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Estimation of the kinetic parameters of the fractal stellar medium in the Solar neighborhood

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M 43 Emission Nebula of Orion

Plan of the talk

- Introduction. The study of Fractal structures in the stellar medium of the Galaxy
- Goal of this work
- Concept of "fractal" and "fractal dimension”
- Main stages in the development of ideas about the fractal structure of the stellar medium of galaxies. Law of Carpenter-Vaucouleurs
- Mandelbrot interpretation and introduction of "fractal dimension"
- The results of numerical calculations of mean stellar density and fractal properties of 200,000 stars in the solar neighborhood
- Estimation of the effective interparticle spacing for the fractal star medium
- Estimation of the impact parameter and correlation length for the studied fractal stellar medium
- Estimation of the coefficient of dynamic friction for the fractal star medium
- Estimation of the relaxation time for the studied fractal stellar medium

- Conclusions

Introduction

The study of Fractal structures in the stellar medium of the Galaxy

→ constructing internally consistent and more appropriate to observational data kinetic theory of the stellar medium
(the kinetic parameters of fractal stellar media differ significantly from the corresponding parameters for quasi-homogeneous medium with limited density fluctuations)

→ refining basic equations of stellar dynamics for fractal stellar media

Our goal – to estimate the kinetic parameters of the fractal distribution of 200,000 stars of all spectral types in the solar neighborhood at a distance of 1 pc to 100 pc from the Sun from observational data of the “GAIA” telescope (DR2, 2018)

The fractality of the structure of stellar medium at distances from 1 pc to 200 pc - from observations of **:**

- young population of galaxies (*Efremov and Elmegrin, 1998; Elias et al., 2009; Elmegrin et al., 2014*)
- F, G – type dwarf stars in the solar neighborhood from observational data of the Geneva–Copenhagen Survey (*Chumak and Rastorguev, 2015)*
- interstellar gas and dust clouds *(Larson, 1981; de Vega et.al, 1998)*

Concept of "fractal" and "fractal dimension"

"Fractal" means fractional, broken, - Mandelbrot called so self-similar geometric figures, each fragment of which is repeated when the scale is reduced, (*Mandelbrot*, *1988).*

Benua Mandelbrot – the founder of fractal analysis who introduced the term "fractal"

→ the figure has the property of *scale invariance* – is the first basic property of fractal objects

Benua Mandelbrot
1924 - 2010

Fractal dimension – is a measure, how the object fills the space of embedding

" Fractal dimension " :

$$D = \lim_{a \to 0} \frac{\log N(a)}{\log(1/a)}$$

a **–** scale unit; ***N*** **–** number of scale units, covering the objects;
D **–** fractal dimension.

If such a limit exists and the *value is a fraction*, then this object is a fractal – is the second basic property of fractal objects

The main stages in development of ideas about fractal structures in stellar medium of galaxies

Carpenter (1938), Vaucouleurs (1970): number density of galaxies in cluster decreases with the growth of characteristic cluster sizes according to the fractional-power law

$$n(r) \sim N/r^3 \sim r^{-1,7}$$

N – number of galaxies in cluster; ***r*** – size of cluster; ***n*** – density of galaxies in cluster

- galactic medium is arranged hierarchically; any observer, included in the hierarchy, will find that the mean density around him decreases with distance; - any large identical volumes have the same mean density, regardless of the position of their centers relative to each other.

Mandelbrot (1988) interpreted results by Carpenter and Vaucouleurs as a special case of stochastic self-similarity of three-dimensional random fractal sets for which the relation holds:

$$n(r) \sim r^{-\alpha}$$

r – characteristic size of the increasing volume around observer included in hierarchy; ***n (r)*** – invariant conditional density ; ***α*** – exponent

D – **is fractal dimension**

$$D = 3 - \alpha \qquad\qquad 0 \le D \le 3$$

The results of our calculations of spatial distribution and fractal properties of 200 000 stars in solar neighborhood from data of "GAIA" (DR2,2018)

200,000 stars of all spectral types at distances from 1 to 100 parsecs from Sun

Mean star density n (r) in spheres of radius *r* around the stars is approximated by power laws of the form:

$$n(r) = hr^{-\alpha}$$

where h = 1,654; α = 0,586; D = 3 - α ≈ 2,41

$$n(r) = 1,654\, r^{-0,586}$$

This law confirms the conclusions by Vaucouleurs and Mandelbrot for fractal structures in gravitating media

Mean stellar density vs. radius of the sphere around the stars

X-axis: **r** in parsecs (pc), Y-axis: stellar density **n** in pc^{-3}.

Red line – **200,000 stars** of all spectral types

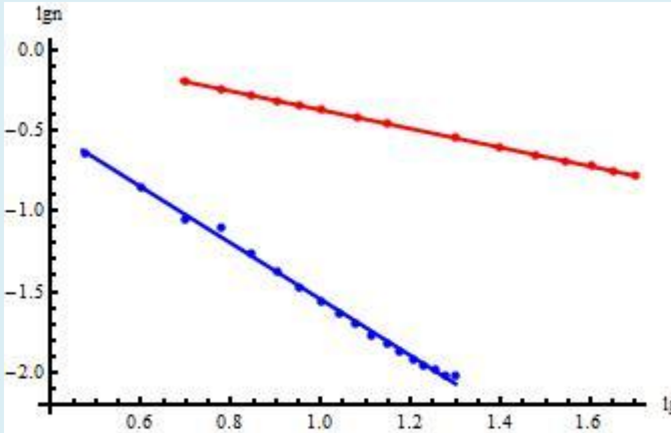
at distances from **1 to 100 parsecs** from Sun

Blue line – **13,000** F, G – type **dwarf stars**

at distances from **1 to 20 parsecs** from Sun from

observational data of the Geneva–Copenhagen Survey

(Chumak and Rastorguev, 2015), h = 1,644; α = 1,769; D ≈ 1,23;



Estimation of the effective interparticle spacing for the fractal stellar medium in the solar neighborhood

The important kinetic parameter in stellar dynamics the effective interparticle spacing r_m for fractal medium is derived from the distribution law of the distance to the nearest neighbor *(Chumak, Rastorguev, 2015)* :

$$w(r)dr = 4\pi h exp\left(-\frac{4\pi h}{3} r^D\right) r^{D-1} dr$$

w* (*r*)** – the distance distribution caused by the nearest neighbor in a spherical layer of radius ***r and thickness ***dr***;

A-priory:

$$r_m = \int_0^\infty r\, w(r)\, dr$$

Then, the effective interparticle spacing:

$$r_m = \frac{3}{D}\left(\frac{D}{4\pi h}\right)^{1/D} \Gamma\left(\frac{D+1}{D}\right)$$

Coefficient **h = 1,654;** fractal dimension **D ≈ 2,41** were taken from our calculations for 200 000 stars: **r_m = 0,49 parsec**

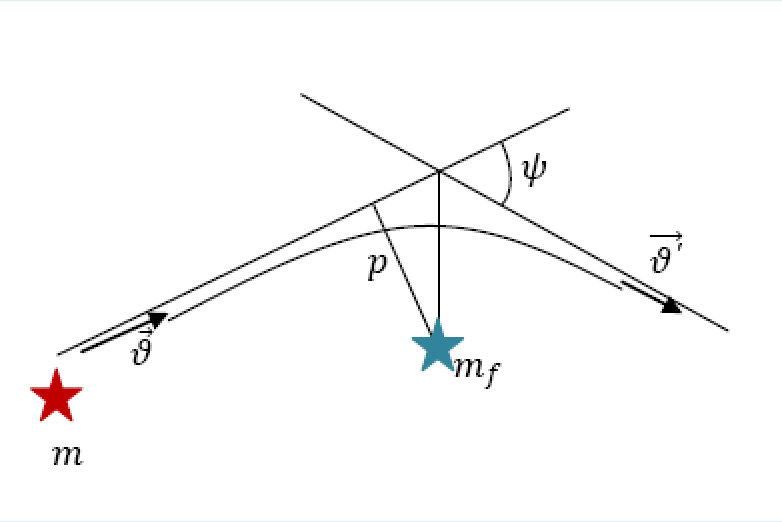
Traditional estimations of the effective interparticle spacing **for homogeneous stellar medium**: **r_m = 1,06 – 1,16 parsec, that is two times more**

This result is consistent with result for F, G – type **dwarf stars** in the Solar vicinity from observational data of the Geneva–Copenhagen Survey*:* **r_m = 0,48 parsec**

(Chumak, Rastorguev, 2015)

Estimation of the impact parameter for the fractal stellar medium in the solar neighborhood

Geometry of paired star encounters



m и m_f – the masses of the test star and the field star,

$\vec{V}$ и $\vec{V}'$ - the relative velocities of the test star before and after encounter,

p - the **impact parameter** – the distance between the fixed star and the asymptote of the velocity vector of the test star before the encounter,

ψ – the angle of deviation of the relative velocity vector of the test star, the magnitude of its relative velocity (the test star) remains constant according to the law of conservation of energy: $|\vec{V}| = |\vec{V}'|$.

Thus, the result of the encounter will be a change in the direction of the relative velocity vector, associated with its rotation.

The relative velocity vector of the test star deviates by an angle ψ , which in the framework of the two-body theory *(Ogorodnikov, 1958):*

$$\mathrm{tg}\frac{\psi}{2} = \frac{G(m+m_f)}{V^2 p} = \frac{p_\perp}{p}$$

where G is the gravitational constant, $p_\perp$ is the **impact parameter** of the paired encounter, at which the vector of the relative velocity of the test star is deflected by an angle of $\psi = \pi/2$ **(close encounter):**

$$p_\perp = \frac{G(m+m_f)}{V^2}$$

For $m = m_f$: $p_\perp = \frac{2Gm}{V^2}$

According to the **virial theorem** the velocity of test star:

$$V^2 \approx \frac{GNm}{r}$$

where N – the number of particles in the system, r – the size of the system.

Expressing the **number of stars** through the **density *n***, we obtain for a homogeneous stellar medium:

$$p_{\perp} = \frac{3}{2n\pi r^2}$$

For the fractal stellar medium we obtain: :

$$p_{\perp} = \frac{3r^{1-D}}{2\pi h}$$

This formula under $D \to 3,\ h \to n$ takes on the classical form of uniform medium

Thus we obtain the relationship:

$$p_{\perp fr} = h^{-1} n r^{(3-D)} p_{\perp}$$

To estimate the impact parameter we use the **correlation length** r_0 from formula for the fractal stellar medium (*Davis and Peebles, 1983*): $n(r_0) = 1$ pc^{-3}.Then:

$r_0 = h^{-\frac{1}{D-3}}$. For 200,000 stars in Solar vicinity r_0 = **2,35 parsec.**

Then under $r = r_0$:

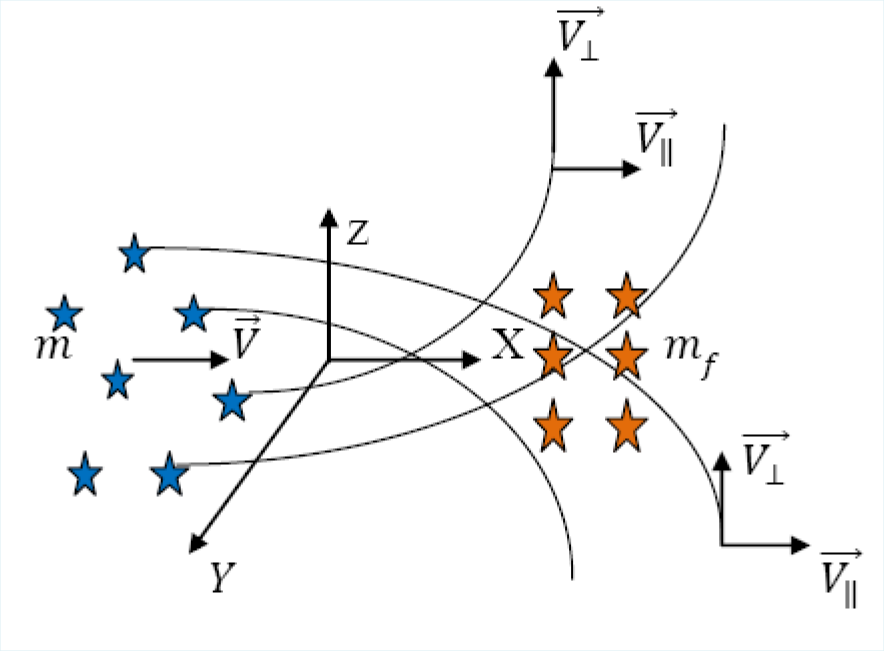
$$p_{\perp fr} = 0.1\, p_{\perp}$$

Thus the impact parameter for fractal stellar medium is ten times less than for the homogeneous stellar medium

Estimation of the dynamic friction coefficient for the fractal stellar medium in the solar neighborhood

Dynamic friction is determined by the paired encounters of stars

We consider the flow of test stars flying into fixed stars of the field, then as a result the test stars will acquire transverse velocity components, and the average flow velocity will decrease. Consequently, in the end, encounters lead to the spreading of the flow of test stars in the transverse direction **(relaxation)** and the deceleration of the flow **(dynamic friction)**.



The coefficient of dynamic friction can be approximately estimated by the formula (*Chumak and Rastorguev, 2017*):

$$a \approx \frac{8\pi G^2 m^2 n \ln \Lambda}{V^3}$$

where $\ln \Lambda = \ln(p_{max}/p_{min})$ is the Coulomb logarithm, m – mass of test star and field star, n – density of field stars, $p_{min} = p_{\perp}$ is the impact parameter of close encounter , p_{max} is the upper limit of the impact parameter for distant interactions (in our case $p_{max} = 2r_m$), V – relative velocity of test star before encounter

For **the fractal stellar medium:**

$$a \approx \frac{8\pi G^2 m^2 h \ln \Lambda}{V^3} r^{D-3}$$

For approximate estimates of the value of ***a***, one can use the above-mentioned **"correlation length"** r_0:

$$a_0 = a(r_0) \approx \frac{8\pi G^2 m^2 h \ln \Lambda}{V^3} r_0^{(D-3)}$$

Estimation of the relaxation time in the fractal stellar medium in the solar neighborhood

- The state of dynamic equilibrium in stellar system disrupted
- Increments in the velocities of the stars under the action of irregular forces reach the values of velocities themselves.
- Irregular forces provide collisionality in stellar system, arise during stellar encounters, act as small perturbations during star regular orbit motion
- Classical stellar dynamics was constructed as stellar dynamics without stellar encounters

→

The main paradox of classical stellar dynamics → the very large relaxation time for most stellar systems, which significantly exceeds the age of the system

- From formula for the characteristic deceleration time of a test star as a result of dynamic friction **in the fractal medium** we obtain:

$$\tau_{rel\,fr} = \frac{V^3}{8\pi G^2 m^2 h \ln \Lambda} r^{(3-D)}$$

For a **homogeneous stellar medium (D→3;h→n)** → $\tau_{rel} = \frac{V^3}{8\pi G^2 m^2 n \ln \Lambda}$

From these last two expressions we obtain the relationship:

$$\tau_{rel\ fr} = nh^{-1}r^{(3-D)}\tau_{rel}$$

Substituting the parameters defined above for the solar neighborhood into this relation and assuming that $r = r_0$, we obtain

$$\boxed{\tau_{rel\ fr} = 0,099\tau_{rel}}$$

Thus, the relaxation time for a fractal medium is approximately ten times shorter than for a homogeneous stellar medium.

M 31 Galaxy Nebula of Andromeda

Conclusions

- Study of spatial distribution of 200,000 stars of all spectral types in the solar neighborhood at a distance of 1 pc to 100 pc from the Sun from observational data of telescope “GAIA” (DR2, 2018) showed the presence of fractal structures with fractal dimension D ≈ 2,41.
- Kinetic parameters for this fractal stellar medium are estimated
- Effective interpartcle (interstellar) distance: r_m = 0,49 parsec

Traditional estimations of the effective interparticle spacing for homogeneous stellar medium: r_m = 1,06 – 1,16 parsec, that is two times more

- The value of "correlation length“ r_0 = 2,35 parsec, that is approximately five times more, than the effective interparticle distance
- The impact parameter for close encounter is ten times less than for the homogeneous stellar medium: $p_{\perp fr} = 0.1\, p_{\perp}$
- Coefficient of the dynamic friction is obtained: $a(r_0) \approx \frac{8\pi G^2 m^2 h \ln \Lambda}{V^3} r_0^{(D-3)}$
-
- The relaxation time for a fractal medium is approximately ten times shorter than for a homogeneous stellar medium: $\tau_{rel\, fr} = 0,099\tau_{rel}$

Conclusion

-
- Stars in solar neighborhood from observational data of telescope "GAIA" form gravitationally bound structured formations, such as clusters, parts of spiral arms, clumps, which have fractal properties and obey the laws of fractal kinetics. This leads to the substantial shortening the collisional relaxation time in our Galaxy.

M 42 Nebula of Orion

Thanks for attention

NGC 3521 Galaxy

For other kinetic parameter – dynamical time – see poster M.O. “Fractal distribution of stars in the Solar neighborhood and dynamical time”

Milky Way Galaxy

Additional slides

Calculation of mean stellar density by the "mass - radius" method

This method consists of determination of number ***N (r; R_i)*** of stars in spheres with increasing radius ***"r "*** with center in the ***i***-th star located at a radial distance ***R_i*** from the observer.

The mean number of stars in spheres of radius ***"r"***:

$$N(r) = \frac{1}{m(r)} \sum_{i=1}^{m(r)} N(r; R_i)$$

N (r) – mean number of stars in spheres with radius "***r*** ";

N (r; R_i) – number of stars in sphere with radius ***"r "*** with center in the ***i***-th star;

R_i – radial distance of ***i***-th star from the observer;

m (r) – number of spheres of radius ***"r"*** is equel to number of studied stars;

Mean stellar density n (r) in spheres of radius "*r* " around the stars:

$$n(r) = N(r)/V(r)$$

V (r) – volume of sphere of radius "***r*** "

The main stages in development of ideas about fractal structures in stellar medium of galaxies

Carpenter (1938) : number density of galaxies in cluster decreases with the growth of characteristic cluster sizes according to the fractional-power law

$$n \sim N/r^3 \sim r^{-1,5}$$

N – number of galaxies in cluster; ***r*** – size of cluster; ***n*** – density of galaxies in cluster

Vaucouleurs (1970) (on base of *new* galaxy clusters data):

$$n(r) \sim r^{-1,7}$$

- galactic medium is arranged hierarchically
- any observer, included in the hierarchy, will find that the mean density around him decreases with distance
- any large identical volumes have the same mean density, regardless of the position of their centers relative to each other. This density can be called the invariant conditional density

Vaucouleurs has extended power law to entire stellar medium

The interpretation of Mandelbrot and fractal dimension

Mandelbrot (1988) interpreted results by Carpenter and Vaucouleurs as a special case of stochastic self-similarity of three-dimensional random fractal sets for which the relation holds:

$$n(r) \sim r^{-\alpha}$$

r – characteristic size of the increasing volume around observer included in hierarchy; $n\,(r)$ – invariant conditional density ; α – exponent

$$D = 3 - \alpha$$

D – **is fractal dimension**

Mandelbrot showed that, depending on the characteristics of the medium in various gravitating media, for fractal structures :

$$0 \leq D \leq 3$$

$\rightarrow$ α – exponent can vary from 0 to 3

The results of our calculations of spatial distribution and fractal properties of 200 000 stars in solar neighborhood from data of "GAIA" (DR2,2018)

200,000 stars of all spectral types at distances from 1 to 100 parsecs from Sun

Mean star density n (r) in spheres of radius *r* around the stars is approximated by power laws of the form:

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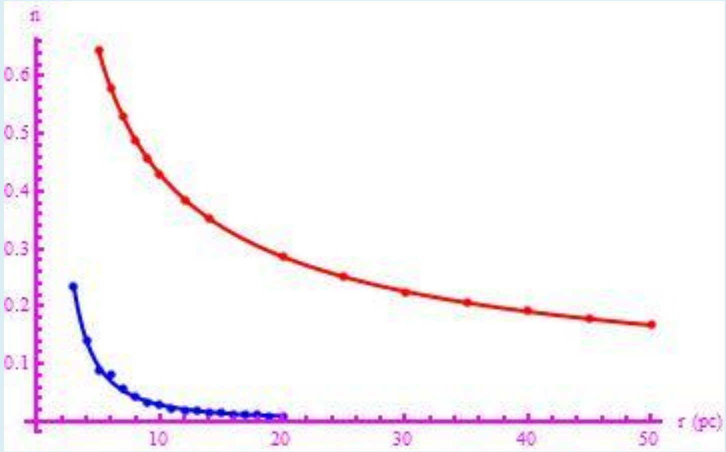
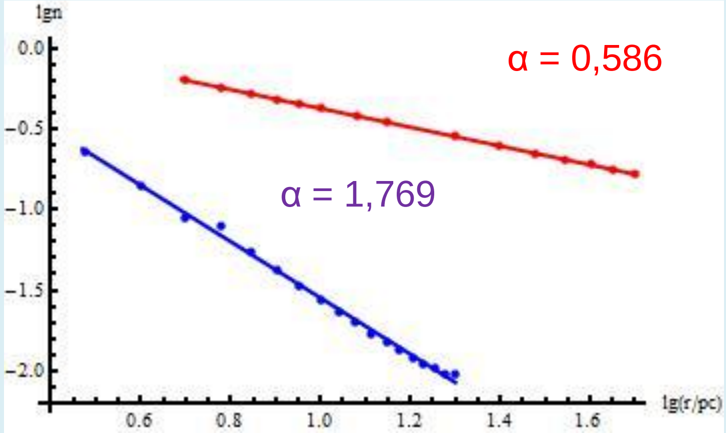
where h = 1,654; α = 0,586; D = 3 - α ≈ 2,41

$$n(r) = 1,654\, r^{-0,586}$$

This law confirms the conclusions by Vaucouleurs and Mandelbrot for fractal structures in gravitating media

Mean stellar density vs. radius of the sphere around the stars





X-axis: **r** in parsecs (pc),
Y-axis: stellar density **n** in pc^{-3}.

Red line – **200,000 stars** of all spectral types at distances from **1 to 100 parsecs** from Sun from observational data of telescope "GAIA" (DR2, 2018)
h = 1,654; α = 0,586; D ≈ 2,41;

Blue line – **13,000** F, G – type **dwarf stars** at distances from **1 to 20 parsecs** from Sun from observational data of the Geneva–Copenhagen Survey (*Chumak and Rastorguev, 2015),* h = 1,644; α = 1,769; D ≈ 1,23;

Telescope “GAIA”

Telescope “GAIA” has a goal – studying coordinates, direction of motion, spectral types of a billion stars in the **Milky Way,** search for exoplanets, asteroids and comets in the Solar System

Launched in 2013 and located at a distance 1, 5 million km from the Earth

DR2 – Data Release 2 (2018)

A-priory:

$$r_m = \int_0^\infty r\, w(r)\, dr$$

Then, the effective interparticle spacing:

$$r_m = \frac{3}{D}\left(\frac{4\pi h}{D}\right)^{-1/D} \int_0^\infty e^{-x} x^{1/D}\, dx = \frac{3}{D}\left(\frac{D}{4\pi h}\right)^{1/D} \Gamma\left(\frac{D+1}{D}\right)$$

coefficient h and fractal dimension D were taken from our calculations.

In the limit fractal dimension D≈3 and mean stellar density h≈n we obtain the well-known estimation from Chandrasekhar (1943) for the mean interparticle spacing for quasi-homogeneous media:

$$r_m = \left(\frac{3}{4\pi n}\right)^{1/3} \Gamma\left(\frac{4}{3}\right) \approx 0{,}554 n^{-1/3}$$

coefficient **h = 1,654;** fractal dimension **D ≈ 2,41** were taken from our calculations for 200 000 stars .

Then, the effective interparticle spacing :

$$r_m$$

A-priory:

$$r_m = \int_0^\infty r\, w(r)\, dr$$

where $w\,(r)$ – the distance distribution, which is presented above;

$w\,(r)dr$ it is better to rewrite in the form:

$$w(x)dx = \frac{3}{D} e^{-x} dx$$

where

$$x = \frac{4\pi h}{D} r^D$$

coefficient h and fractal dimension D were taken from our calculations.

Then, the effective interparticle spacing r_m :

$$r_m = \frac{3}{D}\left(\frac{4\pi h}{D}\right)^{-1/D} \int_0^\infty e^{-x} x^{1/D}\, dx = \frac{3}{D}\left(\frac{D}{4\pi h}\right)^{1/D} \Gamma\left(\frac{D+1}{D}\right)$$

In the limit fractal dimension D≈3 and mean stellar density h≈n we obtain the well-known estimation from Chandrasekhar (1943) for the mean interparticle spacing for quasi-homogeneous media:

$$r_m = \left(\frac{3}{4\pi n}\right)^{1/3} \Gamma\left(\frac{4}{3}\right) \approx 0,554 n^{-1/3}$$

- Study of stellar medium in the solar neighborhood from observational data of telescope “GAIA” (DR2, 2018) showed the presence of fractal structures with fractal dimension D ≈ 2,41 ; h = 1,654 ; r_m = 0,49 parsec
- The result obtained is consistent with result for F, G – type **dwarf stars** from observational data of the Geneva–Copenhagen Survey (*Chumak and Rastorguev, 2015)* (fractal dimension D ≈ 1,23; h =1,644 ; r_m = 0,48 parsec);
- Traditional estimations of the effective interparticle spacing for homogeneous stellar media - r_m = 1,16 parsec ; r_m = 1,06 parsec, that is two times more

ESTIMATION OF THE IMPACT PARAMETER

For $m = m_f$ from (11) we obtain $p_\perp = \dfrac{2Gm}{V^2}$

To determine the velocity of a test star, it is convenient to use the **virial theorem**

$$V^2 \approx \frac{GNm}{r}$$

where N is the number of particles in the system, r is the size of the system.

Substituting formula (13) into (12), we obtain:

$$p_\perp = \frac{2r}{N}$$

Expressing the **number of stars** in formula (14) through the **density** n, we obtain for a homogeneous stellar medium:

$$p_\perp = \frac{3}{2n\pi r^2}$$

ESTIMATION OF THE IMPACT PARAMETER

For a **fractal stellar medium**, taking into account formula (3), we obtain:

$$p_{\perp} = \frac{3r^{\alpha-2}}{2\pi h}$$

Due to the fact that $D = 3 - \alpha$, expression (16) takes the form:

$$p_{\perp} = \frac{3r^{1-D}}{2\pi h}$$

As can be seen from formula (17), it takes on a classical form in the case of a homogeneous distribution of stars with $D \to 3,\ h \to n$.

From formulas (17) and (15) we obtain the relationship:

$$\boxed{p_{\perp fr} = h^{-1} n r^{(3-D)} p_{\perp}}$$

To estimate the impact parameter for a fractal stellar medium, we introduce another important kinetic parameter, the **"correlation length"**

$$r_0 = h^{-\frac{1}{D-3}}$$

ESTIMATION OF THE IMPACT PARAMETER

For our fractal model of stellar distribution, we obtain the value of the **"correlation length"** r_0 **= 2.35 pc**, which is approximately five times larger than the effective interparticle distance.

Substituting the parameters defined above for the solar neighborhood into the relation (18) and assuming that $r = r_0$, we obtain:

$$\boxed{p_{\perp fr} = 0.1\, p_{\perp}}$$

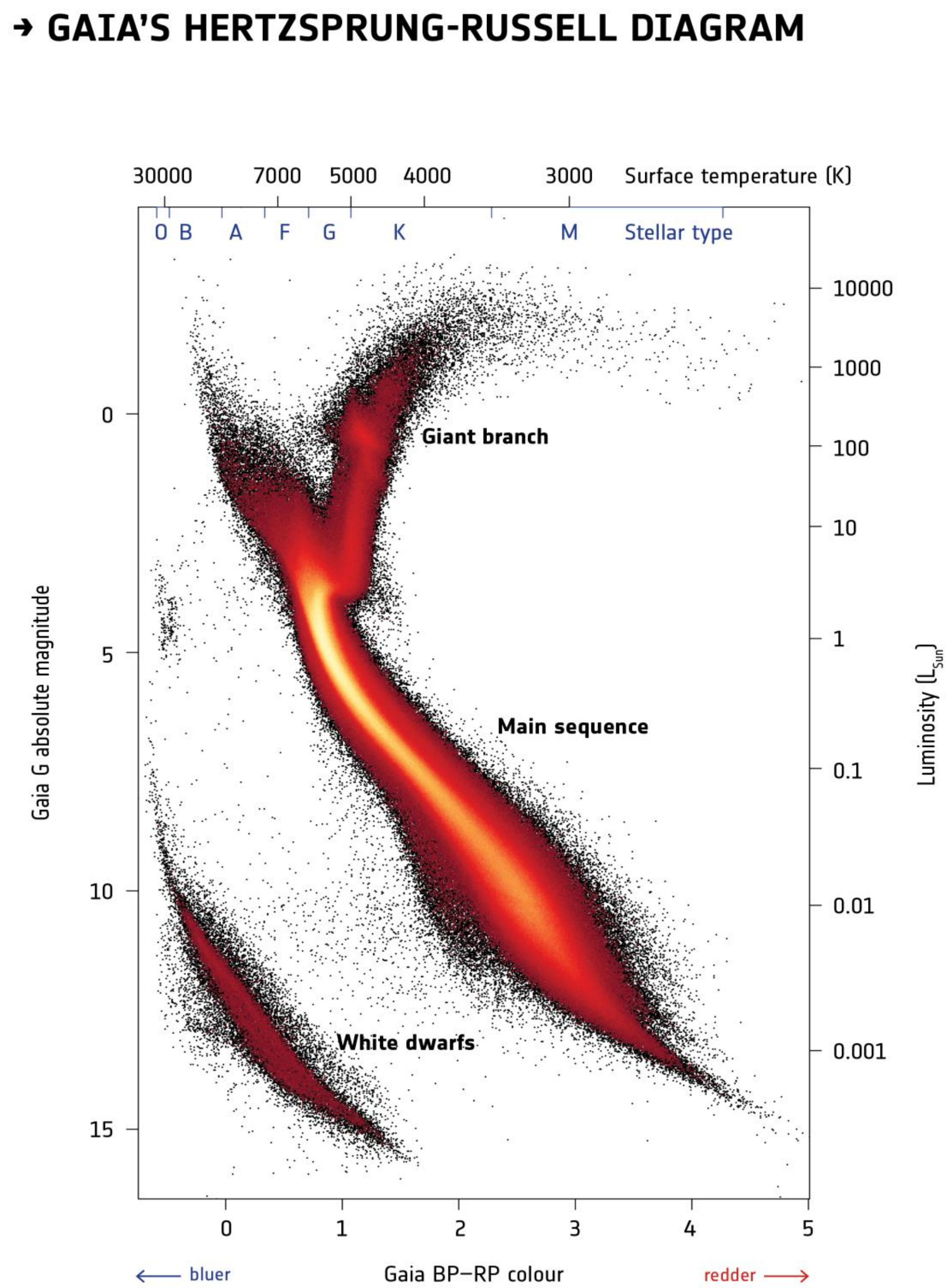
Thus, the impact parameter for the fractal distribution of stars is approximately **ten times smaller** than for the classical homogeneous stellar medium.

- Hertzsprung Russell diagram
- - shows the relationship between
- the *spectral class* and
- the *luminosity* of stars

Main sequence-

the main grouping of stars on the diagram *"spectral class – luminosity"*

It contains most of the stars, since the main sequence corresponds to the *longest stage in the evolution of stars*, at which thermonuclear reactions involving hydrogen take place in the core of the star.



Spectral classes of stars

- Spectral classes of stars - groups of stars distinguished by the nature of their spectra.
- Closely related to the temperature of stellar atmospheres.
- The hottest stars are blue stars of spectral classes O and B (50,000 K)
- The coldest stars are stars of spectral classes M and L (2000 K).
- Sun - spectral class G dwarf
- Dwarf - small main sequence star (up to one solar radius and not exceeding the luminosity of the Sun)
- Luminosity - the radiation power of a celestial body. That is the amount of energy emitted by it per unit of time
- For most main sequence stars, the relation $M^4 \sim R^5 \sim L$
- M – mass of star, R – radius of star, L –luminosity of star

Distances in astronomy

- 1 parsec ~ 30 000 000 000 000 km.
- 1 light year ~ 0,3 parsec.
- 1 astronomical unit ~ 150 000 000 km

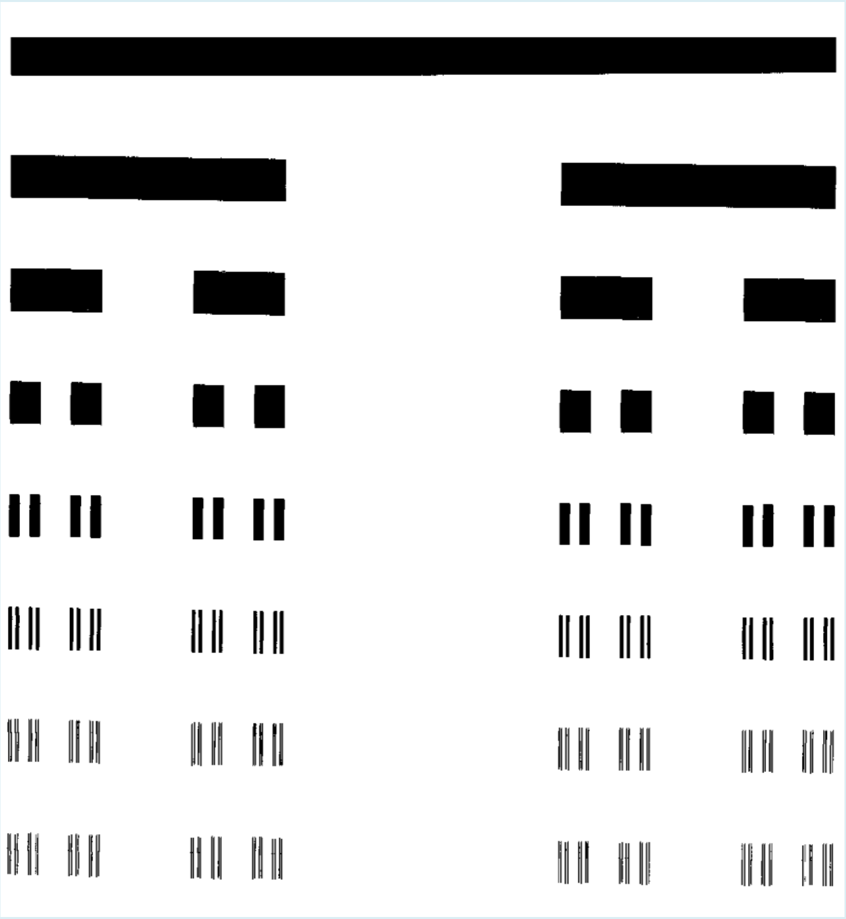
Закон Карпентера – Вокулера и стохастические иерархии в гравитирующих средах - 1

Галактическая среда устроена иерархически, и потому любой наблюдатель, если он связан с объектом, включенным в иерархию, обнаружит, что средняя плотность вокруг него убывает с расстоянием. При этом в галактической среде нет выделенного положения, то есть любые достаточно большие, но *одинаковые* объемы, имеют одинаковую среднюю плотность независимо от положения их центров друг относительно друга. Эту плотность можно назвать инвариантной *условной плотностью (УП)*. Однако, если мы будем синхронно изменять размеры этих (одинаковых) объемов, то УП будет изменяться с характерным размером объема **r** по закону:

$$n(r) \sim r^{-\alpha}$$

где $\alpha = 1,7$. Это и есть закон Карпентера – Вокулера для галактической среды

Пыль Кантора



$$2^3 = 8 \qquad n = 2^N$$

$$1/3 \;\; 1/9 \;\; 1/27$$

$$a = (1/3\,)^N \quad 2^N(1/3\,)^N = (2/3\,)^N$$

$$na = 1, n = 1/a$$

$$na^2 = 1, n = 1/a^2$$

$$n = 1/a^3 \quad n = 1/a^d$$

d – размерность объекта

$$d = \frac{\log n}{\log\left(\frac{1}{a}\right)}$$

$$d = \log(2)/\log(3) = 0,63 \ldots$$

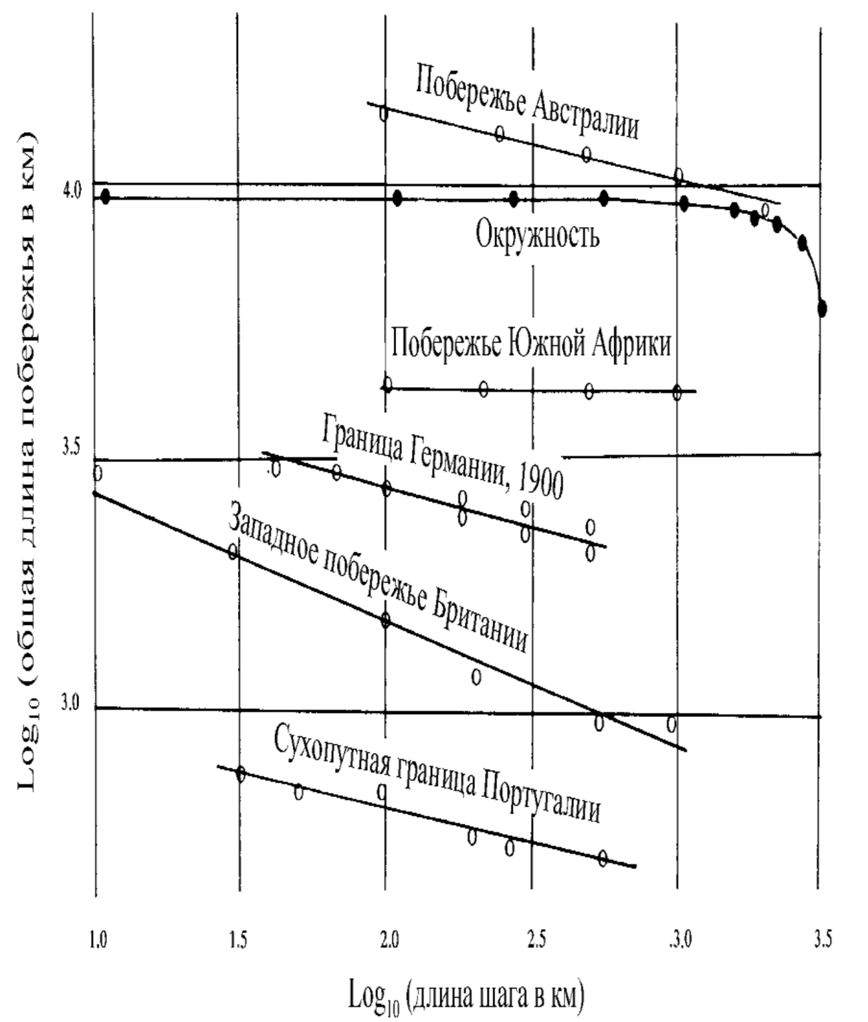
Бокс алгоритм и скейлинг.
Техника получения фрактальной размерности побережья южной части Норвегии

$$L(a) = N(a) * a$$

a – размер клетки

$N(a)$ – число клеток

График Ричардсона. Длина береговых линий



$\log(L(a)) - \log(a)$ - график Ричардсона

$k = 1 - d$ - коэффициент наклона

$$L(a) = b * a^{1-d}$$

$d -$ - фрактальная размерность

$d_N = 1,52 \pm 0,01$ - Норвегия

$d_G = 1,3 \ldots$ - Великобритания

Закон распределения модуля случайной силы для фрактальной среды в приближении ближайшего соседа -1

Как выше отмечалось, Чандрасекар (1943) получил точное решение для распределения флуктуирующей случайной силы в случае некоррелированного (Пуассоновского) случайного статистического пространственного распределения гравитирующих звезд. Это решение имеет вид распределения Хольцмарка:

$$W\left(\left|\vec{F}\right|\right) = H(\beta)a^{2/3}$$

Где

$$H(\beta) = \frac{2}{\pi\beta}\int_0^\infty exp\left[-(x/\beta)^{3/2}\right]x\sin x dx$$

$a = (4/15)(2\pi Gm)^{3/2}\, n$, если все гравитирующие точки имеют равные массы.

Закон распределения модуля случайной силы для фрактальной среды в приближении ближайшего соседа -2

При этом безразмерная сила равна:

$$\beta = \left|\vec{F}\right| / a^{2/3}$$

Далее Чандрасекаром (1943) было показано, что асимптотика распределения Хольцмарка для больших случайных сил в точности совпадает с распределением случайной силы, действующей на пробную звезду единичной массы со стороны ближайшего соседа с массой m, находящегося на расстоянии r :

$$\left|\vec{F}\right| = \frac{Gm}{r^2}$$

Асимптотика для $W(|F|)$ может быть получена из закона распределения расстояния до ближайшего соседа $w(r)$:

$$w(r)dr = 4\pi \exp\left(-4\pi r^3 n/3\right) r^2 n dr$$

Закон распределения ближайшего соседа (задача Герца) - 1

Найдем связь распределения Хольцмарка с распределением ближайшего соседа, $n -$ однородная плотность

$$w(r)dr = P\left\{\text{ближайший сосед} \in \underbrace{(r, r+dr)}_{\substack{\text{сферический слой} \\ \text{радиуса } r \text{ и толщиной } dr}}\right\} =$$

$$= P\{\text{нет звезд ближе } r\}$$

$$* P\{\text{в слое } r + dr \text{ ровно } 1 \text{ звезда}\}$$

Математическое ожидание числа звезд в слое $(r, r + dr)$

$$\lambda = 4\pi r^2 \underbrace{n}_{\text{мала}} \underbrace{dr}_{\text{мала}} \ll 1$$

Закон распределения ближайшего соседа (задача Герца) - 2

Распределение Пуассона

$$\boldsymbol{P}_{\text{п}}(\boldsymbol{k}) = \lambda^{k} \frac{e^{-\lambda}}{k!}$$

В нашем случае $\boldsymbol{k} = \mathbf{1} \Rightarrow \boldsymbol{P}_{\text{п}}(\mathbf{1}) \approx \boldsymbol{\lambda} \left(\boldsymbol{e}^{-\lambda} \text{ близко к } \mathbf{1}, \text{т. к. } \boldsymbol{\lambda} \text{ мало} \right) \Rightarrow$

$$\Rightarrow \boldsymbol{P}\{\text{в слое } \boldsymbol{r} + \boldsymbol{dr} \text{ ровно } \mathbf{1} \text{ звезда}\} = \mathbf{4}\boldsymbol{\pi}\boldsymbol{r}^{2}\boldsymbol{n}\boldsymbol{dr}$$

$$\boldsymbol{P}\{\text{в пределах } \boldsymbol{r} \text{ нет ни одной звезды}\} = \mathbf{1} - \underbrace{\int_{0}^{r} \boldsymbol{w}(\boldsymbol{r})\boldsymbol{dr}}_{\substack{\text{вероятность того,что} \\ \text{есть хотя бы } \mathbf{1} \text{ звезда}}}$$

$$\boldsymbol{w}(\boldsymbol{r})\boldsymbol{dr} = \left[\mathbf{1} - \int_{0}^{r} \boldsymbol{w}(\boldsymbol{r})\boldsymbol{dr} \right] \mathbf{4}\boldsymbol{\pi}\boldsymbol{r}^{2}\boldsymbol{n}\boldsymbol{dr}$$

$$w(r)dr = \left[1 - \int_0^r w(r)dr\right] 4\pi r^2 n dr$$

$$\frac{w(r)}{4\pi r^2 n} = 1 - \int_0^r w(r)dr$$

$$\int_0^r w(r)dr = 1 - \frac{w(r)}{4\pi r^2 n}$$

$$\frac{d}{dr}\left[\int_0^r w(r)dr\right] = \frac{d}{dr}\left[1 - \frac{w(r)}{4\pi r^2 n}\right]$$

$$w(r) = -\frac{d}{dr}\left[\frac{w(r)}{4\pi r^2 n}\right]$$

$$\frac{d}{dr}\left[\frac{w(r)}{4\pi r^2 n}\right] = -w(r)$$

$$\frac{d}{dr}\underbrace{\left[\frac{w(r)}{4\pi r^2 n}\right]}_{y(r)} = \underbrace{\left[\frac{w(r)}{4\pi r^2 n}\right]}_{y(r)} * \left(-4\pi r^2 n\right)$$

$$\frac{dy(r)}{dr} = y(r)\left(-4\pi r^2 n\right)$$

$$\frac{1}{y}\frac{dy}{dr} = -4\pi r^2 n; \quad \frac{d}{dr}\ln y = -4\pi r^2 n$$

$$\frac{d}{dr}\int \ln y \, dr = -\int 4\pi r^2 n dr$$

$$\ln y = -\frac{4\pi r^3 n}{3} \qquad \underbrace{y(r)}_{\frac{w(r)}{4\pi r^2 n}} = C exp\left\{-\frac{4}{3}\pi R^3 n\right\}$$

При $\quad r \to 0: w(r) \underset{r \to 0}{\longrightarrow} 4\pi r^2 n \Rightarrow C = 1$

Закон распределения ближайшего соседа (задача Герца) - 4

Закон для распределения расстояний до ближайшего соседа

$$w(r) = C4\pi r^2 n exp\left\{-\underbrace{\frac{4}{3}\pi R^3}_{V} n\right\},$$

$V * n -$ математическое ожидание числа звезд

Вероятность малых сил $\to 0, w(r) \xrightarrow[r\to\infty]{} 0.$

Точная формула для среднего расстояния между звездами

$$\overline{d} = \int_0^\infty r w(r) dr = \int_0^\infty e^{-4\pi r^3 \frac{n}{3}} * 4\pi r^3 n dr = \frac{1}{\left(\frac{4\pi n}{3}\right)^{\frac{1}{3}}} \int_0^\infty e^{-x} x^{\frac{1}{3}} dx = \frac{\Gamma\left(\frac{4}{3}\right)}{\left(\frac{4\pi n}{3}\right)^{\frac{1}{3}}} \approx$$

$$\approx 0.554 n^{-\frac{1}{3}} \approx \frac{n^{-\frac{1}{3}}}{2} \qquad \boxed{\overline{d} \approx \frac{1}{2n^{\frac{1}{3}}}}$$

Закон распределения модуля случайной силы для фрактальной среды в приближении ближайшего соседа -3

$$w(r)dr = 4\pi \exp\left(-4\pi r^3 n/3\right) r^2 n dr$$

$$W\left(\vec{F}\right)d\left|\vec{F}\right| = 4\pi n (Gm)^{3/2} \times exp\left[-\frac{4\pi n}{3}(Gm)^{3/2} \left|\vec{F}\right|^{-3/2}\right] \left|\vec{F}\right|^{-5/2} d\left|\vec{F}\right| \qquad (1)$$

Получим обобщение распределения $W(|F|)$ для фрактальной пространственной плотности звезд в предположении, что сила, действующая на пробную звезду, так же, в основном, пределяется ближайшим соседом. Для этого при выводе распределения расстояния до ближайшего соседа необходимо учесть зависимость средней плотности от расстояния в виде выражения:

$$n(r) = hr^{-\alpha} \qquad (2)$$

Закон распределения модуля случайной силы для фрактальной среды в приближении ближайшего соседа -4

Итак, $n(r) = hr^{-\alpha}$. Тогда:

$$w(r) = \left[1 - \int_0^r w(r)dr\right] 4\pi r^2 n(r) = 4\pi h \left[1 - \int_0^r w(r)dr\right] r^{D-1}$$

Откуда получаем: $\frac{d}{dr}\left[\frac{w(r)}{4\pi h r^{D-1}}\right] = -4\pi h r^{D-1} \frac{w(r)}{4\pi h r^{D-1}}$

Учитывая, что, согласно этой формуле, при $r \to 0$, $w(r) \to 4\pi h r^{D-1}$, имеем:

$$w(r)dr = 4\pi h exp\left(-\frac{4\pi h}{3} r^D\right) r^{D-1} dr \qquad \textbf{(3)}$$

Это и есть аналог формулы $w(r)dr = 4\pi \exp\left(-4\pi r^3 n/3\right)\ r^2 n dr$, то есть обобщение закона распределения ближайших соседей для случая фрактального распределения масс.

Закон распределения модуля случайной силы для фрактальной среды в приближении ближайшего соседа - 5

Далее, учитывая формулу $\left|\vec{F}\right| = \frac{Gm}{r^2}$

из формулы **(3)** после несложных преобразований получаем обобщение формулы **(1)** на случай степенного закона для плотности **(2)**:

$$W\left(\left|\vec{F}\right| D\right)d\left|\vec{F}\right| = 4\pi h(Gm)^{D/2} * exp\left[\frac{4\pi h}{D}\left(\frac{Gm}{\left|\vec{F}\right|}\right)^{D/2}\right]\left(\left|\vec{F}\right|\right)^{-D+2/2} d\left|\vec{F}\right| \quad \mathbf{(4)}$$

где D - фрактальная размерность.

Из формулы **(4),** переходя к пределу однородной среды $(\alpha \to 0, D \to 3, h = n)$, получаем распределение Хольцмарка **(1).**

В пределе сильных полей $\left(\left|\vec{F}\right| \to \infty\right)$ из **(4)** получаем:

$$W(F|D) \approx 4\pi h(Gm)^{D/2} F^{-\frac{D+2}{2}} \quad \mathbf{(5)}$$

Закон распределения модуля случайной силы для фрактальной среды в приближении ближайшего соседа - 6

Обозначив через x безразмерный показатель экспоненты в формуле (4), запишем (4) в виде:

$$W(F|D)F = 3xe^{-x}$$

Или

$$AW(F|D) = (3x)^{(D+2)/D}e^{-x}$$

Где $A = \dfrac{(4\pi h)^{2/D}}{Gm}$ $\quad x = \dfrac{4\pi h}{3}\left(\dfrac{Gm}{F}\right)^{D/2}$