2}r}(-af_2-af_0)+$$

$$+iM(\sqrt{2}f_0+\sqrt{2}f_2-g_1)=0,$$

$$-i\epsilon g_3-\frac{}{dr}g_3-\frac{1}{r}g_3+\frac{1}{\sqrt{2}r}(-bg_2+bg_0)+iMf_1=0.$$

2 Large and small components, projection operators

Exists the method for performing the nonrelativistic approximation in equations for spin 3/2 particle in Cartesian coordinates [4]:

$$\Big[\Gamma^0\partial_0+i\Gamma^1\partial_1+i\Gamma^2\partial_2+i\Gamma^3\partial_3+iM\Gamma\Big]\Psi^{cart}=0; \qquad (3)$$

After changing notation, this equation is written as

$$\Big[Y^0\partial_0+iY^1\partial_1+Y^2\partial_2+Y^3\partial_3+iM\Big]\Psi^{cart}=0. \qquad (4)$$

On the base of 3-order minimal equation for the matrix Y^0, three projective operators are introduced which allow us to decompose the wave function into the sum of three components: one large and two small. In order to apply the same method in spherical coordinates, we need explicit form for three matrices

$$Y^0 = \Gamma^{-1}\Gamma^0, \quad \Gamma^0 = \gamma^5 \left(\gamma^1 \otimes \mu^{[01]} + \gamma^2 \otimes \mu^{[02]} + \gamma^3 \otimes \mu^{[03]}\right),$$

$$\Gamma = \gamma^5 \Big\{ s_{01} \otimes \mu^{[01]} + s_{02} \otimes \mu^{[02]} + s_{03} \otimes \mu^{[03]}$$

$$+ s_{23} \otimes \mu^{[23]} + s_{31} \otimes \mu^{[31]} + s_{12} \otimes \mu^{[12]} \Big\}.$$

representations for the wave function and the matrix Γ^0 are

$$\Psi = \{ f_0 g_0 h_0 d_0 f_1 g_1 h_1 d_1 f_2 g_2 h_2 d_2 f_3 g_3 h_3 d_3 \}$$

Its minimal equation is $Y_0^2(Y_0^2 - 1) = 0$, we can introduce three projective operators

$$P_0 = 1 - Y_0^2, \quad P_+ = +\frac{1}{2} Y_0^2 (Y_0 + 1), \quad P_- = -\frac{1}{2} Y_0^2 (Y_0 - 1); \quad (5)$$

with the properties $P_0 + P_+ + P_- = 1, P_0^2 = P_0, P_+^2 = P_+, \quad P_-^2 = P_-$. The explicit expressions for these operators are needed ...

$$\Psi_+ = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{2}\left(f_1 + h_1\right) \\ \frac{1}{6}\left(d_1 + \sqrt{2}f_2 + g_1 + \sqrt{2}h_2\right) \\ \frac{1}{2}\left(f_1 + h_1\right) \\ \frac{1}{6}\left(d_1 + \sqrt{2}f_2 + g_1 + \sqrt{2}h_2\right) \\ \frac{1}{6}\left(\sqrt{2}d_1 + 2f_2 + \sqrt{2}g_1 + 2h_2\right) \\ \frac{1}{6}\left(2d_2 + \sqrt{2}f_3 + 2g_2 + \sqrt{2}h_3\right) \\ \frac{1}{6}\left(\sqrt{2}d_1 + 2f_2 + \sqrt{2}g_1 + 2h_2\right) \\ \frac{1}{6}\left(2d_2 + \sqrt{2}f_3 + 2g_2 + \sqrt{2}h_3\right) \\ \frac{1}{6}\left(\sqrt{2}d_2 + f_3 + \sqrt{2}g_2 + h_3\right) \\ \frac{1}{2}\left(d_3 + g_3\right) \\ \frac{1}{6}\left(\sqrt{2}d_2 + f_3 + \sqrt{2}g_2 + h_3\right) \\ \frac{1}{2}\left(d_3 + g_3\right) \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ L_1 \\ L_2 \\ L_1 \\ L_2 \\ L_3 \\ L_4 \\ L_3 \\ L_4 \\ L_5 \\ L_6 \\ L_5 \\ L_6 \end{vmatrix},$$

$$\Psi_0 = \left| \begin{array}{c} f_0 \\ g_0 \\ h_0 \\ d_0 \\ 0 \\ \frac{1}{3}\left(2g_1 - \sqrt{2}f_2\right) \\ 0 \\ \frac{1}{3}\left(2d_1 - \sqrt{2}h_2\right) \\ -\frac{1}{3\sqrt{2}}\left(2g_1 - \sqrt{2}f_2\right) \\ \frac{1}{3}\left(g_2 - \sqrt{2}f_3\right) \\ -\frac{1}{3\sqrt{2}}\left(2d_1 - \sqrt{2}h_2\right) \\ \frac{1}{3}\left(d_2 - \sqrt{2}h_3\right) \\ -\frac{\sqrt{2}}{3}\left(g_2 - \sqrt{2}f_3\right) \\ 0 \\ -\frac{\sqrt{2}}{3}\left(d_2 - \sqrt{2}h_3\right) \\ 0 \end{array} \right| = \left| \begin{array}{r} S_1 \\ S_2 \\ S_3 \\ S_4 \\ 0 \\ S_5 \\ 0 \\ S_6 \\ -\frac{1}{\sqrt{2}}S_5 \\ S_7 \\ -\frac{1}{\sqrt{2}}S_6 \\ S_8 \\ -\sqrt{2}S_7 \\ 0 \\ -\sqrt{2}S_8 \\ 0 \end{array} \right| ,$$

$$
\Psi_- = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{2}\left(f_1 - h_1\right) \\ \frac{1}{6}\left(-d_1 + \sqrt{2}f_2 + g_1 - \sqrt{2}h_2\right) \\ \frac{1}{2}\left(h_1 - f_1\right) \\ \frac{1}{6}\left(d_1 - \sqrt{2}f_2 - g_1 + \sqrt{2}h_2\right) \\ \frac{1}{6}\left(-\sqrt{2}d_1 + 2f_2 + \sqrt{2}g_1 - 2h_2\right) \\ \frac{1}{6}\left(-2d_2 + \sqrt{2}f_3 + 2g_2 - \sqrt{2}h_3\right) \\ \frac{1}{6}\left(\sqrt{2}d_1 - 2f_2 - \sqrt{2}g_1 + 2h_2\right) \\ \frac{1}{6}\left(2d_2 - \sqrt{2}f_3 - 2g_2 + \sqrt{2}h_3\right) \\ \frac{1}{6}\left(-\sqrt{2}d_2 + f_3 + \sqrt{2}g_2 - h_3\right) \\ \frac{1}{2}\left(g_3 - d_3\right) \\ \frac{1}{6}\left(\sqrt{2}d_2 - f_3 - \sqrt{2}g_2 + h_3\right) \\ \frac{1}{2}\left(d_3 - g_3\right) \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ P_1 \\ P_2 \\ -P_1 \\ -P_2 \\ P_3 \\ P_4 \\ -P_3 \\ -P_4 \\ P_5 \\ P_6 \\ -P_5 \\ -P_6 \end{vmatrix} .
$$

As known from the general theory, the component Ψ_+ should be considered as the large one, and two other as small, $P_i << L_i$, $S_i << L_i$. Three projective constituents consist of following variables

$$\Psi_+, \quad \{L_1, ..., L_6\}, \quad \Psi_0, \quad \{S_1, ..., S_8\}, \quad \Psi_-, \quad \{P_1, ..., P_6\}.$$

Since the initial wave function contains 16 components, we should expect existence of 4 restrictions.

After preserving only independent variables among $L_{..}, P_{..}, S_{..}$, and taking into account parity restrictions, we obtain

$$\Psi_{\delta=+1} = \begin{vmatrix} f_0 \\ g_0 \\ g_0 \\ f_0 \\ f_1 \\ g_1 \\ g_3 \\ f_3 \\ f_2 \\ g_2 \\ g_2 \\ f_2 \\ f_3 \\ g_3 \\ g_1 \\ f_1 \end{vmatrix} = \begin{vmatrix} S_1 \\ S_2 \\ +S_2 \\ +S_1 \\ L_1 + y_1 \\ L_2 + y_2 \\ L_1 - y_1 \\ L_2 - y_2 - \sqrt{2}y_4 - \sqrt{2}y_5 \\ \sqrt{2}L_2 + y_4 \\ \sqrt{2}L_2 + y_5 \\ \sqrt{2}L_2 + y_5 \\ \sqrt{2}L_2 + y_4 \\ L_2 - y_2 - \sqrt{2}y_4 - \sqrt{2}y_5 \\ +L_1 - y_1 \\ L_2 + y_2, \\ +L_1 + y_1 \end{vmatrix},$$

$$\Psi_{\delta=-1} = \left|\begin{matrix} f_0 \\ g_0 \\ -g_0 \\ -f_0 \\[4pt] f_1 \\ g_1 \\ -g_3 \\ -f_3 \\[4pt] f_2 \\ g_2 \\ -g_2 \\ -f_2 \\[4pt] f_3 \\ g_3 \\ -g_1 \\ -f_1 \end{matrix}\right| = \left|\begin{matrix} S_1 \\ S_2 \\ -S_2 \\ -S_1 \\[4pt] L_1 + y_1 \\ L_2 + y_2 \\ L_1 - y_1 \\ L_2 - y_2 - \sqrt{2}y_4 + \sqrt{2}y_5 \\[4pt] \sqrt{2}L_2 + y_4 \\ -\sqrt{2}L_2 + y_5 \\ \sqrt{2}L_2 - y_5 \\ -\sqrt{2}L_2 - y_4 \\[4pt] -L_2 + y_2 + \sqrt{2}y_4 - \sqrt{2}y_5 \\ -L_1 + y_1 \\ -L_2 - y_2, \\ -L_1 - y_1 \end{matrix}\right| ;$$

Further, we can pass to truncated 8-component columns (recall that when separating the variables for states with fixed parities, two systems of 8 equations were derived; and they differ only in sign at the mass parameter)

$$\Psi_{\delta=+1} = \left| \begin{array}{c} f_0 \\ g_0 \\ f_1 \\ g_1 \\ g_3 \\ f_3 \\ f_2 \\ g_2 \\ g_2 \\ f_2 \\ f_3 \\ g_3 \end{array} \right| = \left| \begin{array}{c} S_1 \\ S_2 \\ L_1 + y_1 \\ L_2 + y_2 \\ L_1 - y_1 \\ L_2 - y_2 - \sqrt{2}y_4 - \sqrt{2}y_5 \\ \sqrt{2}L_2 + y_4 \\ \sqrt{2}L_2 + y_5 \\ \sqrt{2}L_2 + y_5 \\ \sqrt{2}L_2 + y_4 \\ L_2 - y_2 - \sqrt{2}y_4 - \sqrt{2}y_5 \\ +L_1 - y_1 \end{array} \right| ,$$

$$\Psi_{\delta=-1} = \begin{vmatrix} f_0 \\ g_0 \\ f_1 \\ g_1 \\ -g_3 \\ -f_3 \\ f_2 \\ g_2 \\ -g_2 \\ -f_2 \\ f_3 \\ g_3 \end{vmatrix} = \begin{vmatrix} S_1 \\ S_2 \\ L_1 + y_1 \\ L_2 + y_2 \\ L_1 - y_1 \\ L_2 - y_2 - \sqrt{2}y_4 + \sqrt{2}y_5 \\ \sqrt{2}L_2 + y_4 \\ -\sqrt{2}L_2 + y_5 \\ \sqrt{2}L_2 - y_5 \\ -\sqrt{2}L_2 - y_4 \\ -L_2 + y_2 + \sqrt{2}y_4 - \sqrt{2}y_5 \\ -L_1 + y_1 \end{vmatrix};$$

The truncated 8-dimensional column (it is composed of functions included in the 8-dimensional radial system) turns out to be the same (the difference for the parity will be only in sign at the mass parameter). The inverse transformation has the form

$$L_1 = \frac{1}{2}(f_1 + g_3), L_2 = \frac{1}{6}(\sqrt{2}f_2 + f_3 + g_1 + \sqrt{2}g_2), S_1 = f_0,$$

$$S_2 = g_0, y_1 = \frac{1}{2}(f_1 - g_3), y_2 = \frac{1}{6}(-\sqrt{2}f_2 - f_3 + 5g_1 - \sqrt{2}g_2),$$

$$y_4 = \frac{1}{6}(4f_2 - \sqrt{2}f_3 - \sqrt{2}g_1 - 2g_2), y_5 = \frac{1}{6}(-2f_2 - \sqrt{2}f_3 - \sqrt{2}g_1 + 4g_2).$$

We turn to the system of 8 radial equations (the presence of external Coulomb field is taken into account by formal change $E \Rightarrow E + \alpha/r$), and substitute expressions for functions through independent large and small components.

Then, we should separate the rest energy by the formal change $\epsilon = (M+E), E << M$. Besides, when performing the no-relativistic approximation in the resulting radial system, we should follow the known prescriptions [4] for smallness orders of various quantities

$$L \sim 1, \quad y \sim x, \quad S \sim x, \quad \frac{1}{M}\frac{d}{dr} \sim x, \quad \frac{1}{rM} \sim x,$$

$$rM \sim \frac{1}{x}, \quad r\frac{1}{M}\frac{d^2}{dr^2} \sim x, \quad \frac{E}{M} \sim x^2, \quad rE \sim x, \quad r\frac{d}{dr} \sim 1.$$

Then, after rather long manipulation with the radial equations, in order to correctly take into account existence of large and small variables, and eliminating the all small variables, we can derive two differential equations for two large variables L_1, L_2:

$$\Big[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + 2M(E + \frac{\alpha}{r})\Big] L_1 = \frac{1}{r^2}\Big[b^2 L_1 + 3bL_2\Big],$$

$$\Big[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + 2M(E + \frac{\alpha}{r})\Big] L_2$$
$$= \frac{1}{r^2}\Big[\frac{2a(a+3) + b^2 + 10}{3} L_2 + bL_1\Big].$$

The last system, after performing the relevant linear transformation $\bar{L} = S L$, can be reduced to separate equations for new variables $\bar{L}_1, \bar{L}_2$:

$$\Big[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + 2M(E + \frac{\alpha}{r}) - \frac{(j+2)^2 - 1/4}{r^2}\Big] \bar{L}_1 = 0, \qquad (6)$$

$$\Big[\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}+2M(E+\frac{\alpha}{r})-\frac{j^2-1/4}{r^2}\Big]\bar{L}_2=0. \qquad (7)$$

These equations are solved in hypergeometric functions, they provide us with two energy spectra (let $2Br=-x$)

$$E_2=-\frac{\alpha^2 M}{2N^2}=-\frac{\alpha^2 M}{2(j+\frac{1}{2}+n)^2}, \bar{L}_2=r^{j-1/2}e^{-\sqrt{-2ME}r}F(-n,2j{+}1,x).$$

$$E_1=-\frac{\alpha^2 M}{2N^2}=-\frac{\alpha^2 M}{2(j+\frac{5}{2}+n)^2}, \bar{L}_1=r^{j+3/2}e^{-\sqrt{-2ME}r}F(-n,2j{+}5,x).$$

The initial variables L_1, L_2 are determined by the rule $L = S^{-1}\bar{L}$, and they explicitly read

$$L_1=(b^2-j^2+\frac{1}{4})\bar{L}_1+(b^2-(j+2)^2+\frac{1}{4})\bar{L}_2, L_2=b\bar{L}_1+b\bar{L}_2.$$

The non-relativistic wave function with quantum numbers of the energy, square and the third projection of the total momentum, and parity is given as

$$\Psi_{E,j,m,\delta}(t,r,\theta,\phi) = e^{-iEt} \begin{vmatrix} L_1(r)\, D^j_{-m,-3/2}(\phi,\theta,0) \\ L_2(r)\, D^j_{-m,-1/2}(\phi,\theta,0) \\ \delta L_2(r)\, D^j_{-m,+1/2}(\phi,\theta,0) \\ \delta L_1(r)\, D^j_{-m,+3/2}(\phi,\theta,0) \end{vmatrix}, \quad \delta = \pm 1. \quad (8)$$

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Non-relativistic Approximation for a Spin 3/2 Particle, in Electromagnetic and Gravitational Fields

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The starting relativistic equation

The generally covariant equation for a spin 3/2 particle may be presented the local tetrad form

$$\gamma^5 (\mu^{[ca]})_k{}^n \gamma_c \left[i (D_a)_n{}^l - M \gamma_a \delta_n{}^l \right] \Psi_l = 0, \quad \epsilon_k{}^{can} = (\mu^{[ca]})_k{}^n .$$

where generalized derivative are used

$$D_a = e^{\alpha}_{(a)} (\partial_\alpha + ieA_\alpha) + \frac{1}{2} (\sigma^{ps} \otimes I + I \otimes j^{ps})\, \gamma_{[ps]a} = e^{\alpha}_{(a)} (\partial_\alpha + ieA_\alpha) + \Sigma_a .$$

Large and small non-relativistic components

The wave function may be presented as a matrix (the first index A is bispinor one, the second (n) is 4-vector one)

$$\Psi_{A(n)} = \begin{vmatrix} f_0 & f_1 & f_2 & f_3 \\ g_0 & g_1 & g_2 & g_3 \\ h_0 & h_1 & h_2 & h_3 \\ d_0 & d_1 & d_2 & d_3 \end{vmatrix} .$$

We will present the complete Ψ as a 16-dimensional column.
It is convenient to present the main equation in the matrix form

$$\Big(Y_0 D_0 + Y^1 D_1 + Y^2 D_2 + Y^3 D_3 + iM \Big) \Psi = 0.$$

In accordance with the known general approach, the large and small components in the non-relativistic limit are determined by projective operators constructed through the matrix Y^0. This matrix Y_0 obeys the minimal 4th order equation $Y_0^2(Y_0^2 - I_{16}) = 0$.

Correspondingly, there exist three projective operators (they are found in explicit form)

$$P_0 = I_{16} - Y_0^2,$$

$$P_1 = P_+ = +\frac{1}{2} Y_0^2 (Y + I_{16}),$$

$$P_2 = P_- = -\frac{1}{2} Y_0^2 (Y - I_{16})$$

with the needed properties

$$P_0 + P_+ + P_- = I_{16}, \quad P_0^2 = P_0, \; P_1^2 = P_1, \; P_2^2 = P_2.$$

We find three projective constituents (in each we can see a number of independent variables):

$$\Psi_0 = P_0 \Psi \, , \quad \Psi_+ = \Psi_1 = P_1 \Psi, \quad \Psi_- = \Psi_2 = P_2 \Psi;$$

$$
\Psi_0 = \left| \begin{array}{c} f_0 \\ g_0 \\ h_0 \\ d_0 \\ \frac{1}{3}\left(f_1 + if_2 - g_3\right) \\ \frac{1}{3}\left(f_3 + g_1 - ig_2\right) \\ \frac{1}{3}\left(-d_3 + h_1 + ih_2\right) \\ \frac{1}{3}\left(d_1 - id_2 + h_3\right) \\ \frac{1}{3}\left(-if_1 + f_2 + ig_3\right) \\ \frac{1}{3}i\left(f_3 + g_1 - ig_2\right) \\ \frac{1}{3}\left(i\left(d_3 - h_1\right) + h_2\right) \\ \frac{1}{3}i\left(d_1 - id_2 + h_3\right) \\ \frac{1}{3}\left(f_3 + g_1 - ig_2\right) \\ \frac{1}{3}\left(-f_1 - if_2 + g_3\right) \\ \frac{1}{3}\left(d_1 - id_2 + h_3\right) \\ \frac{1}{3}\left(d_3 - h_1 - ih_2\right) \end{array} \right| = \left| \begin{array}{r} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \\ S_7 \\ S_8 \\ -iS_5 \\ iS_6 \\ -iS_7 \\ iS_8 \\ S_6 \\ -S_5 \\ S_8 \\ -S_7 \end{array} \right| ,
$$

$$
\Psi_- = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{6}\left(-d_3 + 2f_1 - if_2 + g_3 - 2h_1 + ih_2\right) \\ \frac{1}{6}\left(-2d_1 - id_2 - f_3 + 2g_1 + ig_2 + h_3\right) \\ \frac{1}{6}\left(d_3 - 2f_1 + if_2 - g_3 + 2h_1 - ih_2\right) \\ \frac{1}{6}\left(2d_1 + id_2 + f_3 - 2g_1 - ig_2 - h_3\right) \\ \frac{1}{6}i\left(d_3 + f_1 - 2if_2 - g_3 - h_1 + 2ih_2\right) \\ \frac{1}{6}i\left(d_1 + 2id_2 - f_3 - g_1 - 2ig_2 + h_3\right) \\ -\frac{1}{6}i\left(d_3 + f_1 - 2if_2 - g_3 - h_1 + 2ih_2\right) \\ -\frac{1}{6}i\left(d_1 + 2id_2 - f_3 - g_1 - 2ig_2 + h_3\right) \\ \frac{1}{6}\left(d_1 - id_2 + 2f_3 - g_1 + ig_2 - 2h_3\right) \\ \frac{1}{6}\left(-2d_3 + f_1 + if_2 + 2g_3 - h_1 - ih_2\right) \\ \frac{1}{6}\left(-d_1 + id_2 - 2f_3 + g_1 - ig_2 + 2h_3\right) \\ \frac{1}{6}\left(2d_3 - f_1 - if_2 - 2g_3 + h_1 + ih_2\right) \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ P_1 \\ P_2 \\ -P_1 \\ -P_2 \\ P_3 \\ P_4 \\ -P_3 \\ -P_4 \\ P_5 \\ P_6 \\ -P_5 \\ -P_6 \end{vmatrix} .
$$

$$\Psi_+ \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{1}{6}\left(d_3 + 2f_1 - if_2 + g_3 + 2h_1 - ih_2\right) \\ \frac{1}{6}\left(2d_1 + id_2 - f_3 + 2g_1 + ig_2 - h_3\right) \\ \frac{1}{6}\left(d_3 + 2f_1 - if_2 + g_3 + 2h_1 - ih_2\right) \\ \frac{1}{6}\left(2d_1 + id_2 - f_3 + 2g_1 + ig_2 - h_3\right) \\ -\frac{1}{6}i\left(d_3 - f_1 + 2if_2 + g_3 - h_1 + 2ih_2\right) \\ -\frac{1}{6}i\left(d_1 + 2id_2 + f_3 + g_1 + 2ig_2 + h_3\right) \\ -\frac{1}{6}i\left(d_3 - f_1 + 2if_2 + g_3 - h_1 + 2ih_2\right) \\ -\frac{1}{6}i\left(d_1 + 2id_2 + f_3 + g_1 + 2ig_2 + h_3\right) \\ \frac{1}{6}\left(-d_1 + id_2 + 2f_3 - g_1 + ig_2 + 2h_3\right) \\ \frac{1}{6}\left(2d_3 + f_1 + if_2 + 2g_3 + h_1 + ih_2\right) \\ \frac{1}{6}\left(-d_1 + id_2 + 2f_3 - g_1 + ig_2 + 2h_3\right) \\ \frac{1}{6}\left(2d_3 + f_1 + if_2 + 2g_3 + h_1 + ih_2\right) \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 0 \\ L_1 \\ L_2 \\ L_1 \\ L_2 \\ L_3 \\ L_4 \\ L_3 \\ L_4 \\ L_5 \\ L_6 \\ L_5 \\ L_6 \end{vmatrix},$$

While performing the non-relativistic approximation, we should consider Ψ_+ as a large component, and Ψ_-, Ψ_0 as small ones: $P_i << L_i, \S_i << L_i$; in total we have the following 20 variables:

$$\Psi_+, \ \{L_1, ..., L_6\}; \quad \Psi_0, \ \{S_1, ..., S_8\}; \quad \Psi_-, \ \{P_1, ..., P_6\}.$$

1) There exist constraints among the large and small components:

$$iL_3 = L_6 - L_1, \quad iL_4 = L_5 + L_2;$$

$$P_1 + iP_3 - P_6 = 0, \qquad P_2 - iP_4 + P_5 = 0.$$

With new variables $y_1, ..., y_{12}$:

$$S_5 + P_1 = y_1, \quad S_6 + P_2 = y_2, \quad S_7 - P_1 = y_3, \quad S_8 - P_2 = y_4,$$

$$iS_5 + P_3 = y_5, \quad iS_6 + P_4 = y_6, \quad iS_7 - P_3 = y_7, \quad iS_8 - P_4 = y_8,$$

$$S_6 + P_5 = y_9, \quad -S_5 + P_6 = y_{10}, \quad S_8 - P_5 = y_{11}, \quad -S_7 - P_6 = y_{12}.$$

the four variables S_5, S_6, S_7, S_8 may be expressed through 12 y-variables.

The non-relativistic approximation

The non-relativistic approximation is possible only in space-time models with the following structure

$$dS^2 = (dx^0)^2 + g_{ij}(x) dx^i dx^j , \qquad e_{(a)\alpha}(x) = \left| \begin{array}{cc} 1 & 0 \\ 0 & e_{(i)k}(x) \end{array} \right| ;$$

for these space models only four connections differ from zero

$$\Sigma_0 = \frac{1}{2} \, J^{ik} e_{(i)}^{m} \, (\nabla_0 e_{(k)m}), \qquad \Sigma_l = \frac{1}{2} \, J^{ik} e_{(i)}^{m} \, (\nabla_l e_{(k)m}) \; ; \qquad (1)$$

and the contribution of three generators J^{01}, J^{02}, J^{03} is absent.

The basic matrix equation in the nonrelativistic metrics for a spin 3/2 particle has the form

$$\Big(Y^0 \bar{D}_0 + Y^1 \bar{D}_1 + Y^2 \bar{D}_2 + \Gamma^3 \bar{D}_3 + iM \Big) \Psi = 0, \qquad (2)$$

where

$$\bar{D}_0 = (\partial_0 + ieA_0) + (\sigma^{23} \otimes I + I \otimes j^{23})\boldsymbol{\gamma}_{[23]0} +$$

$$+(\sigma^{31} \otimes I + I \otimes j^{31})\boldsymbol{\gamma}_{[31]0} + (\sigma^{12} \otimes I + I \otimes j^{12})\boldsymbol{\gamma}_{[12]0},$$

$$\bar{D}_1 = e^k_{(1)}(\partial_k + ieA_k) + (\sigma^{23} \otimes I + I \otimes j^{23})\boldsymbol{\gamma}_{[23]1} +$$

$$+(\sigma^{31} \otimes I + I \otimes j^{31})\boldsymbol{\gamma}_{[31]1} + (\sigma^{12} \otimes I + I \otimes j^{12})\boldsymbol{\gamma}_{[12]1},$$

$$\bar{D}_2 = e^k_{(2)}(\partial_k + ieA_k) + (\sigma^{23} \otimes I + I \otimes j^{23})\boldsymbol{\gamma}_{[23]2} +$$

$$+(\sigma^{31} \otimes I + I \otimes j^{31})\boldsymbol{\gamma}_{[31]2} + (\sigma^{12} \otimes I + I \otimes j^{12})\boldsymbol{\gamma}_{[12]2},$$

$$\bar{D}_3 = (\partial_k + ieA_k) + (\sigma^{23} \otimes I + I \otimes j^{23})\boldsymbol{\gamma}_{[23]3} +$$

$$+(\sigma^{31} \otimes I + I \otimes j^{31})\boldsymbol{\gamma}_{[31]3} + (\sigma^{12} \otimes I + I \otimes j^{12})\boldsymbol{\gamma}_{[12]3}.$$

We will use special notations for the needed 12 Ricci coefficients

$$G_{10} = \boldsymbol{\gamma}_{230}, \quad G_{20} = \boldsymbol{\gamma}_{310}, \quad G_{30} = \boldsymbol{\gamma}_{120},$$

$$G_{1j} = \boldsymbol{\gamma}_{23j}, \quad G_{21} = \boldsymbol{\gamma}_{31j}, \quad G_{31} = \boldsymbol{\gamma}_{12j}, \quad i = 1, 2, 3.$$

In the basic equation one can distinguish two parts:

$$\left(Y^0 \bar{D}_0 + Y^1 \bar{D}_1 + Y^2 \bar{D}_2 + Y^3 \bar{D}_3 + iM\right)\Psi = 0,$$

$$\left(Y^0 D_0 + Y^1 D_1 + Y^2 D_2 + Y^3 D_3 + iM\right)\Psi +$$

$$+\left(Q^0\Psi + Q^1\Psi + Q^2\Psi + Q^3\Psi\right) = 0,$$

$$D_0 = (\partial_0 + ieA_0), \quad D_1 = e^k_{(1)}(\partial_k + ieA_k),$$

$$D_2 = e^k_{(2)}(\partial_k + ieA_k), \quad \bar{D}_3 = (\partial_k + ieA_k)$$

$$Q^0\Psi = Y^0\left[(\sigma^{23}\Psi + \Psi\tilde{j}^{23})G_{10} + (\sigma^{31}\Psi + \Psi\tilde{j}^{31})G_{20} + (\sigma^{12}\Psi + \Psi\tilde{j}^{12})G_{30}\right],$$

$$Q^1\Psi = Y^1\left[(\sigma^{23}\Psi + \Psi\tilde{j}^{23})G_{11} + (\sigma^{31}\Psi + \Psi\tilde{j}^{31})G_{21} + (\sigma^{12}\Psi + \Psi\tilde{j}^{12})G_{31}\right],$$

$$Q^2\Psi = Y^2\left[(\sigma^{23}\Psi + \Psi\tilde{j}^{23})G_{12} + (\sigma^{31}\Psi + \Psi\tilde{j}^{31})G_{22} + (\sigma^{12}\Psi + \Psi\tilde{j}^{12})G_{32}\right],$$

$$Q^3\Psi = Y^3\left[(\sigma^{23}\Psi + \Psi\tilde{j}^{23})G_{13} + (\sigma^{31}\Psi + \Psi\tilde{j}^{31})G_{23} + (\sigma^{12}\Psi + \Psi\tilde{j}^{12})G_{33}\right].$$

After performing the needed calculations, taking in mind decompositions of all components into large and small parts, we derive 16 rather complicated equations; we omit them ...

It is known, that when performing the non-relativistic approximation, we should assume the following smallness order to the involved quantities

$$L_{..} \sim 1, \quad S_{..}, \qquad y_{..} \sim x, \quad \frac{D_j}{M} \sim x, \quad \frac{G_{ij}}{M} \sim x,$$

$$\frac{D_0}{M} \sim x^2, \quad \frac{G_{j0}}{M} \sim x^2,$$

in the following we will need only equations of orders x и x^2. Taking this in mind we divide equations of order x and order x^2.

Equations of order x permit us to express the small components of order x through derivatives D_0, D_j , acting on the large component L_1, L_2, L_5, L_6 and the Ricci coefficients. Substituting these small components into equations of order x^2, we derive six equations with respect to 4 independent large variables L_1, L_2, L_5, L_6; they contain the terms $2MD_0L_A, \quad A = 1, 2, , 5, 6.$

In can be proved that only 4 equations of them are independent, further we will work with these four equations.

They are rather complicated, by this reason we temporally remove all term containing the Ricci rotation coefficients (thereby we effectively turn back the the case of Cartesian coordinates in Minkowski space).

Further, we transform this system of 4 equations to other variables

$$\Psi_1 = L_1 + L_6, \quad \Psi_2 = L_2 - L_5, \quad \Psi_3 = 2L_1 - L_6, \quad \Psi_4 = 2L_2 + L_5;$$

in this way we arrive at four equations.

With the following notations

$$D_2 D_3 - D_3 D_2 = D_{23}, \quad D_3 D_1 - D_1 D_3 = D_{31}, \quad D_1 D_2 - D_2 D_3 = D_{12},$$

$$\Delta = (D_1 D_1 + D_2 D_2 + D_3 D_3), \quad D_0 = \partial_0 + ieA_0, \quad D_j = e^k_{(j)}(\partial_k + ieA_k).$$

they read

$$M i D_0 \Psi_1 + \frac{1}{2}\Delta \Psi_1$$

$$-i\frac{1}{6}D_{12}\Psi_1 + \frac{1}{2}D_{31}\Psi_2 - \frac{i}{6}D_{23}\Psi_2 + \frac{i}{3}D_{12}\Psi_3 + \frac{i}{3}D_{23}\Psi_4 + \ldots = 0,$$

$$M i D_0 \Psi_2 + \frac{1}{2}\Delta \Psi_2$$

$$+\frac{i}{6}D_{12}\Psi_2 - \frac{1}{2}D_{31}\Psi_1 - \frac{i}{6}D_{23}\Psi_1 + +\frac{i}{3}D_{23}\Psi_3 - \frac{i}{3}D_{12}\Psi_4 + \ldots = 0,$$

$$M i D_0 \Psi_3 + \frac{1}{2}\Delta \Psi_3$$

$$+\frac{i}{2}D_{12}\Psi_3 + \frac{1}{3}D_{31}\Psi_2 + \frac{i}{3}D_{23}\Psi_2 - \frac{1}{6}D_{31}\Psi_4 - \frac{i}{6}D_{23}\Psi_4 + \ldots = 0,$$

$$M i D_0 \Psi_4 + \frac{1}{2}\Delta \Psi_4$$

$$-\frac{i}{2}D_{12}\Psi_4 - \frac{1}{3}D_{31}\Psi_1 + \frac{i}{3}D_{23}\Psi_1 + \frac{1}{6}D_{31}\Psi_3 - \frac{i}{6}D_{23}\Psi_3 + \ldots = 0.$$

This system may be presented in the matrix form

$$\Psi = \left| \begin{array}{c} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \end{array} \right|, \qquad iMD_0\Psi + \frac{1}{2}\Delta\Psi + (S_1D_{23} + S_2D_{31} + S_3D_{12})\Psi = 0,$$

$$S_1 = \left| \begin{array}{cccc} 0 & \frac{i}{2} & 0 & -i \\ \frac{i}{2} & 0 & -i & 0 \\ 0 & -i & 0 & \frac{i}{2} \\ -i & 0 & \frac{i}{2} & 0 \end{array} \right|, S_2 = \left| \begin{array}{cccc} 0 & -\frac{3}{2} & 0 & 0 \\ \frac{3}{2} & 0 & 0 & 0 \\ 0 & -1 & 0 & \frac{1}{2} \\ 1 & 0 & -\frac{1}{2} & 0 \end{array} \right|,$$

$$S_3 = \left| \begin{array}{cccc} \frac{i}{2} & 0 & -i & 0 \\ 0 & -\frac{i}{2} & 0 & i \\ 0 & 0 & -\frac{3i}{2} & 0 \\ 0 & 0 & 0 & \frac{3i}{2} \end{array} \right|.$$

they obey the needed commutators $S_1S_2 - S_2S_1 = S_3, \ldots$

By using a linear transformation we determine new spin matrices

$$\bar{S}_1 = \left| \begin{array}{cccc} 0 & -\frac{1}{2} & 0 & 0 \\ -\frac{3}{2} & 0 & -1 & 0 \\ 0 & -1 & 0 & -\frac{3}{2} \\ 0 & 0 & -\frac{1}{2} & 0 \end{array} \right| , \quad \bar{S}_2 = \left| \begin{array}{cccc} 0 & -\frac{i}{2} & 0 & 0 \\ \frac{3i}{2} & 0 & -i & 0 \\ 0 & i & 0 & -\frac{3i}{2} \\ 0 & 0 & \frac{i}{2} & 0 \end{array} \right| ,$$

$$\bar{S}_3 = \left| \begin{array}{cccc} -3/2 & 0 & 0 & 0 \\ 0 & -1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 3/2 \end{array} \right| .$$

Performing the similar calculations now with the presence all terms with Ricci coefficients, we arrive at the complicated system of 4 equations, it may be presented in matrix form

$$2M(D_0 + A_0)\Psi + (S_1 D_{23} + S_2 D_{31} + S_3 D_{12})\Psi +$$

$$+(D_1 A_1)\Psi + (A_1 + B_1) D_1 \Psi + (D_2 A_2)\Psi + (A_2 + B_2) D_2 \Psi +$$

$$+(D_3 A_3)\Psi + (A_3 + B_3) D_3 \Psi + \Delta\Psi = 0.$$

The matrices A_0, A_1, A_2, A_3 and B_1, B_2, B_3 depend linearly on 9 Ricci coefficients; the matrix Δ depends on the products of Ricci coefficient $G_{..} \times G_{..}$.

In turns out that all these matrices may be decomposed in linear combination of 9 independent $4\times$ basic elements (spin matrices and their products)

$$S_1 = t_1, \quad S_2 = t_2, \quad S_3 = t_3,$$

$$S_1^2 = t_4, \quad S_2^2 = t_5, \quad S_3^2 = t_6,$$

$$S_2 S_3 = t_7, \quad S_3 S_1 = t_8, \quad S_1 S_2 = t_9,$$

Similar decompositions exist also for the matrices A_0 and Δ.

These additional geometrical terms depend on the Ricci rotation coefficients G_{ab}, the Ricci scalar $R(x)$, and the tetrad components of the Ricci tensor

$$R_{\alpha\beta}(x) \quad \Longrightarrow \quad R_{ab}(x) = e^{\alpha}_{(a)} e^{\beta}_{(b)} R_{\alpha\beta}(x).$$

Example 1: the particle in electric and magnetic fields

Cylindrical coordinates $x^\alpha = (t, r, \phi, z)$,

$$dS^2 = dt^2 - dr^2 - r^2 d\phi^2 - dz^2, \quad e^\alpha_{(a)} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix},$$

$$0 \Longrightarrow t, \quad 1 \Longrightarrow r, \quad 2 \Longrightarrow \phi, \quad 3 \Longrightarrow z, \quad \gamma_{122} = \frac{1}{r},$$

$$D_0 = \partial_0 + ieA_0, \quad D_1 = \partial_r + ieA_r, \quad D_2 = \frac{1}{r}(\partial_\phi + ieA_\phi), \quad D_3 = \partial_z + ieA_z,$$

Let us consider the presence of uniform magnetic and electric fields along the exes z;

$$D_0 = \partial_t + ieEz, \quad D_1 = \partial_r, \; D_2 = \frac{1}{r}\left(\partial_\phi + ie\frac{Br^2}{2}\right), \; D_3 = \partial_z,$$

$$A_0 = 0, \quad \Delta = \frac{\partial^2}{\partial r^2} + \left(\frac{1}{r}\partial_\phi + \frac{ieB}{2} r\right)^2 + \frac{\partial^2}{\partial z^2},$$

$$D_{23} = 0, \quad D_{31} = 0, \quad D_{12} = -\frac{1}{r^2}\left(\partial_\phi - ie\frac{Br^2}{2}\right),$$

The basic Pauli like equation reads

$$i(\partial_t + ieEz)\bar{\Psi} = -\frac{1}{2M}\left[\frac{\partial^2}{\partial r^2} + \left(\frac{1}{r}\partial_\phi + \frac{ieB}{2} r\right)^2 + \frac{\partial^2}{\partial z^2}\right]\bar{\Psi} +$$

$$+\frac{i}{3M}\frac{1}{r^2}\left(\partial_\phi - ie\frac{Br^2}{2}\right)\bar{S}_3\bar{\Psi} -$$

$$-\frac{i}{2M}\left[(\partial_r \bar{A}_1) + (\bar{A}_1 + \bar{B}_1)\partial_r + (\bar{A}_2 + \bar{B}_2)\frac{1}{r}\left(\partial_\phi + ie\frac{Br^2}{2}\right) +\right.$$

$$\left. +(\bar{A}_3 + \bar{B}_3)\partial_z + \bar{\Sigma}\right]\bar{\Psi} = 0,$$

The involved matrices are

$$(\partial_r \bar{A}_1) = \frac{i}{r^2} \begin{vmatrix} 9 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 9 \end{vmatrix},$$

$$\bar{A}_1 + \bar{B}_1 = -\frac{6i}{r} I, \ \bar{A}_2 + \bar{B}_2 = -\frac{8}{r} \begin{vmatrix} -3/2 & 0 & 0 & 0 \\ 0 & -1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 3/2 \end{vmatrix},$$

$$\bar{A}_3 + \bar{B}_3 = 0, \quad \bar{\Sigma} = \frac{3i}{r^2} \begin{vmatrix} 9 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 9 \end{vmatrix},$$

$$\bar{S}_3 = \begin{vmatrix} -3/2 & 0 & 0 & 0 \\ 0 & -1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 3/2 \end{vmatrix}.$$

This structure means that the non-relativistic equation in cyclic basis consists of 4 unlinked equations with similar structure, so the Hamiltonian is rather simple one.

Example 2: Particle in the Coulomb field

Let us consider the case of spherical coordinates $x^\alpha = (t, r, \theta, \phi)$:

$$dS^2 = dt^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2 - dr^2 \, , e^\alpha_{(0)} = (1, 0, 0, 0) \, ,$$

$$e^\alpha_{(3)} = (0, 1, 0, 0) \, , \;\; e^\alpha_{(1)} = (0, 0, \frac{1}{r}, 0) \, , e^\alpha_{(2)} = (1, 0, 0, \frac{1}{r \sin\theta}) \, .$$

$$\gamma_{ab1} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{r} \\ 0 & 0 & 0 & 0 \\ 0 & +\frac{1}{r} & 0 & 0 \end{vmatrix} , \gamma_{ab2} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & +\frac{\cot\theta}{r} & 0 \\ 0 & -\frac{\cot\theta}{r} & 0 & -\frac{1}{r} \\ 0 & 0 & +\frac{1}{r} & 0 \end{vmatrix} .$$

$$0 \Longrightarrow t, \quad 1 \Longrightarrow \theta, \quad 2 \Longrightarrow \phi, \quad 3 \Longrightarrow r, \quad i(D_0 + ieA_0) \Longrightarrow (E + \frac{\alpha}{r}),$$

$$D_1 = \frac{1}{r}\partial_\theta, \quad D_2 = \frac{1}{r\sin\theta}\partial_\phi, \quad D_3 = \partial_r \quad \Delta = \frac{1}{r^2}\partial_\theta^2 + \frac{1}{r^2\sin^2\theta}\partial_\phi^2 + \partial_r^2$$

The Pauli-like equation reads

$$(\epsilon + \frac{\alpha}{r})\Psi = -\frac{1}{2M}\left[\partial_r^2 + \frac{1}{r^2}\partial_\phi^2 + \frac{\partial^2}{\partial z^2}\right]\Psi -$$

$$-\frac{i}{2M}\Big[\frac{1}{3}(\partial_\theta A_1) + 0 + (\partial_3 A_3) +$$

$$+\frac{1}{r}(A_1 + B_1)\partial_\theta \; + \frac{1}{r\sin\theta}(A_2 + B_2)\partial_\phi + (A_3 + B_3)\partial_r + \Sigma\Big]\Psi,$$

where the involved matrices explicitly read

$$A_1 = \frac{i}{r}\begin{vmatrix} -9\cot\theta & 6 & 0 & 0 \\ -6 & -\cot\theta & 8 & 0 \\ 0 & -8 & -\cot\theta & 6 \\ 0 & 0 & -6 & -9\cot\theta \end{vmatrix},$$

$$A_2 = \frac{1}{r}\left|\begin{array}{cccc} 9\cot\theta & -6 & 0 & 0 \\ -6 & 3\cot\theta & -8 & 0 \\ 0 & -8 & -3\cot\theta & -6 \\ 0 & 0 & -6 & -9\cot\theta \end{array}\right|,$$

$$A_3 = \frac{i}{r}\left|\begin{array}{cccc} -6 & \cot\theta & 0 & 0 \\ 9\cot\theta & -14 & -2\cot\theta & 0 \\ 0 & 2\cot\theta & -14 & -9\cot\theta \\ 0 & 0 & -\cot\theta & -6 \end{array}\right|$$

$$(\partial_\theta A_1) = \ldots\ldots, \qquad \partial_\phi A_2 = 0,$$

$$\partial_r A_3 = -\frac{i}{r^2}\left|\begin{array}{cccc} -6 & \cot\theta & 0 & 0 \\ 9\cot\theta & -14 & -2\cot\theta & 0 \\ 0 & 2\cot\theta & -14 & -9\cot\theta \\ 0 & 0 & -\cot\theta & -6 \end{array}\right|,$$

$$B_1 = \frac{i}{r}\left|\begin{array}{cccc} 3\cot\theta & -2 & 0 & 0 \\ -6 & -5\cot\theta & 0 & 0 \\ 0 & 0 & -5\cot\theta & 6 \\ 0 & 0 & 2 & 3\cot\theta \end{array}\right|,$$

$$B_2 = \frac{1}{r}\begin{vmatrix} 3\cot\theta & 2 & 0 & 0 \\ -6 & \cot\theta & 0 & 0 \\ 0 & 0 & -\cot\theta & -6 \\ 0 & 0 & 2 & -3\cot\theta \end{vmatrix},$$

$$B_3 = \frac{i}{r}\begin{vmatrix} -6 & -\cot\theta & 0 & 0 \\ -9\cot\theta & 2 & 2\cot\theta & 0 \\ 0 & -2\cot\theta & 2 & 9\cot\theta \\ 0 & 0 & \cot\theta & -6 \end{vmatrix}$$

$$A_1 + B_1 = \frac{2i}{r}\begin{vmatrix} -3\cot\theta & 2 & 0 & 0 \\ -6 & -3\cot\theta & 4 & 0 \\ 0 & -4 & -3\cot\theta & 6 \\ 0 & 0 & -2 & -3\cot\theta \end{vmatrix},$$

$$A_2 + B_2 = \frac{4}{r}\begin{vmatrix} 3\cot\theta & -1 & 0 & 0 \\ -3 & \cot\theta & -2 & 0 \\ 0 & -2 & -\cot\theta & -3 \\ 0 & 0 & -1 & -3\cot\theta \end{vmatrix},$$

$$A_3 + B_3 = \begin{vmatrix} -\frac{12i}{r} & 0 & 0 & 0 \\ 0 & -\frac{12i}{r} & 0 & 0 \\ 0 & 0 & -\frac{12i}{r} & 0 \\ 0 & 0 & 0 & -\frac{12i}{r} \end{vmatrix} = -\frac{12i}{r} I$$

$$\Sigma = \frac{3}{r^2} \times$$

$$\begin{vmatrix} 3i\left(3\cot^2\theta - 4\right) & -4i\cot\theta & 0 & 0 \\ -36i\cot\theta & i\left(\cot^2\theta + 12\right) & 8i\cot\theta & 0 \\ 0 & -8i\cot\theta & i\left(\cot^2\theta + 12\right) & 36i\cot\theta \\ 0 & 0 & 4i\cot\theta & 3i\left(3\cot^2\theta - 4\right) \end{vmatrix}$$

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