Basis invariants in two-higgs-doublet model: Hilbert series, syzygies, and renormalization-group equations

Alexander Bednyakov (BLTP JINR)

South Africa - JINR Workshop on
Theoretical and Computational Physics

based on [JHEP 06 (2025) 0817],
[JHEP11(2018) 154], and [JHEP04(2021) 233]

Outline

- Introduction/Motivations
  - Quantum corrections: to run or not to run?
  - Fundamental scalars: one doublet or two?
  - Basis redundancies: friend or foe?
- Two-higgs-doublet model: scalar sector
  - Ring of basis invariants: generating set and syzygies
  - Counting basis invariants: Hilbert series, etc
  - RG for basis invariants at one loop and beyond
- Conclusions and Outlook

Quantum corrections and running parameters

Predictions in QFT involving fields Φ_i depend on parameters a_j

entering a Lagrangian $\mathcal{L}(\Phi_i, a_j)$

High-order quantum corrections:

Regularization and renormalization introduce aux scale μ

$$a_j \to a_j(\mu) \leftarrow \text{running Lagrangian parameters}$$

Renormalization-group (RG) equations for running parameters:

$$da_j/dt \equiv \dot{a}_j = \beta_j(a_k), \quad t = \ln(\mu/\mu_0), \quad a_j(\mu_0) = \tilde{a}_j$$

(e.g., in $\overline{\mathrm{MS}}$ scheme with $\beta_j(a_k)$ polynomial in a_k)

On importance of high-order RG: vacuum stability

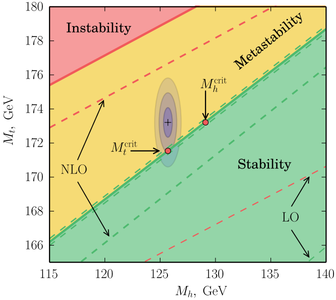
- Three-loop beta-functions
  for gauge, Yukawa, and self-coupling
  [Mihaila,Salomon,Steinhauser'12]
  [AB,Pikelner,Velizhanin'13;Chetyrkin&Zoller'13]

$$V_{\rm eff}(h \gg v) \simeq \frac{\lambda(\mu = h)}{4} h^4$$

$$(4\pi)^2 \frac{d\lambda}{dt} = 12\lambda + 6y_t^2\lambda - 3y_t^4 + \dots$$

$$(4\pi)^2 \frac{dy_t}{dt} = \frac{9}{4} y_t^3 - 4{g_s}^2 y_t + + \dots$$

[AB,Kniehl,Pikelner,Veretin'15]



- The SM: a single Higgs self-coupling λ
- What about extended Higgs sector?

Scalar sector of 2HDM

- 2HDM [Lee'73] is one of the simplest SM extensions (see, e.g., [Branco…'12,Ivanov'17])
- Predicts three more scalar states $(H_0, A_0, H^\pm)$ in the spectrum
- Additional sources for CP (and flavor) violation

$$V_H = m_{11}^2 \Phi_1^\dagger \Phi_1 + m_{22}^2 \Phi_2^\dagger \Phi_2 - \left(m_{12}^2 \Phi_1^\dagger \Phi_2 + \text{h.c.} \right), \quad \Phi_{1,2} \text{ are higgs doublets}$$

$$+ \frac{1}{2}\lambda_1 \left(\Phi_1^\dagger \Phi_1\right)^2 + \frac{1}{2}\lambda_2 \left(\Phi_2^\dagger \Phi_2\right)^2 + \lambda_3 \left(\Phi_1^\dagger \Phi_1\right)\left(\Phi_2^\dagger \Phi_2\right) + \lambda_4 \left(\Phi_1^\dagger \Phi_2\right)\left(\Phi_2^\dagger \Phi_1\right)$$

$$+ \left[\frac{1}{2}\lambda_5 \left(\Phi_1^\dagger \Phi_2\right)^2 + \lambda_6 \left(\Phi_1^\dagger \Phi_1\right)\left(\Phi_1^\dagger \Phi_2\right) + \lambda_7 \left(\Phi_2^\dagger \Phi_2\right)\left(\Phi_1^\dagger \Phi_2\right) + \text{h.c.} \right]$$

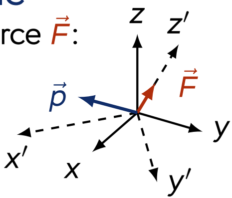
m_{11}^2, m_{22}^2, $\lambda_{1,2,3,4}$ are real, m_{12}^2 and $\lambda_{5,6,7}$ can be complex

- 14 real parameters in V_H instead of 2 in the SM
- All of them are "physical"?
- No: there is a freedom to redefine higgs basis

$$\Phi_a \to U_{ab}\Phi_b, \quad U \in \text{SU}(2)$$

Basis Redundancies: a simple example

- A particle experiencing constant external force $\vec{F}$:



$$L = K - V, \qquad K = \frac{\dot{\vec{r}}^{\,2}}{2}, \quad V = -\vec{F} \cdot \vec{r}$$

- General case (3+3+3 parameters):

$$\vec{r} = (r_x, r_y, r_z), \qquad \vec{p} \equiv \dot{\vec{r}} = (p_x, p_y, p_z), \qquad \vec{F} = (F_x, F_y, F_z)$$

- A basis **choice** (3+2+1 parameters): via a rotation

$$\vec{r} = (r'_x, r'_y, r'_z), \qquad \vec{p} = (p'_x, 0, p'_z), \qquad \vec{F} = (0, 0, F')$$

can be **different** for different people!

- Rotation/basis invariants (3+2+1 parameters):

$$I_1 = \vec{p} \cdot \vec{p}, \qquad I_2 = \vec{p} \cdot \vec{F}, \qquad I_3 = \vec{F} \cdot \vec{F}$$

$$I_4 = \vec{r} \cdot \vec{r}, \qquad I_5 = \vec{r} \cdot \vec{p}, \qquad I_6 = \vec{F} \cdot \vec{r},$$

$$\underbrace{I_7 = \vec{r} \cdot (\vec{p} \times \vec{F}),}_{\text{odd under parity}} \quad \underbrace{I_7^2 = P_3(I_1, \ldots, I_6)}_{\text{polynomial relation}} = I_1 I_3 I_4 - \cdots$$

Basis redundancies: Yukawa couplings in the SM

- Three generations of quarks and leptons

$$-\mathcal{L}_Y = \bar{Q}^i(\Phi Y_d^{ij})d_R^j + \bar{Q}^i(\tilde{\Phi} Y_u^{ij})u_R^j + \bar{L}^i(\Phi Y_l^{ij})l_R^j + \mathrm{h.c.}$$

- Complex quark Yukawa matrices: $2 \cdot 2 \cdot 3 \times 3 = 36$ parameters

$$18_{\text{re}} + 18_{\text{im}} \quad \Rightarrow \quad \underbrace{U_Q(3) \times U_u(3) \times U_d(3)}_{\text{change basis in quark sector}} \quad \Rightarrow \quad 9_{\text{re}} + 1_{\text{im}}$$

- Basis redundancies

$$Q^i \to U_Q^{ij} Q^j \qquad d_R^i \to U_d^{ij} d_R^j \qquad u_R^i \to U_u^{ij} u_R^j$$

can obscure physical questions, like CP-violation?

- "Physical" parameters:

6 quark masses,
3 angles and 1 phase of 3×3 CKM matrix

Basis redundancies: Yukawa couplings in the SM

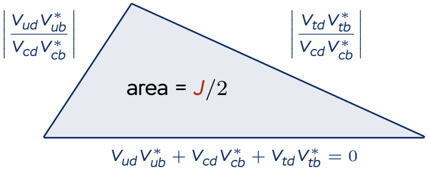
- Three generations of quarks and leptons (CP-violation?)

$$-\mathcal{L}_Y = \bar{Q}^i(\Phi Y_d^{ij})d_R^j + \bar{Q}^i(\tilde{\Phi} Y_u^{ij})u_R^j + \bar{L}^i(\Phi Y_l^{ij})l_R^j + \mathrm{h.c.}$$

- Invariants under $Y_d \to U_Q^\dagger Y_d U_d$, $Y_u \to U_Q^\dagger Y_u U_u$: [Jenkins,Manohar'09]

$$\underbrace{\mathrm{Tr}(Y_u Y_u^\dagger), \mathrm{Tr}(Y_d Y_d^\dagger), \text{etc},}_{\text{11 CP-even}}$$

$$\underbrace{\det[Y_u Y_u^\dagger, Y_d Y_d^\dagger] \propto J}_{\text{1 CP-odd}}$$



- [Jarlskog'85] invariant $J = \mathrm{Im}(V_{ud}V_{cs}V_{us}^*V_{cd}^*)$ $\quad J \neq 0$: ~~CP~~!

- Ultimate goal: **Observables** in terms of basis invariants.

Scalar sector of 2HDM (once again)

- 2HDM [Lee'73] is one of the simplest SM extensions

(see, e.g., [Branco...'12,Ivanov'17])

$$V_H = m_{11}^2 \Phi_1^\dagger \Phi_1 + m_{22}^2 \Phi_2^\dagger \Phi_2 - \left(m_{12}^2 \Phi_1^\dagger \Phi_2 + \text{h.c.} \right), \quad \Phi_{1,2} \text{ are higgs doublets}$$
$$+ \frac{1}{2}\lambda_1 \left(\Phi_1^\dagger \Phi_1\right)^2 + \frac{1}{2}\lambda_2 \left(\Phi_2^\dagger \Phi_2\right)^2 + \lambda_3 \left(\Phi_1^\dagger \Phi_1\right)\left(\Phi_2^\dagger \Phi_2\right) + \lambda_4 \left(\Phi_1^\dagger \Phi_2\right)\left(\Phi_2^\dagger \Phi_1\right)$$
$$+ \left[\frac{1}{2}\lambda_5 \left(\Phi_1^\dagger \Phi_2\right)^2 + \lambda_6 \left(\Phi_1^\dagger \Phi_1\right)\left(\Phi_1^\dagger \Phi_2\right) + \lambda_7 \left(\Phi_2^\dagger \Phi_2\right)\left(\Phi_1^\dagger \Phi_2\right) + \text{h.c.}\right]$$

m_{11}^2, m_{22}^2, $\lambda_{1,2,3,4}$ are real, m_{12}^2 and $\lambda_{5,6,7}$ can be complex

- Freedom to redefine higgs basis $\Phi_a \to U_{ab}\Phi_b, \quad U \in \mathrm{SU}(2)$

14 (parameters in V_H) - 3 = 11 (physical parameters)

- What about basis invariant (combination of parameters)?

How to construct?
How do they run?

NB: in what follows we consider the limit of vanishing gauge and Yukawa couplings

Bilinear formalism

Pack λ_i, m_{ij}^2 into $SO(3)$ multiplets [Ivanov'05-07]: $2 \otimes \bar{2} = 3 \oplus 1$

$$\Phi_a \Phi_b^\dagger = \frac{1}{2}\left[\ r_0\ \right]\delta_{ab} + \frac{1}{2}\left[\ \vec{r}\ \right]\vec{\sigma}_{ab}, \qquad V_H = M_\mu r^\mu + \Lambda_{\mu\nu} r^\mu r^\nu$$

$$r_\mu = \{r_0, \vec{r}\}$$

Λ_{00}, $\vec{\Lambda}$, Λ

$$\Lambda_{\mu\nu} = \begin{pmatrix} \frac{\lambda_1+\lambda_2}{2} + \lambda_3 & \mathrm{Re}\,(\lambda_6+\lambda_7) & -\mathrm{Im}\,(\lambda_6+\lambda_7) & \frac{\lambda_1-\lambda_2}{2} \\ \mathrm{Re}\,(\lambda_6+\lambda_7) & \lambda_4 + \mathrm{Re}\,(\lambda_5) & -\mathrm{Im}\,(\lambda_5) & \mathrm{Re}\,(\lambda_6-\lambda_7) \\ -\mathrm{Im}\,(\lambda_6+\lambda_7) & -\mathrm{Im}\,(\lambda_5) & \lambda_4 - \mathrm{Re}\,(\lambda_5) & -\mathrm{Im}\,(\lambda_6-\lambda_7) \\ \frac{\lambda_1-\lambda_2}{2} & \mathrm{Re}\,(\lambda_6-\lambda_7) & -\mathrm{Im}\,(\lambda_6-\lambda_7) & \frac{\lambda_1+\lambda_2}{2} - \lambda_3 \end{pmatrix}$$

M_0, $\vec{M}$

$$M_\mu = \{m_{11}^2 + m_{22}^2, \; -2\mathrm{Re}\, m_{12}^2, 2\mathrm{Im}\, m_{12}^2, m_{11}^2 - m_{22}^2\}$$

Bilinear formalism: SO(3) covariants

- $U \in \mathrm{SU}(2)$ is mapped to rotations $\mathrm{SO}(3)$, $R_{ij} = 1/2\mathrm{Tr}(U^\dagger \sigma_i U \sigma_j)$:

$$\Lambda_{00} \to \Lambda_{00},\ M_0 \to M_0, \quad \vec{\Lambda} \to R \cdot \vec{\Lambda},\ \vec{M} \to R \cdot \vec{M}, \quad \Lambda \to R \cdot \Lambda \cdot R^T,$$

Λ_{00}, M_0 [and $\mathrm{tr}\Lambda$] – singlets,
$\vec{\Lambda}$, and $\vec{M}$ – triplets (vectors),
$\tilde{\Lambda} \equiv \Lambda - \frac{1}{3}\mathrm{tr}\Lambda$ – five-plet (traceless symmetric matrix)

- Use group theory to find singlet combinations - invariants
- Examples: $\vec{\Lambda} \cdot \vec{\Lambda}$ and $\vec{\Lambda} \cdot [(\tilde{\Lambda} \cdot \vec{\Lambda}) \times (\tilde{\Lambda}^2 \cdot \vec{\Lambda})]$ are basis invariants.

NB: for Generalized CP-transormation $\Phi_a \to X_{ab} \Phi_b^*$
$X \in \mathrm{SU}(2)$ is mapped to $\tilde{R}_{ij} = 1/2\mathrm{Tr}[X^\dagger \sigma_i X \sigma_j^T]$ with $\det \tilde{R} = -1$
Invariants that involve odd number of triplets are CP-odd [Trautner'18].

Ring of basis invariants $\mathcal{R}$ as graded polynomial ring

- Algebraically independent basis invariants $\{f_i\}$:

$$\forall \mathcal{I} \in \mathcal{R}, \quad \exists P \in \mathbf{k}[x_1, \ldots, x_{N+1}] : \quad P(\mathcal{I}, f_1, \ldots, f_N) = 0, \quad N = 11$$

- Generating set of invariants $\{g_n\}$:

$$\forall \mathcal{I} \in \mathcal{R}, \quad \exists \mathcal{P} \in \mathbf{k}[x_1, \ldots, x_M] : \quad \mathcal{I} = \mathcal{P}(g_1, \ldots, g_M), \quad M = 22$$

- Relations between $\{g_n\}$ ("syzygies") form an ideal (R-module)

$$I_{2HDM} = \langle r_1, \ldots, r_K \rangle, \qquad K = 63$$

in polynomial ring $R \equiv \mathbf{k}[g_1, \ldots, g_M]$ generated by $r_k \in R$

$$r_k(g_1, \ldots, g_M) \xrightarrow{\phi} 0, \quad \phi - \text{substitutes } g_i \text{ by their expressions}$$

- Quotient ring of basis invariants $\mathcal{R} = R/I_{2HDM}$:

$$p_1 \sim p_2 : p_1 - p_2 \in I_{2HDM}$$

- Gröbner basis $G = \langle \mathfrak{g}_1 \ldots \mathfrak{g}_J \rangle$ for I_{2HDM} – membership problem:

$$p \in I_{2HDM} \Longleftrightarrow p = \mathrm{a}_1 \mathfrak{g}_1 + \ldots + \mathrm{a}_J \mathfrak{g}_J, \ \mathrm{a}_i \in R \Longleftrightarrow (p \bmod G) = 0$$

Ring of basis invariants $\mathcal{R}$ as graded polynomial ring

- Algebraically independent basis invariants $\{f_i\}$:

$$\forall \mathcal{I} \in \mathcal{R}, \quad \exists P \in \mathbf{k}[x_1, \ldots, x_{N+1}] : \quad P(\mathcal{I}, f_1, \ldots, f_N) = 0, \quad N = 11$$

- Generating set of invariants $\{g_n\}$:

$$\forall \mathcal{I} \in \mathcal{R}, \quad \exists \mathcal{P} \in \mathbf{k}[x_1, \ldots, x_M] : \quad \mathcal{I} = \mathcal{P}(g_1, \ldots, g_M), \quad M = 22$$

- Relations between $\{g_n\}$ ("syzygies") form an ideal (R-module)

$$I_{2HDM} = \langle r_1, \ldots, r_K \rangle, \qquad K = 63$$

in polynomial ring $R \equiv \mathbf{k}[g_1, \ldots, g_M]$ generated by $r_k \in R$

$$r_k(g_1, \ldots, g_M) \xrightarrow{\phi} 0, \quad \phi - \text{substitutes } g_i \text{ by their expressions}$$

- Quotient ring of basis invariants $\mathcal{R} = R/I_{2HDM}$:

$$p_1 \sim p_2 : p_1 - p_2 \in I_{2HDM}$$

- Gröbner basis $G = \langle \mathfrak{g}_1 \ldots \mathfrak{g}_J \rangle$ for I_{2HDM} – membership problem:

$$p \in I_{2HDM} \Longleftrightarrow p = \mathrm{a}_1 \mathfrak{g}_1 + \ldots + \mathrm{a}_J \mathfrak{g}_J, \ \mathrm{a}_i \in R \Longleftrightarrow (p \bmod G) = 0$$

Ring of basis invariants $\mathcal{R}$ as graded polynomial ring

- (Multi)grading of $\mathcal{R}$ [n - (multi) degree]

$$\mathcal{R} = \bigoplus_{n=0}^{\infty} \mathcal{R}_n = \mathcal{R}_0 \oplus \mathcal{R}_1 \oplus \mathcal{R}_2 \oplus \cdots \qquad \mathcal{R}_i \mathcal{R}_j \subseteq \mathcal{R}_{i+j}$$

  - $n \to$ total number of λ_i and m_{ij}^2 entering an invariant
  - $n \to \{\alpha, \beta\}$: homogeneous element $\mathcal{I}_{\alpha\beta} \in \mathcal{R}_{\alpha\beta}$: [convenient for RG]

$$\mathcal{I}_{\alpha\beta} \to x^\alpha y^\beta \cdot \mathcal{I}_{\alpha\beta}, \quad \text{if } \lambda_i \to x \cdot \lambda_i,\ m_{ij}^2 \to y \cdot m_{ij}^2$$

  - $n \to [a, b, c]$: homogeneous element $T_{abc} \in \mathcal{R}_{abc}$: [Trautner'18]

$$T_{abc} \to x^a y^b z^c \cdot T_{abc}, \quad \text{if } \tilde{\Lambda} \to x \cdot \tilde{\Lambda},\ \vec{M} \to y \cdot \vec{M},\ \vec{\Lambda} \to z \cdot \vec{\Lambda}$$

- Examples:

$$\Lambda_{00}, \mathrm{tr}\Lambda \in \mathcal{R}_{10} \subset \mathcal{R}_1, \quad M_0 \in \mathcal{R}_{01} \subset \mathcal{R}_1, \quad \vec{\Lambda} \cdot \vec{M} \in \mathcal{R}_{011} \subset \mathcal{R}_{11} \subset \mathcal{R}_2$$

Hilbert Series and Plethystic Logarithm

see [Hanany...'09,Jenkins...'09,Hanany...'10,Lehman...'15]

- Tool to count linear independent elements of $\mathcal{R}_n$:

$$H(t) = \sum_n (\dim \mathcal{R}_n) t^n$$ [AB'18]

$$H(q, y, t) = \sum_{abc} (\dim \mathcal{R}_{abc}) q^a y^b t^c$$ [Trautner'18]

$$H(\lambda, m) = \sum_{\alpha\beta} (\dim \mathcal{R}_{\alpha\beta}) \lambda^\alpha m^\beta$$ [AB'25]

$$H(t) = H(t,t), \quad H(\lambda, m) = \frac{1}{\underbrace{(1-\lambda)^2}_{\Lambda_{00}, \mathrm{tr}\,\Lambda}\underbrace{(1-m)}_{M_0}} H(\lambda, m, \lambda)$$

$$H(t) = \frac{1 + t^3 + 4y^4 + 2t^5 + 4t^6 + t^7 + t^{10}}{\underbrace{(1-t)^3(1-t^2)^4(1-t^3)^3(1-t^4)}_{\text{degree and number of } \{f_i\}}} = \frac{7/864}{\underbrace{(1-t)^{11}}_{N=11}} + \ldots$$

Hilbert Series and Plethystic Logarithm

see [Hanany…'09,Jenkins…'09,Hanany…'10,Lehman…'15]

- Tool to enumerate generators of the invariants ring and of the syzygy module:

$$PL[H(q,y,t)] \equiv \sum_{k=1}^{\infty} \frac{\mu(k) \ln H[q^k, y^k, t^k]}{k}, \quad \mu(k) - \text{Möbius function}$$

$$PL[H(t)] \equiv \sum_{k=1}^{\infty} \frac{\mu(k) \ln H[t^k]}{k} = \sum_{n=1}^{\infty} d_n t^{a_n}, \qquad \sum_{k|m} \mu(k) = \delta_{m,1}$$

$$= 3t + 4t^2 + 4t^3 + 5t^4 + 2t^5 + [4-1]t^6$$

$$- 3t^7 - 12t^8 - 12t^9 - [17-2]t^{10} - [8-14]t^{11} - [10-38]t^{12} + \dots$$

- First positive terms – ring generators and their degree: $M = 22$
- First negative terms – syzygies and their degree: $K = 63$
- Explicitly construct generators $\{g_n\}$ and syzygies $\{r_k\}$?

Generating set of invariants

- Full set $\{g_n\}$ was constructed via group theory in [Trautner'18]:

$$
\begin{aligned}
3t &: \quad Z_{1_{(1)}}, Z_{1_{(2)}}, Y_1, \\
4t^2 &: \quad T_{200}, T_{020}, T_{002}, T_{011}, \\
4t^3 &: \quad T_{300}, T_{120}, T_{102}, T_{111}, \\
5t^4 &: \quad T_{220}, T_{202}, T_{211}, T_{112}, T_{121}, \\
2t^5 &: \quad T_{212}, T_{221}, \\
4t^6 &: \quad T_{312}, T_{321}, T_{303}, T_{330}
\end{aligned}
$$

expressed in terms of 14 real parameters (related to λ_i and m_{ij}^2)

- We re-express $\{g_n\}$ in terms of SO(3) covariants, e.g.,

$$8 \cdot T_{200} = \mathrm{tr}\tilde{\Lambda}^2, \; 8 \cdot T_{111} = (\vec{\Lambda} \cdot \tilde{\Lambda} \cdot \vec{M}), \qquad \text{CP-even}$$

$$-16i \cdot T_{112} = \vec{M} \cdot [\vec{\Lambda} \times (\tilde{\Lambda} \cdot \vec{\Lambda})] \qquad \text{CP-odd}$$

- Algebraically independent elements $\{f_i\}$: [Trautner'18,Bento et al'20]

$$\underbrace{Z_{1_{(1)}}, Z_{1_{(2)}}, Y_1}_{\text{degree 1}}, \underbrace{T_{200}, T_{002}, T_{020}, T_{011}}_{\text{degree 2}}, \underbrace{T_{300}, T_{102}, T_{120}}_{\text{degree 3}}, \underbrace{T_{211}}_{\text{degree 4}} .$$

Syzygies: How to find?

see, e.g., textbook [Cox, Little, O'Shea'2015]

- Consider an auxiliary polynomial ring over a field $\mathbf{k}$

$$\tilde{R} = \mathbf{k}[x_1, \dots, x_N, g_1, \dots, g_M]$$

NB: convenient choice of x_j can simplify/speed up calculations

- Consider an ideal

$$\mathcal{I} = \langle g_1 - g_1(x_1, \dots, x_N), \dots, g_M - g_M(x_1, \dots, x_N)\rangle$$

- The syzygy ideal is the elimination ideal

$$I_{2HDM} = \mathcal{I} \cap \mathbf{k}[g_1, \dots, g_M]$$

- syzygies up to certain degree belonging to I_{2HDM} can be found by constructing a Gröbner basis for $\mathcal{I}$ [AB'25]
- We use [Macaulay2] package to implement the algorithm and find minimal set of relations (generators r_k of syzygy module)

NB: The form of generators r_k depends on the monomial ordering!

Generators of syzygy module

[Trautner'18,Bento et al'20,AB'25]

- 3 "Even×Even" syzygies allowing one to get rid of products

$$T_{111}{}^2 \quad T_{111}T_{211} \quad T_{211}{}^2$$
$$^{6}[222] \quad ^{7}[322] \quad ^{8}[422]$$

Example: $[222]$ - syzygy

$$T_{111}{}^2 = 2T_{011}T_{211} + 3T_{102}T_{120} + (T_{002}T_{020} - T_{011}^2)T_{200} - T_{020}T_{202} - T_{002}T_{220}$$

- 24 "Odd×Even" syzygies: get rid, e.g., of products

$$T_{111} \cdot \{T_{112}, T_{212}, T_{312}, T_{303}\}, \quad T_{211} \cdot \{T_{112}, T_{212}, T_{312}, T_{303}\},$$
$$T_{303} \cdot \{T_{011}, T_{020}, T_{120}, T_{220}\},$$

$[acb]$ – syzygy from $[abc]$ one

- 36 "Odd ×Odd" syzygies: any product of two odd invariants

$$\{T_{112}, T_{121}, T_{212}, T_{221}, T_{312}, T_{321}, T_{303}, T_{330}\}$$

can be expressed in terms of even elements

Ring of invariants and RG

- Given $\{g_n\}$ and $\{r_k\}$ we explicitly construct bases* $\{I^{(i)}_{\alpha,\beta}\}$ of $\mathcal{R}_{\alpha\beta}$ needed to find L-loop beta function of an element $I_{\alpha\beta} \in \{g_n\}$

$$\frac{d}{dt} I_{\alpha\beta} = \sum_{l=1}^{L} h^l \cdot \beta^{(l)}_{I_{\alpha\beta}}, \quad \beta^{(l)}_{I_{\alpha\beta}} = \underbrace{\sum_{i=1}^{d_{\alpha+l,\beta}} c^{(i)}_{\alpha,\beta;l} I^{(i)}_{\alpha+l,\beta}}_{\text{element of } \mathcal{R}_{\alpha+l,\beta}}, \qquad h \equiv 1/(16\pi^2)$$

$$d_{\alpha,\beta} = (\dim R_{\alpha\beta}) = d^{even}_{\alpha,\beta} + d^{odd}_{\alpha,\beta} \qquad \text{[from Hilbert Series } H(\lambda, m)]$$

*We use a Gröbner basis G for I_{2HDM} to test if a monomial $m_{\alpha\beta} = g_1^{\alpha_1} \ldots g_M^{\alpha_M}$ with multidegree $\{\alpha\beta\}$ belongs to basis of $\mathcal{R}_{\alpha\beta}$: normal form $(m_{\alpha\beta} \bmod G) = m_{\alpha\beta}$

Ring of invariants and RG

- Given $\{g_n\}$ and $\{r_k\}$ we explicitly construct bases* $\{I^{(i)}_{\alpha,\beta}\}$ of $\mathcal{R}_{\alpha\beta}$ needed to find L-loop beta function of an element $I_{\alpha\beta} \in \{g_n\}$

$$\left[\beta^{(l)}_{\lambda_i}\partial_{\lambda_i} + \beta^{(l)}_{m^2_{ij}}\partial_{m^2_{ij}}\right] I_{\alpha\beta}(\lambda_k, m^2_{pq}) = \sum_{i=1}^{d_{\alpha+l,\beta}} c^{(i)}_{\alpha,\beta;l} I^{(i)}_{\alpha+l,\beta}(\lambda_k, m^2_{pq})$$

$$d_{\alpha,\beta} = (\dim R_{\alpha\beta}) = d^{even}_{\alpha,\beta} + d^{odd}_{\alpha,\beta} \qquad \text{[from Hilbert Series } H(\lambda, m)\text{]}$$

- Numerical coefficients $c^{(i)}_{\alpha,\beta;l}$ are found by linear algebra: we express LHS and RHS in terms of λ_i, and m^2_{ij} and use 6-loop beta functions for the latter derived in [AB,Pikelner'21]

*We use a Gröbner basis G for I_{2HDM} to test if a monomial $m_{\alpha\beta} = g_1^{\alpha_1} \dots g_M^{\alpha_M}$ with multidegree $\{\alpha\beta\}$ belongs to basis of $\mathcal{R}_{\alpha\beta}$: normal form $(m_{\alpha\beta} \bmod G) = m_{\alpha\beta}$

Ring of invariants and RG

- Given $\{g_n\}$ and $\{r_k\}$ we explicitly construct bases* $\{I^{(i)}_{\alpha,\beta}\}$ of $\mathcal{R}_{\alpha\beta}$ needed to find L-loop beta function of an element $I_{\alpha\beta} \in \{g_n\}$

$$\left[\beta^{(l)}_{\lambda_i}\partial_{\lambda_i} + \beta^{(l)}_{m^2_{ij}}\partial_{m^2_{ij}}\right] I_{\alpha\beta}(\lambda_k, m^2_{pq}) = \sum_{i=1}^{d_{\alpha+l,\beta}} c^{(i)}_{\alpha,\beta;l} I^{(i)}_{\alpha+l,\beta}(\lambda_k, m^2_{pq})$$

$d_{\alpha,\beta} = (\dim R_{\alpha\beta}) = d^{even}_{\alpha,\beta} + d^{odd}_{\alpha,\beta}$ [from Hilbert Series $H(\lambda, m)$]

- Numerical coefficients $c^{(i)}_{\alpha,\beta;l}$ are found by linear algebra: we express LHS and RHS in terms of λ_i, and m^2_{ij} and use 6-loop beta functions for the latter derived in [AB,Pikelner'21]
- Beta function for an arbitrary invariant $p \in \mathcal{R}$:

$$\beta_p = \left[\sum_{n=1}^{M} \frac{\partial p}{\partial g_n}\beta_{g_n}\right] \mod G$$

*We use a Gröbner basis G for I_{2HDM} to test if a monomial $m_{\alpha\beta} = g_1^{\alpha_1}\dots g_M^{\alpha_M}$ with multidegree $\{\alpha\beta\}$ belongs to basis of $\mathcal{R}_{\alpha\beta}$: normal form $(m_{\alpha\beta} \mod G) = m_{\alpha\beta}$

Results: one-loop RG (examples)

$$\beta^{(1)}_{Z_{1_{(1)}}} = 28T_{200} + 48T_{002} + \frac{29}{3}Z^2_{1_{(1)}} + 2Z_{1_{(1)}}Z_{1_{(2)}} + 3Z^2_{1_{(2)}}, \quad d^{even}_{2,0} = 5$$

$$\beta^{(1)}_{T_{303}} = 12[6Z_{1_{(1)}} - Z_{1_{(2)}}]T_{303}, \quad d^{odd}_{7,0} = 2$$

$$\beta^{(1)}_{Y_1} = 2[3Z_{1_{(1)}} + Z_{1_{(2)}}]Y_1 + 24T_{011}, \quad d^{even}_{1,1} = 3$$

$$\beta^{(1)}_{T_{112}} = \frac{2}{3}[65Z_{1_{(1)}} - 9Z_{1_{(2)}}]T_{112} - 28T_{212}, \quad d^{odd}_{4,1} = 3$$

$$\beta^{(1)}_{T_{020}} = 4[Z_{1_{(1)}} - Z_{1_{(2)}}]T_{020} + 24T_{120} + 12T_{011}Y_1, \quad d^{even}_{1,2} = 6$$

$$\beta^{(1)}_{T_{121}} = \frac{8}{3}[11Z_{1_{(1)}} - 3Z_{1_{(2)}}]T_{121} - 6T_{112}Y_1 - 4T_{221}, \quad d^{odd}_{3,2} = 4$$

$$\beta^{(1)}_{T_{330}} = 2(17Z_{1_{(1)}} - 9Z_{1_{(2)}})T_{330} + 24[T_{120}T_{112} - T_{020}T_{212}] - 72T_{011}T_{221} + 6T_{321}Y_1, \quad d^{odd}_{4,3} = 21$$

We derived all RG functions up to 6 loops and provided [Macaulay2] routines to compute normal forms in [AB'25]

Conclusions and outlook

Summary:

- Basis/Reparamerization invariants
  – "unambiguous" parametrization of physics
- Hilbert Series, Plethystic Logarithms
  – convenient counting tools
- Construction of invariants (via group theory)
  and finding syzygies (via Gröbner bases)

Conclusions and outlook

DONE: Polynomial ring of basis invariants in 2HDM [AB'25]

- Convenient representation for all $M = 22$ generators $\{g_i\}$
- Gröbner bases: $K = 63$ non-trivial relations $\{r_k\}$ of I_{2HDM}
- Derived 6-loop beta functions for elements of the ring
- Checked RG invariance* of syzygies $\{r_k\}$

TODO:

- Gauge and Yukawa interactions :

  possible CP non-conservation for real 2HDM potential

  discussed recently in literature [de Lima,Logan'24]

Thank you for attention!

Backup

Hilbert Series: Molien-Weyl formula

- Plethystic exponent

$$\mathrm{PE}[z, t; \mathrm{r}] = \exp\left[\sum_{i=1}^{\infty} \frac{t^i \chi_{\mathrm{r}}(z^i)}{i}\right]$$

- Characters of $[\mathrm{SU}(2)]$ representations $\mathrm{r} = \{3, 5\}$

$$\chi_3(z) = z^2 + 1 + \frac{1}{z^2}, \quad \chi_5(z) = z^4 + z^2 + 1 + \frac{1}{z^2} + \frac{1}{z^4}$$

- (Multi-graded) Hilbert Series (via Molien-Weyl formula)

$$H(q, y, t) = \int d\mu_{\mathrm{SU}(2)}(z) \cdot \underbrace{\mathrm{PE}[z, q, 5]}_{\tilde{\Lambda}} \cdot \underbrace{\mathrm{PE}[z, y, 3]}_{\vec{M}} \cdot \underbrace{\mathrm{PE}[z, t, 3]}_{\vec{\Lambda}}$$

$$\int d\mu_{\mathrm{SU}(2)} = \frac{1}{2\pi i} \oint_{|z|=1} \frac{dz}{z}(1 - z^2)$$

Simple [Mathematica] routine based on [LieART] package

Hilbert Series: MacMahon Omega calculus

see [Bento'21] for application to 2HDM and NHDM

- An operator $\underset{=}{\Omega}$ that extracts constant term of a series:

$$\underset{=}{\Omega} \sum_{j_1=-\infty}^{\infty} \cdots \sum_{j_n=-\infty}^{\infty} a_{j_1,\ldots,j_n} \lambda_1^{j_1} \ldots \lambda_n^{j_n} := a_{0,\ldots,0}$$

- Fast evaluation based on

$$H(q, y, t) = \underset{=}{\Omega} \left[(1 - z^2) \cdot \underbrace{\mathrm{PE}[z, q, 5]}_{\tilde{\Lambda}} \cdot \underbrace{\mathrm{PE}[z, y, 3]}_{\vec{M}} \cdot \underbrace{\mathrm{PE}[z, t, 3]}_{\vec{\Lambda}} \right]$$

[Omega] Mathematica package by [Andrews et al'01]

[Maple] code by [G.Xin'04]