# Relativistic generalization of the Faddeev-Yakubovsky equation

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BLTP JINR

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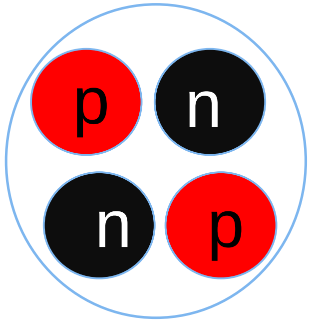
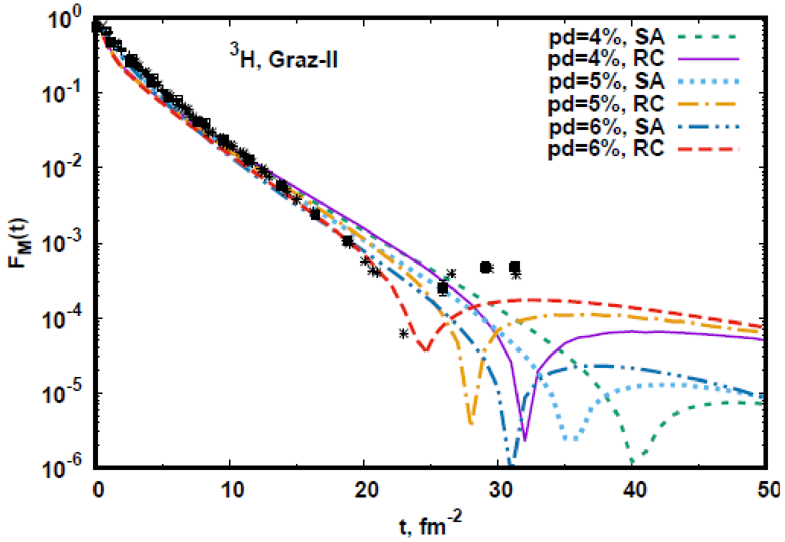
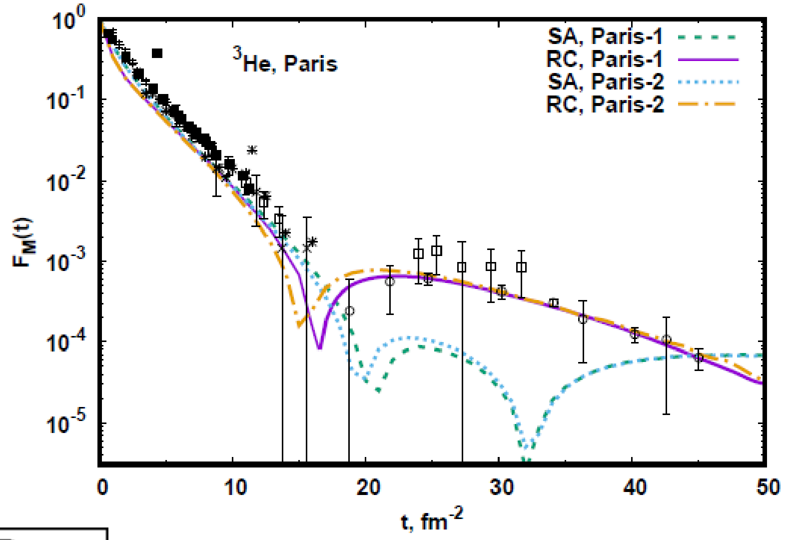
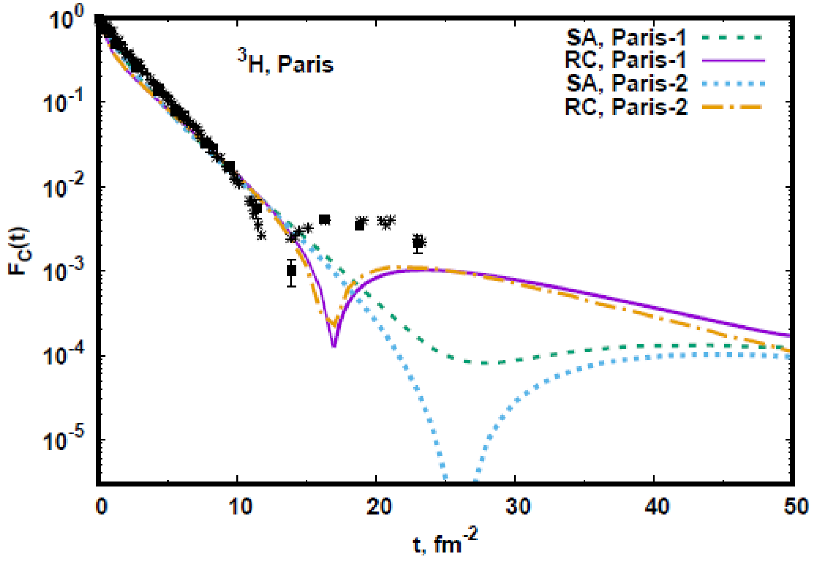
- **Object** - Four-particle system ( Helium-4 )




- **Goal** - Research within the framework of relativistic formalism
- **Motivation** - Recent successful results of the relativistic study of three-nucleon nuclei




| Potential | NR | R |
|---|---|---|
| GRAZ-II $p_D = 4$ % | 8.334 | 8.617 |
| GRAZ-II $p_D = 5$ % | 7.934 | 8.217 |
| GRAZ-II $p_D = 6$ % | 7.548 | 7.831 |

S.Yurev, PhD thesis, (**2022**); S.G. Bondarenko, DSc thesis (**2024**); S.G. Bondarenko , V.V. Burov, S.A. Yurev. Nucl. Phys. A., 1004 (**2020**), 122065 ;1014 (**2021**), 122251

| | nonrelativistic case | relativistic case |
|---|---|---|
| **2** particles case | **Lippmann-Schwinger equation** $$t = V + \int d^3k V g_0 t$$ | **Bethe-Salpeter equation** $$t = V + \int d^4k V G_0 t$$ |
| **3** particles case | **Faddeev equation** $$M^{(i)} = t_i + \int d^3k t_i g_0 [M^{(j)} + M^{(k)}]$$ L. Faddeev, JETP 39 (1960) 1459 | **Relativistic Faddeev equation (Bethe-Salpeter-Faddeev)** $$M^{(i)} = t_i + \int d^4k t_i G_0 [M^{(j)} + M^{(k)}]$$ |
| **4** particles case | **Faddeev-Yakybovsky equation** $$T^{(ijk,l)} = \int d^3 M_{ij,ij} g_0 \left[T^{(jkl,i)} + T^{(ikl,j)}\right] + \int d^3 M_{ij,jk} g_0 \left[T^{(kil,j)} + T^{(jil,k)}\right] + \int d^3 M_{ij,ki} g_0 \left[T^{(ijl,k)} + T^{(kjl,i)}\right]$$ O. A. Yacubovsky, Sov. J. Nucl. Phys. 5 (1967) 1312<br>В. Ф. Харченко. ЭЧАЯ. т. 10, Вып. 4 (1979) [review]<br>V. F. Kharchenko and V. E. Kuzmichev Nucl. Phys. A 183 (1972) 606 | **Relativistic generalization of the Faddeev-Yacubovsky equation** |

# Bethe-Salpeter-Faddeev approach

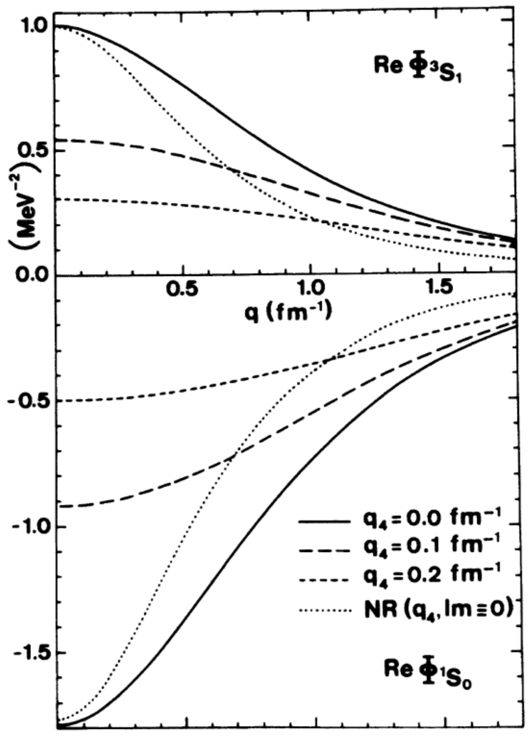


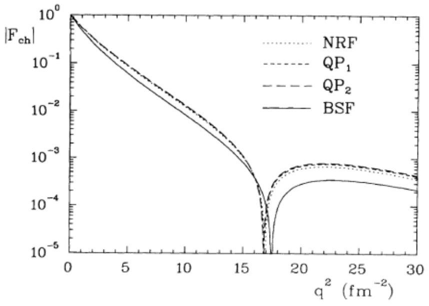
A. Ahmadzadeh, J. A. Tjon, Phys. Rev., v. 147, 4 (**1966**), 1111

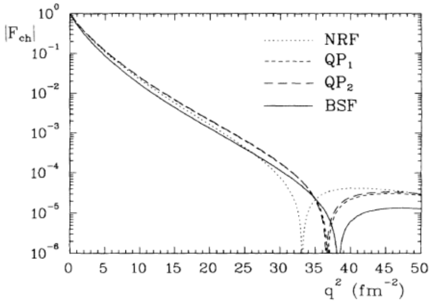
G. Rupp, J. A. Tjon, Phys. Rev. C 45 (**1992**), 2133

| | NRF | $QP_1$ | $QP_2$ | BSF |
|---|---|---|---|---|
| $P_D^d = 4\%$ | 8.369 | 8.539 | 8.606 | 8.701 |
| $P_D^d = 5\%$ | 7.961 | 8.129 | 8.188 | 8.290 |
| $P_D^d = 6\%$ | 7.566 | 7.735 | 7.787 | 7.892 |

| | NRF | $QP_1$ | $QP_2$ | BSF |
|---|---|---|---|---|
| Paris-1 | 7.245 | 7.389 | 7.435 | 7.535 |
| Paris-2 | 7.183 | 7.363 | 7.408 | 7.474 |
| Bonn | 7.822 | 8.083 | 8.140 | 8.201 |




S.A. Yurev, PhD thesis, (**2022**);

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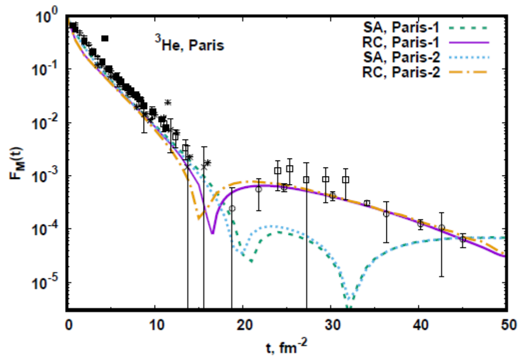
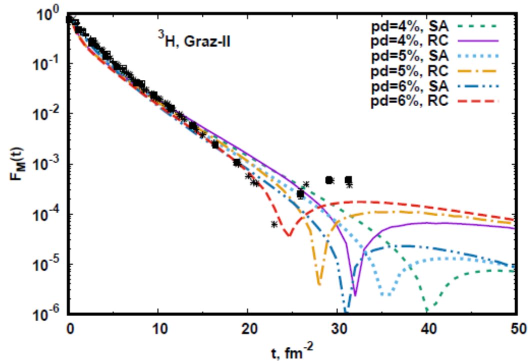
S.G. Bondarenko , V.V. Burov, S.A. Yurev. Nucl. Phys. A., 1004 (**2020**), 122065 ; 1014 (**2021**), 122251

| $p_D$ | $^1S_0 - {}^3S_1$ | $^3D_1$ | $^3P_0$ | $^1P_1$ | $^3P_1$ |
|---|---|---|---|---|---|
| 4 | 9.221 | 9.294 | 9.275 | 9.295 | 9.321 |
| 5 | 8.819 | 8.909 | 8.891 | 8.910 | 8.933 |
| 6 | 8.442 | 8.545 | 8.528 | 8.545 | 8.567 |

| | $^1S_0, {}^3S_1$ | $^1S_0, {}^3S_1, {}^3D_1$ | $^1S_0, {}^3S_1$ | $^1S_0, {}^3S_1, {}^3D_1$ |
|---|---|---|---|---|
| GRAZ-II $p_D = 4$ % | 8.372 | 8.334 | 8.628 | 8.617 |
| GRAZ-II $p_D = 5$ % | 7.964 | 7.934 | 8.223 | 8.217 |
| GRAZ-II $p_D = 6$ % | 7.569 | 7.548 | 7.832 | 7.831 |

| | |
|---|---|
| Paris-1 | 7.535 |
| Paris-2 | 7.474 |



# FY equation in operator (symbolic) form

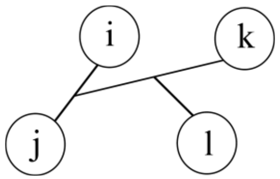
$$T = V + VGT$$

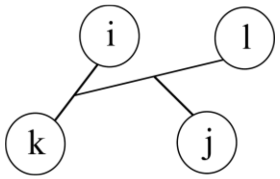
Bethe-Salpeter equation for a four-particle T matrix

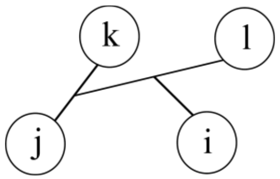
$V = V_{12} + V_{23} + V_{31} + V_{14} + V_{24} + V_{34} +$ ~~$V_{123}$~~ $+ \dots = \sum_{(ij)} V_{ij}$

Only pair interaction

$$T = T^{(ijk,l)} + T^{(ikl,j)} + T^{(kjl,i)} + \;..... + T^{(ij,kl)} + T^{(ik,lj)} + \;.....$$







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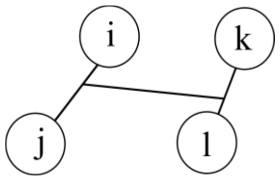
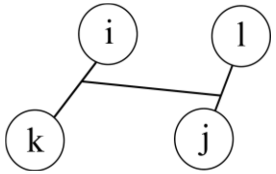




+ .....

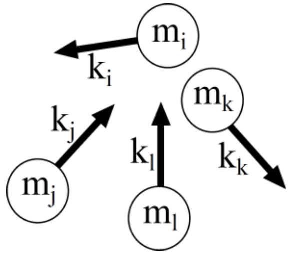
**12** «3+1»

**6** «2+2»

# Jacobi variables for a four-particle system

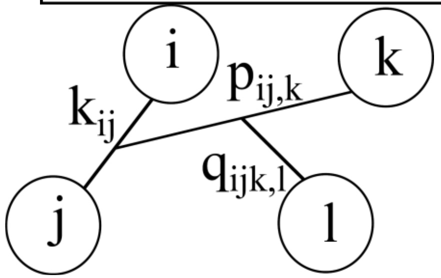


$$k_i \quad k_j \quad k_k \quad k_l$$



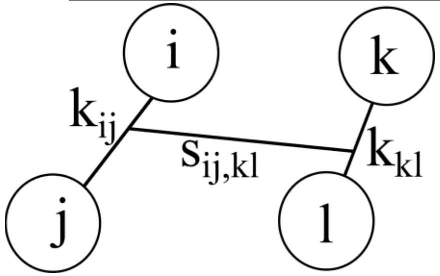

$$if \quad m_i = m_j = m_k = m_l$$

$$k_{ij} \quad p_{ij,k} \quad q_{ijk,l} \quad K$$




$$k_{ij} = \frac{1}{2}(k_i - k_j)$$
$$p_{ij,k} = \frac{1}{3}(k_i + k_j) - \frac{2}{3}k_k$$
$$q_{ijk,l} = \frac{1}{4}(k_i + k_j + k_k) - \frac{3}{4}k_l$$
$$K = k_i + k_j + k_k + k_l$$

$$k_{ij} \quad k_{kl} \quad s_{ij,kl} \quad K$$




$$k_{ij} = \frac{1}{2}(k_i - k_j)$$
$$k_{kl} = \frac{1}{2}(k_k - k_l)$$
$$s_{ij,kl} = \frac{1}{2}(k_i + k_j - k_k - k_l)$$
$$K = k_i + k_j + k_k + k_l$$

$$T^{(ijk,l)}(k_{ij}\ ,\ p_{ij,k}\ ,\ q_{ijk,l}) = T^{(kjl,i)}(k_{kj}\ ,\ p_{kj,l}\ ,\ q_{kjl,i}) = T^{(lji,k)}(k_{lj}\ ,\ p_{lj,i}\ ,\ q_{lji,k}) = .... = \underline{T_M(k\ ,\ p\ ,\ q)}$$

$$T^{(ij,kl)}(k_{ij}\ ,\ k_{kl}\ ,\ s_{ij,kl}) = T^{(lj,ik)}(k_{lj}\ ,\ k_{ik}\ ,\ s_{lj,ik}) = T^{(kj,li)}(k_{kj}\ ,\ k_{li}\ ,\ s_{kj,li}) = .... = \underline{T_N(k\ ,\ \kappa\ ,\ s)}$$

# System integral equation



System integral equation for $T_M$ and $T_N$

$$T_M(k,p,q) = \int \frac{dk'}{(2\pi)^4}\frac{dq'}{(2\pi)^4} S(K/4-q')S(k'+q/2+\frac{1}{2}q'+K/4)$$
$$[M(k,p;\mathcal{K},\mathcal{P})T_M(k',q+\frac{1}{3}q',q') + M(k,p;\mathcal{K}_1,\mathcal{P}_1)T_N(k',q-\frac{1}{2}q',q')]$$
$$T_N(k,\kappa,s) = \int \frac{dk'}{(2\pi)^4}\frac{dq'}{(2\pi)^4} S(K/4-q')S(K/4+q'+s)$$
$$N(k,\kappa;\frac{1}{2}s+q',k')T_M(k',-s-\frac{2}{3}q',q')$$

S -- nucleon propagator

t -- two-particle T matrix

$$\mathcal{K} = -\frac{1}{2}k' - \frac{1}{4}q - \frac{3}{4}q' \qquad \mathcal{K}_1 = \mathcal{K} - \frac{3}{4}q$$
$$\mathcal{P} = k' - \frac{1}{6}q - \frac{1}{2}q' \qquad \mathcal{P}_1 = \mathcal{P} + \frac{1}{2}q$$

$$Z_{qp} = \frac{1}{2}K + \frac{2}{3}q + p$$

$$Z_s = \frac{1}{2}K + s$$

Integral equation for M

$$M(k,p;k',p') = 2\,(2\pi)^4\,t(k,k',Z_{qp})\delta(p-p') +$$
$$\int \frac{d^4p''}{(2\pi)^4} S(\frac{1}{4}K+\frac{1}{3}q-p'')S(\frac{1}{4}K+\frac{1}{3}q+p+p'')$$
$$2t(k,\frac{1}{2}p+p'',Z_{qp})M(p+\frac{1}{2}p'',p'';k',p')$$

Integral equation for N

$$N(k,\kappa;k',\kappa') = (2\pi)^4\,2\,t(k,k',Z_s)[\delta(\kappa-\kappa')+\delta(\kappa+\kappa')] +$$
$$\int \frac{d^4k''}{(2\pi)^4} S(\frac{1}{4}K+\frac{1}{2}s-k'')S(\frac{1}{4}K+\frac{1}{2}s+k'')$$
$$t(k,k'',Z_s)N(\kappa,k'';k',\kappa')$$

Integral equation for two-particle t matrix

$$t = v + \int vGt$$

# Difference between relativistic and non-relativistic cases



- replacement of nonrelativistic Green's functions by scalar nucleon propagators

$$g_0(Z) = (Z - H_0)^{-1} \longrightarrow G_{ij}(Z) = (k_i^2 - m_N^2)^{-1}(k_j^2 - m_N^2)^{-1}$$

- replacement of 3-momenta by 4-momenta

$$t(\boldsymbol{k_1}; \boldsymbol{k_2}) \longrightarrow t(k_1^0, \boldsymbol{k_1}; k_2^0, \boldsymbol{k_2})$$
$$v(\boldsymbol{k_1}; \boldsymbol{k_2}) \longrightarrow v(k_1^0, \boldsymbol{k_1}; k_2^0, \boldsymbol{k_2})$$
$$M(\boldsymbol{k_1}; \boldsymbol{k_2}; \boldsymbol{k_3}) \longrightarrow M(k_1^0, \boldsymbol{k_1}; k_2^0, \boldsymbol{k_2}; k_3^0, \boldsymbol{k_3})$$

- replacement of 3-momentum integration by 4-momentum integration

$$\int d^3k \longrightarrow \int d^4k$$

# Separable potential of NN interaction



$$v(k,k') = \lambda g(k) g(k')$$

$$g(k) = 1/(k^2 - \beta^2 + i0)$$ monopole

$$g(k) = 1/(k^2 - \beta^2 + i0)^2$$ dipole

| | Parameter | Y | Y2 |
|---|---|---|---|
| $^1S_0$ | $\beta$ (GeV) | 0.228302 | 0.336 |
| | $\lambda$ (GeV$^4$) | -1.12087 | -0.071436[a] |
| $^3S_1$ | $\beta$ (GeV) | 0.279731 | 0.4 |
| | $\lambda$ (GeV$^4$) | -3.1548 | -0.3857451[a] |

[a] GeV$^8$

# Separable two-particle matrix

$$t = v + \int vGt \longrightarrow t(k,k',s) = \tau(s) g(k) g(k')$$

$$\tau(s) = [\frac{1}{\lambda} - \frac{i}{4\pi^3}\int_{-\infty}^{\infty} dk_0 \int_0^{\infty} k^2 dk g^2(k_0,k) G(k_0,k;s)]^{-1}$$

# Equation with the separable potential



$$T_M(k,p,q) = g(k)Q(p,q)$$
$$T_N(k,\kappa,s) = g(k)R(\kappa,s)$$

$$M(k,p;k',p') = 2\tau(Z_{qp})g(k)g(k')[(2\pi)^4\delta(p-p') + \tau(Z_{qp'})X(p,p')]$$
$$N(k,\kappa;k',\kappa') = 2\tau(Z_s)g(k)g(k')[(2\pi)^4[\delta(\kappa-\kappa')+\delta(\kappa+\kappa')] + \tau(Z_s)Y(\kappa,\kappa')]$$

$$Q(p,q) = \tau(Z_{qp})$$
$$\int \frac{dq'}{(2\pi)^4}[X(p,\frac{1}{3}q+q')Q(q+\frac{1}{3}q',q') + X(p,-\frac{2}{3}q+q')R(q-\frac{1}{2}q',q')]$$
$$R(\kappa,s) = 2\tau(Z_s)$$
$$\int \frac{dq'}{(2\pi)^4}Y(\kappa,\frac{1}{2}s+q')Q(-s-\frac{2}{3}q',q')$$

System of integral equations for Q and R

$$X(p,p') = U(p,p') + \int \frac{d^4p''}{(2\pi)^4}U(p,p'')\tau(Z_{qp''})X(p'',p')$$
$$Y(\kappa,\kappa') = W(\kappa,\kappa') + \int \frac{d^4\kappa''}{(2\pi)^4}W(\kappa,\kappa'')\tau(Z_s)Y(\kappa'',\kappa')$$

Integral equations for X and Y

$$U(p,p') = S(\frac{1}{4}K+\frac{1}{3}q-p')S(\frac{1}{4}K+\frac{1}{3}q+p+p')g(\frac{1}{2}p+p')2g(p+\frac{1}{2}p')$$
$$W(\kappa,\kappa') = S(\frac{1}{4}K+\frac{1}{2}s-\kappa')S(\frac{1}{4}K+\frac{1}{2}s+\kappa')g(\kappa)g(\kappa')$$

# Partial states



$$Q_i(p,q) = \tau_i(Z_{qp})$$

$$\sum_j \int \frac{dq'}{(2\pi)^4}[X_{ij}(p,\frac{1}{3}q+q')Q_j(q+\frac{1}{3}q',q') + X_{ij}(p,-\frac{2}{3}q+q')R_j(q-\frac{1}{2}q',q')]$$

$$R_i(\kappa,s) = 2\tau_i(Z_s)$$

$$\int \frac{dq'}{(2\pi)^4} Y_{ii}(\kappa,\frac{1}{2}s+q')Q_i(-s-\frac{2}{3}q',q')$$

$$i,j = {}^1S_0, {}^3S_1$$

$$X_{ij}(p,p') = U_{ij}(p,p') + \sum_k \int \frac{d^4p''}{(2\pi)^4} U_{ik}(p,p'')\tau_k(Z_{qp''})X_{kj}(p'',p')$$

$$Y_{ii}(\kappa,\kappa') = W_{ii}(\kappa,\kappa') + \int \frac{d^4\kappa''}{(2\pi)^4} W_{ii}(\kappa,\kappa'')\tau_i(Z_s)Y_{ii}(\kappa'',\kappa')$$

$$U_{ij}(p,p') = C_{ij}S(\frac{1}{4}K+\frac{1}{3}q-p')S(\frac{1}{4}K+\frac{1}{3}q+p+p')g_i(\frac{1}{2}p+p')2g_j(p+\frac{1}{2}p')$$

$$W_{ii}(\kappa,\kappa') = S(\frac{1}{4}K+\frac{1}{2}s-\kappa')S(\frac{1}{4}K+\frac{1}{2}s+\kappa')g_i(\kappa)g_i(\kappa')$$

$$C_{ij} = \begin{pmatrix} \frac{1}{4} & -\frac{3}{4} \\ -\frac{3}{4} & \frac{1}{4} \end{pmatrix}$$

Spin-isospin recoupling coefficient

# Singularities. Wick rotation

Poles from propagators S

$$p_0''^{1,2} = \frac{1}{4}K_0 + \frac{1}{3}q_0 \pm [E(|\frac{1}{3}\boldsymbol{q} - \boldsymbol{p}''|) - i\epsilon]$$

$$p_0''^{3,4} = -\frac{1}{4}K_0 - \frac{1}{3}q_0 - p_0 \pm [E(|\frac{1}{3}\boldsymbol{q} + \boldsymbol{p} + \boldsymbol{p}''|) - i\epsilon]$$

Poles from potential (g)

$$p_0''^{5,6} = -2p_0 \pm 2[E_\beta(|\boldsymbol{p} + \frac{1}{2}\boldsymbol{p}''|) - i\epsilon]$$

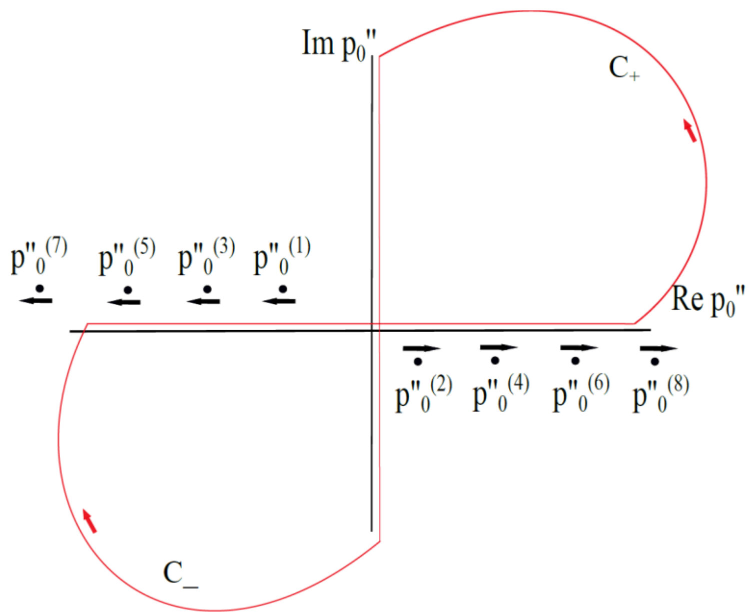
$$p_0''^{7,8} = -\frac{1}{2}p_0 \pm \frac{1}{2}[E_\beta(|\frac{1}{2}\boldsymbol{p} + \boldsymbol{p}''|) - i\epsilon]$$

Poles from tau

$$p_0''^{9,10} = -\frac{1}{2}K_0 - \frac{2}{3}q_0 - p_0 \pm [|\frac{2}{3}\boldsymbol{q} + \boldsymbol{p}''| + M_d^2 - i\epsilon]^{\frac{1}{2}}$$

Cuts from tau

$$p_0''^{11,12} = -\frac{1}{2}K_0 - \frac{2}{3}q_0 - p_0 \pm [|\frac{2}{3}\boldsymbol{q} + \boldsymbol{p}''| + 4m^2 - i\epsilon]^{\frac{1}{2}}$$




$$p_0'' \longrightarrow ip_4''$$

$$p_0 \longrightarrow ip_4$$

$$q_0 \longrightarrow iq_4$$

# Method of solution



$$Q(p,q) = \int \frac{dq'}{(2\pi)^4} \tau(K_{qp}) X(p, q/3 + q', K_q) Q(q + \tfrac{1}{3}q', q')$$

$$X(p, p''; K_q) = U(p, p''; K_q) + \int dp' U(p, p'; K_q) \tau(K_{qp'}) X(p', p''; K_q)$$

## Iteration method

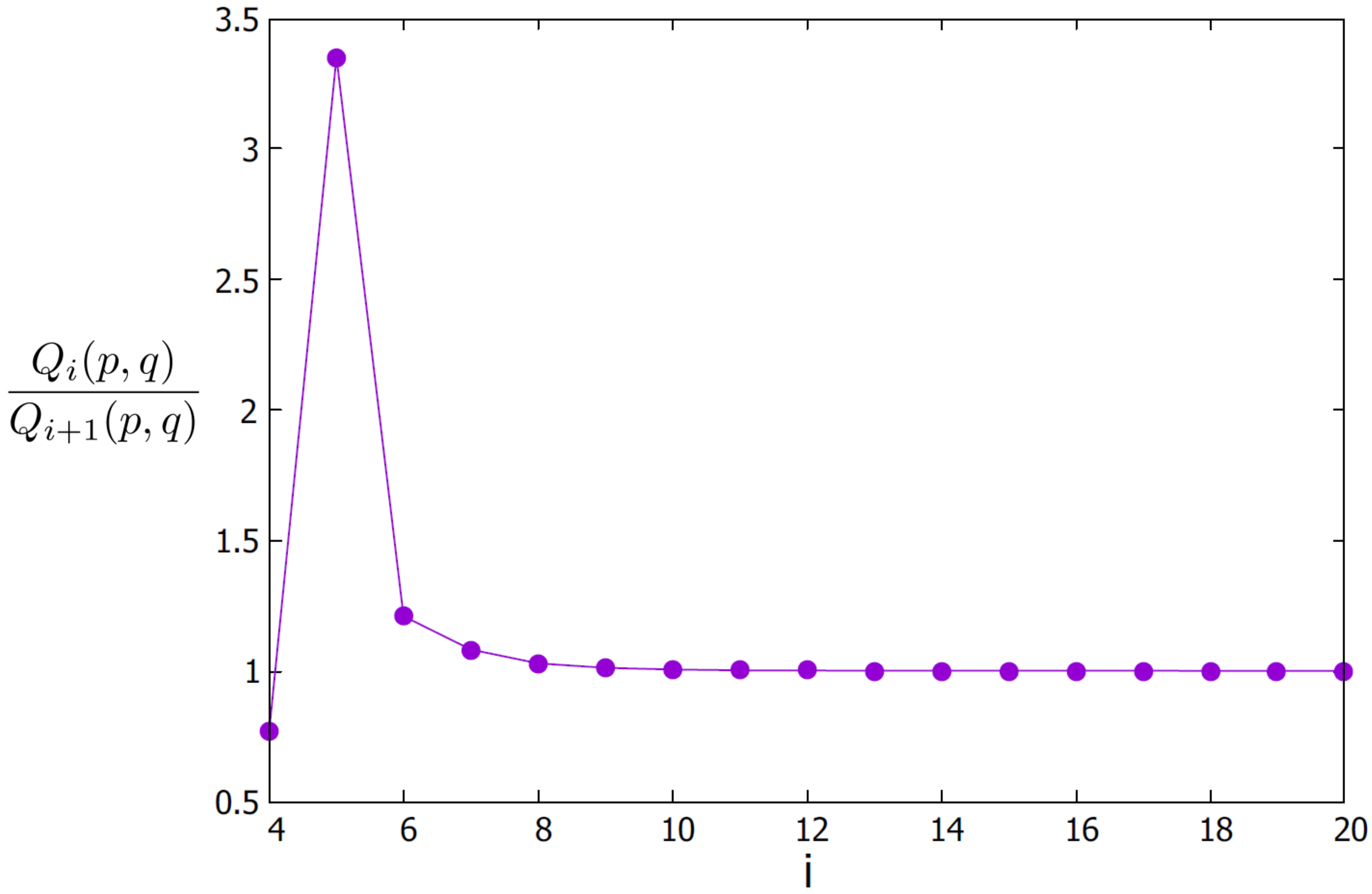
Homogeneous integral equation with parameter
in our case parameter - bound state energy

solvability condition

$$\lim_{i \to \infty} \frac{Q_i(p,q)}{Q_{i+1}(p,q)} = 1$$

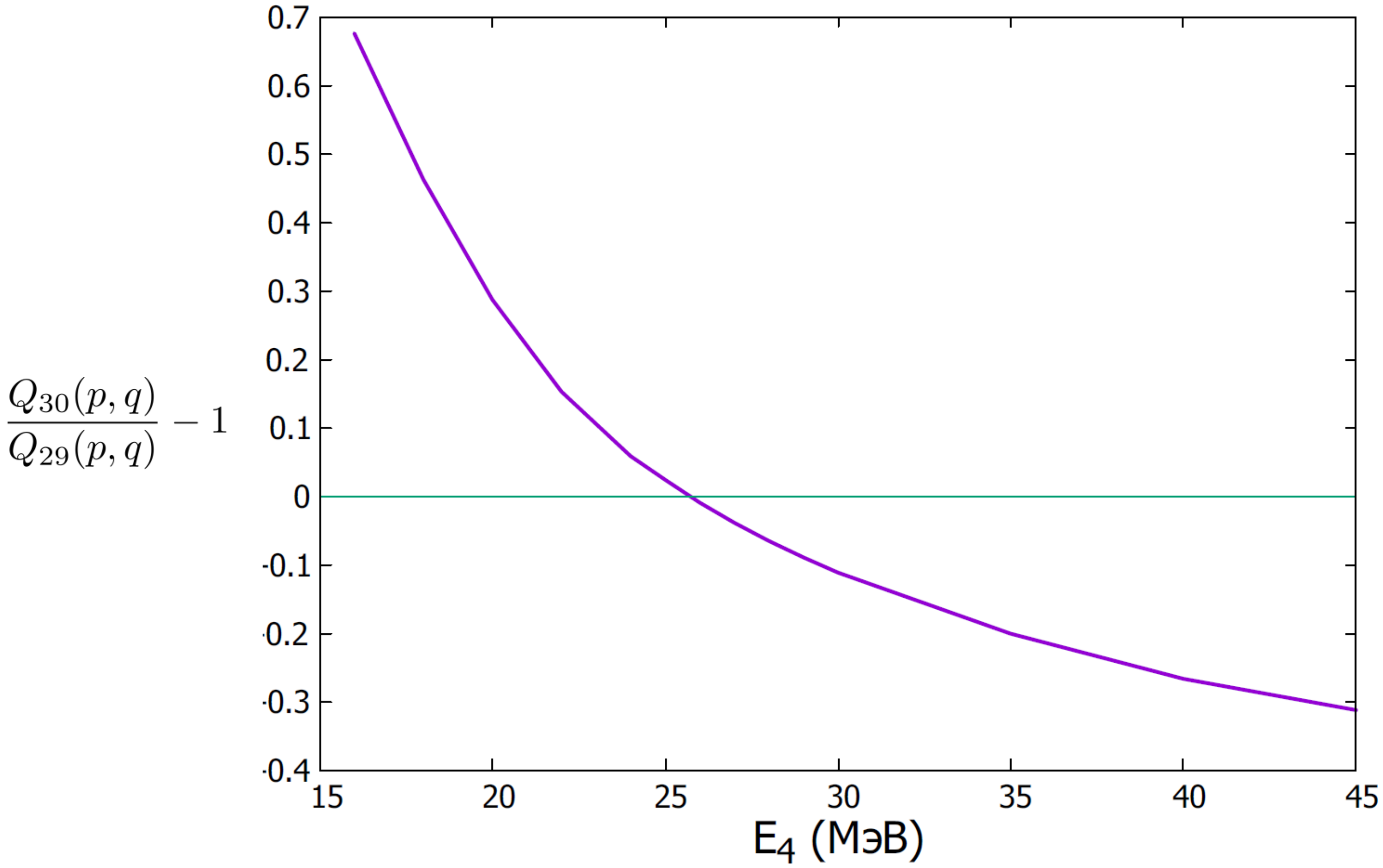
# Convergence of iteration method






Convergence of the ratio of two subsequent iterations with an increase in the iteration number (i)

# Search of bound state energy



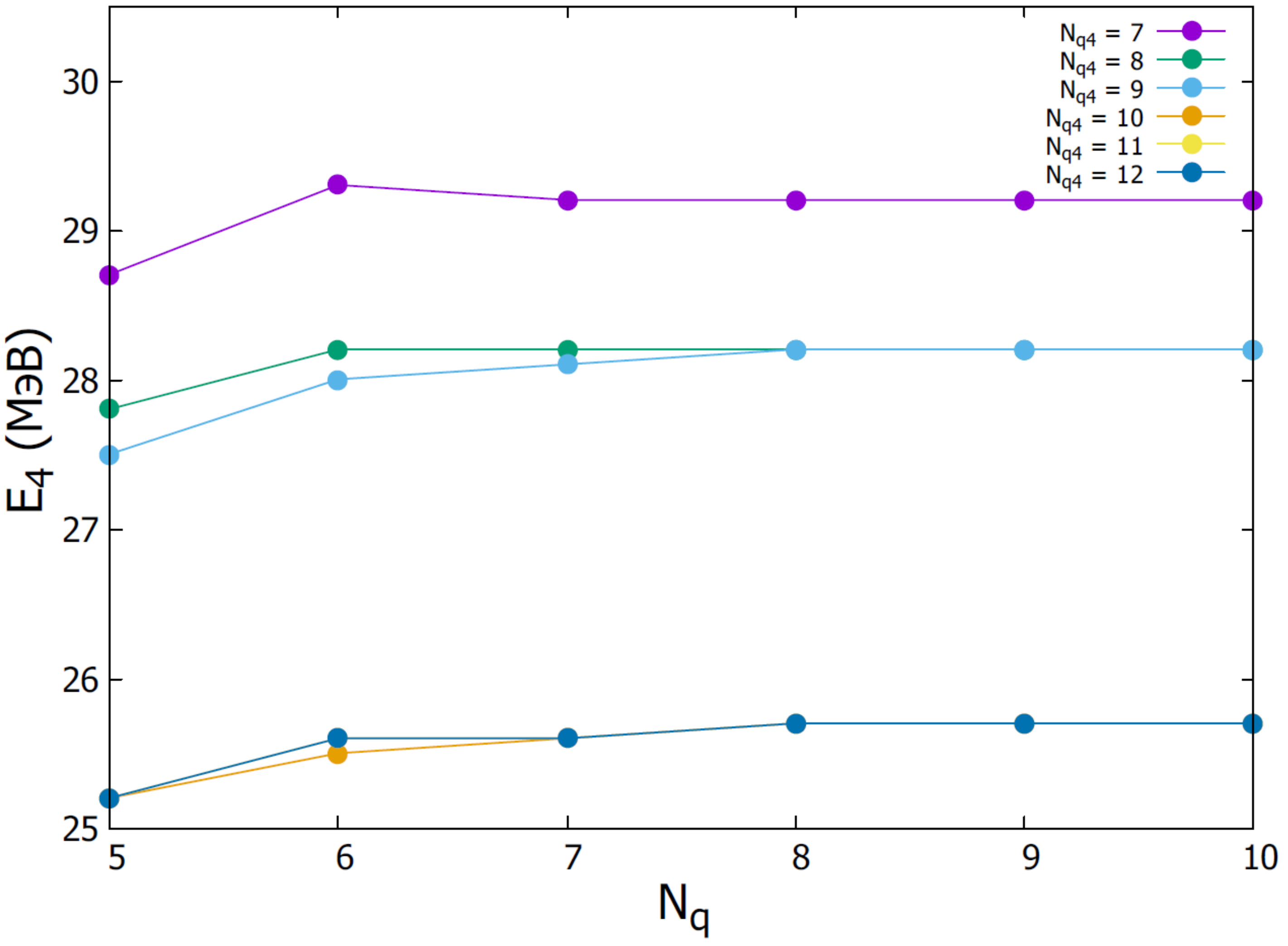

Ratio of two subsequent iterations as a function of parameter value( bound state energy in our case)

Parameter value(bound state energy) at which:

$$\lim_{i\to\infty} \frac{Q_i(p,q)}{Q_{i+1}(p,q)} = 1$$

gives the true binding energy of the nucleus

# q ($q_4$) convergence




$$\int_a^b f(q)dq \rightarrow \sum_{i=1}^{N} f(q_i)a_i$$

$q_i$,$a_i$ - nodes and weights of Gaussian quadrature

# Result of calculation

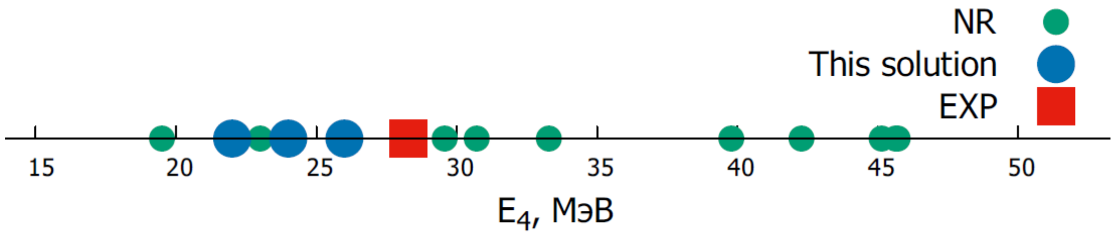
NN potential **Y**

| State | Non-relativistic calculation | Relativistic calculation |
|---|---|---|
| $^3S_1$ w/o "2+2" | 47 | 58 |
| $^1S_0$, $^3S_1$ w/o "2+2" | 19 | 26 |
| $^3S_1$ with "2+2" | 75 | 51 |
| $^1S_0$, $^3S_1$ with "2+2" | 34 | 24 |

NN potential **Y2**

| State | Relativistic calculation |
|---|---|
| $^3S_1$ w/o "2+2" | 51 |
| $^1S_0$, $^3S_1$ w/o "2+2" | 24 |
| $^3S_1$ with "2+2" | 45 |
| $^1S_0$, $^3S_1$ with "2+2" | 22 |

Experiment:
28.3 MeV




Triton binding energy
(Exp = 8.48 MeV)

| State | Y | Y2 |
|---|---|---|
| $^3S_1$ | 25.26 | 22.99 |
| $^1S_0$, $^3S_1$ | 11.04 | 10.24 |

# Summary

- The Faddeev-Yakubovsky equation was generalized to the relativistic case;
- The equation was solved numerically by the iteration method;
- The binding energy value of helium 4 has been obtained.

## Outlook:

- Using the found amplitudes of the state, calculate the form factors of the helium-4 nucleus;
- Calculation using more accurate multi-rank potentials.

# Thank you for your attention