Heavy tetraquarks in the hyperspherical approach

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Experimental review

Discovering of tetraquark candidates

- The first candidate tetraquark (with mass 3872 MeV) discovering by Belle:

  S. K. Choi, S. L. Olsen, K. Abe, T. Abe, et al. (Belle Collaboration), Phys. Rev. Lett. 91, 262001 (2003)

- Observation of $Z_{cs}(3985)$ in $e^+e^- \to K^+(D_s^- D^{*0} + D_s^{*-} D^0)$ at BESIII:

  M. Ablikim, et al. (BESIII Collaboration), Phys. Rev. Lett. 126, 102001 (2021)

- Observation of $(cc\bar{c}\bar{c})$ candidate $X(6900)$ in proton - proton collision at LHCb:

  R. Aaij et al. [the LHCb Collaboration], Sci. Bull., 65, 1983, (2020)

- Observation of $X(6900)$ in the $J/\Psi\, J/\Psi$, $J/\Psi\, \Psi(2S)$ mass spectrum at ATLAS detector:

  G. Aad et al. [the ATLAS Collaboration], Phys. Rev. Lett., 131, 151902, (2023)

- Observation of $X(6600)$, $X(6900)$, $X(7200)$ in the $J/\Psi\, J/\Psi$ mass spectrum at CMS:

  A. Hayrapetyan et al. [the CMS Collaboration], Phys. Rev. Lett., 132, 111901, (2024)

  The CMS Collaboration, CMS PAS BPH-24-003

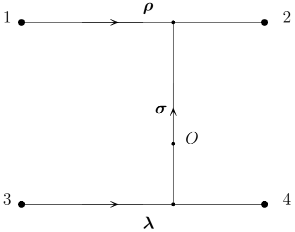
Theoretical review

Hyperspherical approach in the study of multiparticle systems:

- Study of baryons and pentaquarks
  - A. P. Martynenko, Phys. Lett. B 663, 317 (2008).
  - I. M. Narodetskii, M. A. Trusov, Phys. Atom. Nucl. 67, 762 (2004).
  - I. M. Narodetskii, Yu. A. Simonov, M. A. Trusov, A. I. Veselov, Phys. Lett. B 578, 318 (2004).
  - K. Thakkar, et al., Pramana 77. 6, 1053-1067 (2011)
  - A. M. Badalian, Yu. A. Simonov, Sov. J. Nucl. Phys. 3, 755 (1966).
- Study of three - body exotic atoms:
  - M.A. Khan, Eur. Phys. J. D 66, 83 (2012)
  - M. A. Khan, M. Hasan, Phys. Scripta 98, 7, 075407 (2023)
  - Fatehizadeh H., Gheisari R., Falinejad H. Annals of Physics. 385, 512-521 (2017)
- $(cc\bar{c}\bar{c})$, $(cs\bar{c}\bar{s})$, $(ss\bar{s}\bar{s})$ P - wave 1^{--} state:
  - A. M. Badalyan, B. L. Ioffe, A. V. Smilga, Phys. B 281, 85 (1987)

Four particle bound state in Jacobi coordinates

This work represents calculation of heavy tetraquark $(cc\bar{c}\bar{c})$, $(bb\bar{b}\bar{b})$, $(bb\bar{c}\bar{c})$, $(cc\bar{b}\bar{b})$ mass spectrum within the hyperspherical approach.



Jacobi coordinates ρ, λ, σ

$$\mathbf{r}_1 = -\frac{m_2}{m_{12}}\boldsymbol{\rho} + \frac{m_{34}}{m_{1234}}\boldsymbol{\sigma}, \quad \mathbf{r}_2 = \frac{m_1}{m_{12}}\boldsymbol{\rho} + \frac{m_{34}}{m_{1234}}\boldsymbol{\sigma},$$

$$\mathbf{r}_3 = -\frac{m_4}{m_{34}}\boldsymbol{\lambda} - \frac{m_{12}}{m_{1234}}\boldsymbol{\sigma}, \quad \mathbf{r}_4 = \frac{m_3}{m_{34}}\boldsymbol{\lambda} - \frac{m_{12}}{m_{1234}}\boldsymbol{\sigma}.$$

Particles $1-2$ are quarks c, b, particles $3-4$ are antiquarks $\bar{c}, \bar{b}$. $m_c = 1.55\ GeV$, $m_b = 4.88\ GeV$

By changing the numbering of the particles, we can obtain different sets of coordinates $(\boldsymbol{\rho}_{ij},\ \boldsymbol{\lambda}_{kl},\ \boldsymbol{\sigma}_{(ij)(kl)})$. All ways of introducing of Jacobi coordinates are equivalent

$$d\boldsymbol{\rho}_{12} d\boldsymbol{\lambda}_{34} d\boldsymbol{\sigma}_{(12)(34)} = d\boldsymbol{\rho}_{13} d\boldsymbol{\lambda}_{24} d\boldsymbol{\sigma}_{(13)(24)} = d\boldsymbol{\rho}_{14} d\boldsymbol{\lambda}_{23} d\boldsymbol{\sigma}_{(14)(23)} = \dots$$

$$\boldsymbol{\rho}_{12}^2 + \boldsymbol{\lambda}_{34}^2 + \boldsymbol{\sigma}_{(12)(34)}^2 = \boldsymbol{\rho}_{13}^2 + \boldsymbol{\lambda}_{24}^2 + \boldsymbol{\sigma}_{(13)(24)}^2 = \boldsymbol{\rho}_{14}^2 + \boldsymbol{\lambda}_{23}^2 + \boldsymbol{\sigma}_{(14)(23)}^2 = \dots$$

Theoretical framework

The aim of the work is to find tetraquark wave function, binding energy and mass by solving the Schrödinger equation

$$\hat{H}|\Psi\rangle = E|\Psi\rangle$$

Four particle bound state Hamiltonian

$$\hat{H} = \hat{T} + \hat{U},$$

In terms of basis $\boldsymbol{\rho} = \boldsymbol{\rho}_{12}$, $\boldsymbol{\lambda} = \boldsymbol{\lambda}_{34}$, $\boldsymbol{\sigma} = \boldsymbol{\sigma}_{(12)(34)}$ (here and further):

$$\hat{T} = \sum_{i=1}^{4} \frac{\boldsymbol{p}_i^2}{2m_i} = -\frac{\boldsymbol{\nabla}_\rho^2}{2\mu_{12}} - \frac{\boldsymbol{\nabla}_\lambda^2}{2\mu_{34}} - \frac{\boldsymbol{\nabla}_\sigma^2}{2\mu_{(12)(34)}}.$$

where $\mu_{ij} = \frac{m_i m_j}{m_i + m_j}$, i, j indicate particles $1 - 4$. Introducing of auxiliary parameter μ ($[\mu] = GeV$), the dependence on which disappears in the final expression for the binding energy, allows to rewrite the operator of kinetic energy in the form

$$\boldsymbol{\rho} \to \sqrt{\frac{\mu}{\mu_{12}}}\boldsymbol{\rho}, \quad \boldsymbol{\lambda} \to \sqrt{\frac{\mu}{\mu_{34}}}\boldsymbol{\lambda}, \quad \boldsymbol{\sigma} \to \sqrt{\frac{\mu}{\mu_{(12)(34)}}}\boldsymbol{\sigma},$$

$$\hat{T} = -\frac{1}{2\mu}\left(\boldsymbol{\nabla}_\rho^2 + \boldsymbol{\nabla}_\lambda^2 + \boldsymbol{\nabla}_\sigma^2\right)$$

Theoretical framework

Potential energy of four particle bound state includes nonrelativistic terms, hyperfine structure potential and corrections:

$$\hat{U} = \hat{U}_{coul} + \hat{U}_{conf} + \hat{U}_{hfs} + \underbrace{\hat{U}_{rel} + \hat{U}_{rel-rec} + \hat{U}_{cont} + \hat{U}_{conf,hfs}}_{corrections}.$$

The Coulomb potential is determined by the sum of pairwise Coulomb interactions

$$\hat{U}_{coul} = \underbrace{-\frac{2}{3}\left(\frac{\alpha_s}{|\boldsymbol{r}_2-\boldsymbol{r}_1|}+\frac{\alpha_s}{|\boldsymbol{r}_4-\boldsymbol{r}_3|}\right)}_{qq\ \ interaction}\underbrace{-\frac{4}{3}\left(\frac{\alpha_s}{|\boldsymbol{r}_3-\boldsymbol{r}_1|}+\frac{\alpha_s}{|\boldsymbol{r}_4-\boldsymbol{r}_1|}+\frac{\alpha_s}{|\boldsymbol{r}_3-\boldsymbol{r}_2|}+\frac{\alpha_s}{|\boldsymbol{r}_4-\boldsymbol{r}_2|}\right)}_{q\bar{q}\ \ interaction}$$

Strong coupling constant is taken to be $\alpha_{s,cc} = 0.314$, $\alpha_{s,bb} = 0.207$, $\alpha_{s,cb} = 0.264$ as for the pair interaction of quarks and antiquarks in mesons.

K.G. Chetyrkin, B.A. Kniehl, M. Steinhauser, arXiv:hep-ph/9706430v1 20 Jun 1997

Confinement potential as the sum of pairwise confinement potentials for the quark-quark, quark-antiquark subsystems:

$$\hat{U}_{conf} = \frac{A}{2}\left(|\boldsymbol{r}_2-\boldsymbol{r}_1|+|\boldsymbol{r}_4-\boldsymbol{r}_3|\right) + A\left(|\boldsymbol{r}_3-\boldsymbol{r}_1|+|\boldsymbol{r}_4-\boldsymbol{r}_1|+|\boldsymbol{r}_3-\boldsymbol{r}_2|+|\boldsymbol{r}_4-\boldsymbol{r}_2|\right) + B$$

Constants in confinement potential are: $A = 0.18\ GeV^2$, $B = -0.8\ GeV$

Hyperspherical coordinates

Transition to hyperspace $\{\boldsymbol{\rho},\ \boldsymbol{\lambda},\ \boldsymbol{\sigma}\} \to \{R,\ \theta,\ \phi,\ \theta_1,\ \theta_2,\ \theta_3,\ \phi_1,\ ,\phi_2,\ \phi_3\}$

$$\rho = R\sin\theta\cos\phi,\ \ \lambda = R\sin\theta\sin\phi,\ \ \sigma = R\cos\theta,\ \ \theta,\phi\in(0,\pi/2),\ \ R=\sqrt{\rho^2+\lambda^2+\sigma^2}.$$

$$\begin{aligned}
\rho_x &= \rho\sin\theta_1\cos\phi_1. & \lambda_x &= \lambda\sin\theta_2\cos\phi_2. & \sigma_x &= \sigma\sin\theta_3\cos\phi_3.\\
\rho_y &= \rho\sin\theta_1\sin\phi_1. & \lambda_y &= \lambda\sin\theta_2\sin\phi_2. & \sigma_y &= \sigma\sin\theta_3\sin\phi_3.\\
\rho_z &= \rho\cos\theta_1. & \lambda_z &= \lambda\cos\theta_2. & \sigma_z &= \sigma\cos\theta_3.
\end{aligned}$$

$$\theta_i\in(0,\pi),\quad \phi_i\in(0,2\pi).$$

The volume element in hyperspace of dimension nine has the form

$$d\boldsymbol{\rho}d\boldsymbol{\lambda}d\boldsymbol{\sigma} = \mathcal{J}_9 dRd\theta d\phi d\theta_1 d\phi_1 d\theta_2 d\phi_2 d\theta_3 d\phi_3,$$
$$\mathcal{J}_9 = R^8\sin^5\theta\cos^2\theta\sin^2\phi\cos^2\phi\sin\theta_1\sin\theta_2\sin\theta_3.$$

The volume of 8-dimensional sphere is

$$\Omega_8 = (4\pi)^3\int_0^{\pi/2}\sin^2\phi\cos^2\phi d\phi\int_0^{\pi/2}\sin^5\theta\cos^2\theta d\theta = \frac{32\pi^4}{105}.$$

Hyperradial approximation

Any coordinate wave function can be expanded in the complete set of hyperspherical functions $Y_{[K]}(\Omega)$

$$\Psi(\boldsymbol{\rho},\boldsymbol{\lambda},\boldsymbol{\sigma}) = \sum_K \Psi_{[K]}(R)\cdot Y_{[K]}(\Omega).$$

Operator of kinetic energy in hyperspherical coordinates:

$$\hat{T} = -\frac{1}{2\mu}\left(\frac{\partial^2}{\partial R^2} + \frac{8}{R}\frac{\partial}{\partial R} + \frac{\mathcal{K}^2(\Omega)}{R^2}\right),$$

where operator of 9 - dimensional angular momentum

$$\mathcal{K}^2(\Omega) Y_{[K]}(\Omega) = K(K+7) Y_{[K]}(\Omega).$$

In the hyperradial approximation (for S - energy states) the wave function depends only on hyperradius R.

$$\Psi(\boldsymbol{\rho},\boldsymbol{\lambda},\boldsymbol{\sigma}) = \Psi(R), \quad \int d\Omega \; Y_{[0]}(\Omega) Y^*_{[0]}(\Omega) = 1, \quad K = 0.$$

$$\hat{T}(R) = -\frac{1}{2\mu}\left(\frac{\partial^2}{\partial R^2} + \frac{8}{R}\frac{\partial}{\partial R}\right), \quad \hat{U}(R) = \langle \hat{U} \rangle_\Omega$$

Schrödinger equation in hyperradial approximation

In the nonrelativistic approximation potential energy is sum of Coulomb potential and confinement potential. After averaging over angle variables

$$\hat{U}_{coul}(R) = \langle \hat{U}_{coul} \rangle_\Omega = \frac{a}{\sqrt{\mu}\, R}, \quad \hat{U}_{conf}(R) = \langle \hat{U}_{conf} \rangle_\Omega = -b\,\sqrt{\mu}\, R.$$

Averaging over angle variables can be accomplished using integration of the form:

$$\langle \frac{1}{\rho_{ij}} \rangle = \frac{(4\pi)^3}{R\,\Omega_8} \int_0^{\pi/2} d\theta \int_0^{\pi/2} d\phi \sin^2\phi \cos\phi \sin^4\theta \cos^2\theta = \frac{35}{16}\frac{1}{R}\sqrt{\frac{\mu_{ij}}{\mu}} = \langle \frac{1}{\lambda_{ij}} \rangle$$

$$\langle \rho_{ij} \rangle = \frac{R(4\pi)^3}{\Omega_8} \int_0^{\pi/2} d\theta \int_0^{\pi/2} d\phi \sin^2\phi \cos^3\phi \sin^6\theta \cos^2\theta = \frac{35}{64} R \sqrt{\frac{\mu}{\mu_{ij}}} = \langle \lambda_{ij} \rangle.$$

Parameters of Coulomb and confinement terms after averaging are

$$a = \frac{35\,\alpha_s}{24} \left[\sqrt{\mu_{12}} + \sqrt{\mu_{34}} + 2\left(\sqrt{\mu_{13}} + \sqrt{\mu_{14}} + \sqrt{\mu_{23}} + \sqrt{\mu_{24}}\right)\right],$$

$$b = \frac{35\,A}{128} \left[\frac{1}{\sqrt{\mu_{12}}} + \frac{1}{\sqrt{\mu_{34}}} + 2\left(\frac{1}{\sqrt{\mu_{13}}} + \frac{1}{\sqrt{\mu_{14}}} + \frac{1}{\sqrt{\mu_{23}}} + \frac{1}{\sqrt{\mu_{24}}}\right)\right].$$

Schrödinger equation in hyperradial approximation

It is convenient to introduce function $\chi(R) = R^4\,\Psi(R)$
and new variable $x = \sqrt{\mu}R$ ($[x] = GeV^{-1/2}$)

Finally the Schrödinger equation for function $\chi(x)$ takes the form of one-dimensional equation:

$$\frac{d^2\chi(x)}{dx^2} + 2\left[E - B + \frac{a}{x} - \frac{6}{x^2} - b\,x\right]\chi(x) = 0$$

To solve such an equation, a variational approach is usually used

The trial wave function is usually chosen in the following general form

$$\Psi(x) = N\,e^{-p\,x^q}$$

where N is normalization factor, p is variational parameter. The degree of the variable in the exponent can be chosen for different reasons.

Hydrogen atom function form

$$\chi(x) = N\,x^4\,e^{-p\,x}$$

Gaussian form

$$\chi(x) = N\,x^4\,e^{-p\,x^2}$$

Asymptotic $x \to \infty$ leads to well known Airy equation, the solution of which gives

$$\chi(x) = N\,x^4\,e^{-p\,x^{3/2}}$$

Solving the Schrödinger equation

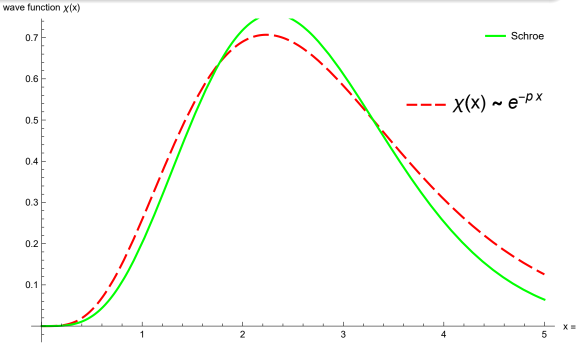
Taking the wave function in the form of hydrogen atom wave function, the result is:

$$\chi(x) = \frac{2\,p^{9/2}}{3\sqrt{35}}\,x^4\,e^{-p\,x}, \quad E_0 = B + \frac{9\,b}{2\,p} + \frac{p\,(2p-a)}{4}$$

The binding energy and variation parameter are found by calculating the minimum of the function $E_0(p)$:

$$E_0 = 0.404 \text{ GeV}$$

$$p_0 = 1.79 \text{ GeV}^{1/2}$$



For comparison and analysis we use the result of numerical solution of the Schrödinger equation in "Schroe" program in Wolfram Mathematica

$$E_{0,\ Schroe} = 0.380 \text{ GeV}$$

Solving the Schrödinger equation

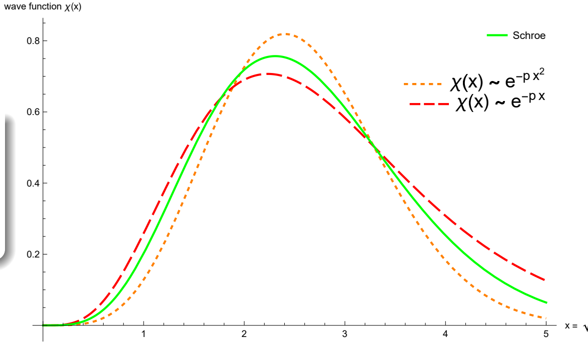
In an attempt to obtain a better agreement with the numerical solution, we also consider other variants of trial wave functions.

$$\chi(x) = \frac{16\, 2^{3/4}\, p^{9/4}}{\sqrt{105\sqrt{\pi}}}\, x^4\, e^{-p\,x^2}, \quad E_0 = B + \frac{9\,p}{2} + \frac{64\, b\sqrt{2}}{35\sqrt{p\pi}} - \frac{32\, a\sqrt{2p}}{35\sqrt{\pi}}$$

Trial wave function in Gaussian form

$$E_0 = 0.414 \text{ GeV}$$

$$p_0 = 0.346 \text{ GeV}$$



Solving the Schrödinger equation

In an attempt to obtain a better agreement with the numerical solution, we also consider other variants of trial wave functions.

Trial wave function in Gaussian form

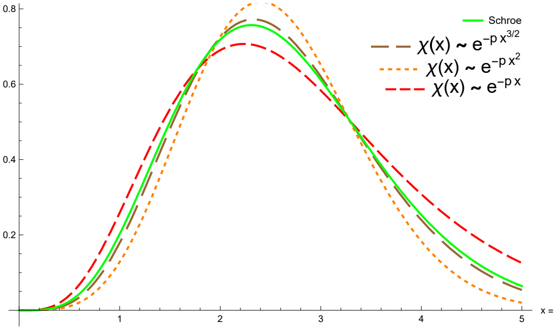
$$E_0 = 0.414 \text{ GeV}$$

$$p_0 = 0.346 \text{ GeV}$$

Trial wave function $\chi(x) \sim e^{-p\,x^{3/2}}$

$$E_0 = 0.382 \text{ GeV}$$

$$p_0 = 0.75 \text{ GeV}^{3/4}$$



$$\chi(x) = \frac{2\,p^3}{\sqrt{5}}\,x^4\,e^{-p\,x^{3/2}}, \quad E_0 = B + \frac{119\,p^2\,\Gamma\left[\frac{14}{3}\right] - 2^{10/3}\,a\,p^{4/3}\Gamma\left[\frac{16}{3}\right] + 4\,b\,\Gamma\left[\frac{20}{3}\right]}{480\,2^{2/3}\,p^{2/3}}$$

Tetraquark ($cc\bar{c}\bar{c}$) wave function and binding energy

The best agreement with the numerical solution, as well as the closest binding energy value to the variational approach, is achieved by introducing an additional parameter into the trial wave function.

$$\chi(x) = N\, x^4\, e^{-p^2 x^q}$$

$$N = \sqrt{\frac{q 2^{\frac{9}{q}} p^{\frac{18}{q}}}{\Gamma\left[\frac{9}{q}\right]}}$$

Solving the Schrödinger equation gives for binding energy:

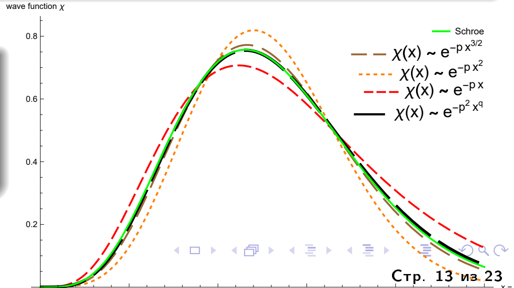
$$E_0 = \frac{-a\, 2^{\frac{1}{q}} p^{2/q} \Gamma\left[\frac{8}{q}\right] + b\, 2^{-1/q} p^{-2/q} \Gamma\left[\frac{10}{q}\right] + B\, \Gamma\left[\frac{9}{q}\right] + 7\; 2^{\frac{2}{q}-3} (q+7) p^{4/q} \Gamma\left[\frac{7}{q}\right]}{\Gamma\left[\frac{9}{q}\right]}$$

Result for binding energy and variation parameters

$$E_0 = 0.380 \text{ GeV}$$

$$p_0 = 0.97 \text{ GeV}^{q/4}, \quad q_0 = 1.36$$

$$E_{0,\ Schroe} = 0.380 \text{ GeV},$$



Tetraquark ($cc\bar{c}\bar{c}$) wave function and binding energy

This work — analytic result for wave function and binding energy.

$$\Psi(\boldsymbol{\rho},\boldsymbol{\lambda},\boldsymbol{\sigma}) = \sum_K \Psi_{[K]}(R)\cdot Y_{[K]}(\Omega) \to \Psi(R).$$

$$E_{bind} = 0.380 \text{ GeV}$$

Numerical calculation of binding energy within the variation approach with Gauss basis wave functions

A. V. Eskin, A. P. Martynenko and F. A. Martynenko, Mass spectrum of heavy tetraquarks in variational approach, arXiv:2505.05993 [hep-ph]

$$\Psi(\boldsymbol{\rho},\boldsymbol{\lambda},\boldsymbol{\sigma}) = \sum_{l=1}^{K} C_l e^{-\frac{1}{2}\left[A_{11}(l)\boldsymbol{\rho}^2+2A_{12}(l)\boldsymbol{\rho}\boldsymbol{\lambda}+A_{22}(l)\boldsymbol{\lambda}^2+2A_{13}(l)\boldsymbol{\rho}\boldsymbol{\sigma}+2A_{23}(l)\boldsymbol{\lambda}\boldsymbol{\sigma}A_{33}(l)\boldsymbol{\sigma}^2\right]}$$

$$E_{bind} = 0.347 \text{ GeV}$$

Discrepancy is about 0.03 *GeV* (~ 10 %).
That difference may mean that real tetraquark wave function has a small dependence on angles.

Tetraquark $(bb\bar{b}\bar{b})$, $(cc\bar{b}\bar{b})$, $(bb\bar{c}\bar{c})$ wave function and binding energy

Taking the trial wave function with two variation parameters

$$\chi(x) = N\, x^4\, e^{-p^2\, x^q}, \quad N = \sqrt{\frac{q 2^{\frac{9}{q}} p^{\frac{18}{q}}}{\Gamma\left[\frac{9}{q}\right]}}$$

for tetraquarks with b quarks we obtain

Tetraquark $(bb\bar{b}\bar{b})$

$$E_0 = -0.538 \text{ GeV}$$

$$p_0 = 1.02 \text{ GeV}^{q/4}, \quad q_0 = 1.25$$

Tetraquark $(cc\bar{b}\bar{b})$, $(bb\bar{c}\bar{c})$

$$E_0 = 0.022 \text{ GeV}$$

$$p_0 = 0.98 \text{ GeV}^{q/4}, \quad q_0 = 1.33$$

HFS

The hyperfine structure of S states tetraquark spectrum is determined by the pair potentials of the spin-spin interactions of the quarks and antiquarks that form the tetraquark.

$$\kappa_{ij} = \frac{32\,\pi\alpha_{s,ij}}{9\,m_i m_j}, \quad \alpha_{s,ij} = \alpha_s(2\mu_{ij}), \quad \boldsymbol{r}_{ij} = |\boldsymbol{r}_i - \boldsymbol{r}_j|.$$

$$\Delta V_{hfs} = \frac{1}{2}\left(\underbrace{\kappa_{12}\,(\boldsymbol{S}_1\boldsymbol{S}_2)\,\delta(\boldsymbol{r}_{12})}_{qq\ \ interaction} + \underbrace{\kappa_{34}\,(\boldsymbol{S}_3\boldsymbol{S}_4)\,\delta(\boldsymbol{r}_{34})}_{\bar{q}\bar{q}\ \ interaction}\right) +$$
$$+\underbrace{\kappa_{13}\,(\boldsymbol{S}_1\boldsymbol{S}_3)\,\delta(\boldsymbol{r}_{13}) + \kappa_{14}\,(\boldsymbol{S}_1\boldsymbol{S}_4)\,\delta(\boldsymbol{r}_{14}) + \kappa_{23}\,(\boldsymbol{S}_2\boldsymbol{S}_3)\,\delta(\boldsymbol{r}_{23}) + \kappa_{24}\,(\boldsymbol{S}_2\boldsymbol{S}_4)\,\delta(\boldsymbol{r}_{24})}_{q\bar{q}\ \ interaction}$$

Contribution to the energy spectrum can be obtained by using following expression for integration of Dirac delta function

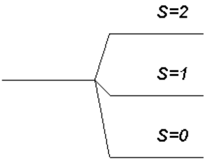
$$I_{hfs} = \int \Psi(\boldsymbol{\rho},\boldsymbol{\lambda},\boldsymbol{\sigma})\delta(\boldsymbol{\rho}_{ij})\Psi(\boldsymbol{\rho},\boldsymbol{\lambda},\boldsymbol{\sigma})d\boldsymbol{\rho}_{ij}d\boldsymbol{\lambda}_{lk}d\boldsymbol{\sigma}_{(ij)(lk)} = \frac{105\,2^{\frac{3}{q}-5}p^{\frac{6}{q}}\Gamma\left[\frac{6}{q}\right]}{\pi\,\Gamma\left[\frac{9}{q}\right]}\mu_{ij}^{3/2}$$

HFS

The spin wave function of the tetraquark $\chi^{S_{12}S_{34}}_{SS_z}$ is symmetric with respect to the permutation of two quarks 1 and 2 and two antiquarks 3 and 4. The charge and spatial parities of a tetraquark are:

$$C_T = (-1)^{S_T+L_T}, \quad P_T = (-1)^{L_T}.$$

$$|0^{++}>_T = |S_{cc}=1, S_{\bar{c}\bar{c}}=1, S_T=0, L_T=0>_{J_T=0},$$
$$|1^{+-}>_T = |S_{cc}=1, S_{\bar{c}\bar{c}}=1, S_T=1, L_T=0>_{J_T=1},$$
$$|2^{++}>_T = |S_{cc}=1, S_{\bar{c}\bar{c}}=1, S_T=2, L_T=0>_{J_T=2}.$$



Hyperfine splitting in general form, when quarks 1 and 2 are the same, antiquarks 3 and 4 are the same

$$\Delta E^{hfs} = \frac{35\, 2^{\frac{3}{q}-3} p^{\frac{6}{q}} \Gamma\left[\frac{6}{q}\right]}{3\,\Gamma\left[\frac{9}{q}\right]} \cdot \left[\frac{\alpha_{s,12}\mu_{12}^{3/2}}{m_1 m_2} + \frac{\alpha_{s,34}\mu_{34}^{3/2}}{m_3 m_4} + 4\frac{\alpha_{s,13}\mu_{13}^{3/2}}{m_1 m_3}\left(S_T(S_T+1)-4\right)\right]$$

Hyperfine splitting of $(cc\bar{c}\bar{c})$

$$\Delta E^{hfs} = \begin{cases} -\frac{7}{2}\tilde{\kappa}, & S_T = 0 \\ -\frac{3}{2}\tilde{\kappa}, & S_T = 1 \\ +\frac{5}{2}\tilde{\kappa}, & S_T = 2 \end{cases}$$

$$\tilde{\kappa} = \frac{35\, 2^{\frac{3}{q}-\frac{5}{2}}\, p^{\frac{6}{q}} \alpha_s \Gamma\left[\frac{6}{q}\right]}{3\sqrt{m}\,\Gamma\left[\frac{9}{q}\right]}$$

Relativistic corrections

The correction for relativity is one of main corrections in the energy spectrum. This applies in particular to the motion of c-quarks. General expression for this correction in the Hamiltonian of the system is equal to the sum of terms for each quark and antiquark of the following form:

$$\hat{T}_{rel} = -\frac{\mathbf{p}_1^4}{8m_1^3} - \frac{\mathbf{p}_2^4}{8m_2^3} - \frac{\mathbf{p}_3^4}{8m_3^3} - \frac{\mathbf{p}_4^4}{8m_4^3}.$$

When calculating the corresponding matrix elements for quarks and antiquarks of the same mass, we express the momenta $\mathbf{p}_i$ through derivatives with respect to the Jacobi coordinates:

$$\nabla_{\mathbf{r_1}} = \nabla_{\rho} - \frac{1}{2}\nabla_{\sigma},\ \nabla_{\mathbf{r_2}} = \nabla_{\rho} + \frac{1}{2}\nabla_{\sigma},\ \nabla_{\mathbf{r_3}} = \nabla_{\lambda} + \frac{1}{2}\nabla_{\sigma},\ \nabla_{\mathbf{r_4}} = \nabla_{\lambda} - \frac{1}{2}\nabla_{\sigma}.$$

Calculating analytically the matrix element we obtain the following result:

$$\Delta E_{rel} = -\frac{5 \cdot 2^{(\frac{4}{q}-9)}(90q^3 + 675q^2 + 1250q + 20337)p^{\frac{8}{q}}\Gamma(\frac{5}{q})}{33m\Gamma(\frac{9}{q})}.$$

Contact interaction correction

Contact interaction correction from Breit potential

$$\Delta V^{cont} = -\frac{2\pi}{3}\Big[\frac{\alpha_{s,12}\delta(\boldsymbol{r}_{12})}{m_1\, m_2} + \frac{\alpha_{s,34}\delta(\boldsymbol{r}_{34})}{m_3\, m_4} + 2\Big(\frac{\alpha_{s,13}\delta(\boldsymbol{r}_{13})}{m_1\, m_3} + \frac{\alpha_{s,14}\delta(\boldsymbol{r}_{14})}{m_1\, m_4} + \frac{\alpha_{s,23}\delta(\boldsymbol{r}_{23})}{m_2\, m_3} + \frac{\alpha_{s,24}\delta(\boldsymbol{r}_{24})}{m_2\, m_4}\Big)\Big]$$

After calculation of matrix element with wave function of tetraquark

$$\Delta E^{cont} = \frac{35\, 2^{\frac{3}{q}-4}\, p^{\frac{6}{q}}\, \Gamma\left[\frac{6}{q}\right]}{\Gamma\left[\frac{9}{q}\right]}\Big[\frac{\alpha_{s,12}\mu_{12}^{3/2}}{m_1\, m_2} + \frac{\alpha_{s,34}\mu_{34}^{3/2}}{m_3\, m_4} + 2\Big(\frac{\alpha_{s,13}\mu_{13}^{3/2}}{m_1\, m_3} + \frac{\alpha_{s,14}\mu_{14}^{3/2}}{m_1\, m_4} + \frac{\alpha_{s,23}\mu_{23}^{3/2}}{m_2\, m_3} + \frac{\alpha_{s,24}\mu_{24}^{3/2}}{m_2\, m_4}\Big)\Big]$$

Resulting expression can be simplified in the case of all same quarks and antiquarks

$$\Delta E^{cont}_{(cc\bar{c}\bar{c})} = \frac{175\, 2^{\frac{3}{q}-\frac{9}{2}}\, \alpha_s\, p^{\frac{6}{q}}\, \Gamma\left[\frac{6}{q}\right]}{\sqrt{m}\,\Gamma\left[\frac{9}{q}\right]}$$

Relativistic recoil correction

Relativistic recoil correction, represented in terms of relative momenta of particles $\mathbf{p}_{ij}$:

$$\delta H_{rel-rec} = -\sum_{i,j} c_{ij} \frac{\alpha_s \ _{ij}}{3 m_i m_j r_{ij}} [\mathbf{p}_{ij}^2 + \frac{\mathbf{r}_{ij}(\mathbf{r}_{ij}\mathbf{p}_{ij})\mathbf{p}_{ij}}{r_{ij}^2},$$

where the coefficients c_{ij} are equal to 1 for a pair of quarks or a pair of antiquarks i, j and 2 in the case of a quark-antiquark pair. Calculating matrix element of that operator, which is also performed analytically, yields the result:

$$\Delta E_{rel-rec} = -\frac{175\alpha_s 2^{(\frac{3}{q}-\frac{15}{2})} q^2 p^{\frac{6}{q}} \Gamma(2+\frac{6}{q})}{3\sqrt{m}\Gamma(\frac{9}{q})}.$$

Confinement with HFS

To clarify the value of the hyperfine splitting in quarkonia, a nonperturbative potential with spin-spin interaction of the form is used

S. F. Radford and W. W. Repko, Potential model calculations and predictions for heavy quarkonium, Phys. Rev. D 75, 074031 (2007).

$$\Delta V^{hfs}_{conf} = \sum_{i,j} f_v \frac{A}{8 r_{ij}} \left(\frac{1}{m_i^2} + \frac{1}{m_j^2} + \frac{16}{3\, m_i m_j} (\boldsymbol{S}_i \boldsymbol{S}_j) \right), \quad f_v = 0.9$$

Contribution to the energy spectrum of tetraquark ($cc\bar{c}\bar{c}$)

$$\Delta E^{hfs}_{conf} = \begin{cases} \lambda, & S_T = 0 \\ \frac{7}{3}\lambda, & S_T = 1 \\ 5\,\lambda, & S_T = 2 \end{cases}, \quad \lambda = \frac{35\, 2^{\frac{1}{q}-7}\, A\, f_v\, q\, p^{\frac{2}{q}}\, \Gamma\left[\frac{q+8}{q}\right]}{(2\,m)^{3/2}\, \Gamma\left[\frac{9}{q}\right]}$$

Masses of heavy tetraquarks $(cc\bar{c}\bar{c})$, $(bb\bar{b}\bar{b})$, $(cc\bar{b}\bar{b})$, $(bb\bar{c}\bar{c})$

State	$(cc\bar{c}\bar{c})$	[1]	[6]	[5]	[2]	[3]	[7]	[8]	[4]	[9]
0^{++}	5.86	6.10	6.190	6.477	5.966	5.9694	6.797	5.883	6.435	6.503
1^{+-}	6.02	6.26	6.271	6.528	6.051	6.0209	6.899	6.120	6.515	6.517
2^{++}	6.35	6.57	6.367	6.573	6.223	6.1154	6.956	6.246	6.543	6.544

State	$(bb\bar{b}\bar{b})$	[1]	[6]	[2]	[7]	[8]	[4]
0^{++}	18.63	18.81	19.314	18.754	20.155	18.748	19.201
1^{+-}	18.76	18.86	19.320	18.808	20.212	18.828	19.251
2^{++}	19.02	18.97	19.330	18.916	20.243	18.900	19.262

State	$(cc\bar{b}\bar{b})$	[1]	[6]	[7]	[8]
0^{++}	12.44	12.77	12.846	13.496	12.445
1^{+-}	12.47	12.78	12.859	13.560	12.536
2^{++}	12.52	12.80	12.883	13.595	12.614

[1]arXiv:2505.05993 [hep-ph]

[2]Phys. Rev. D 86, 034004 (2012)

[3]Chinese Physics C 43, 1, 013105 (2019)

[4]Phys. Rev. D 97, 094015 (2018)

[5]Phys. Rev. D 70, 014009 (2004)

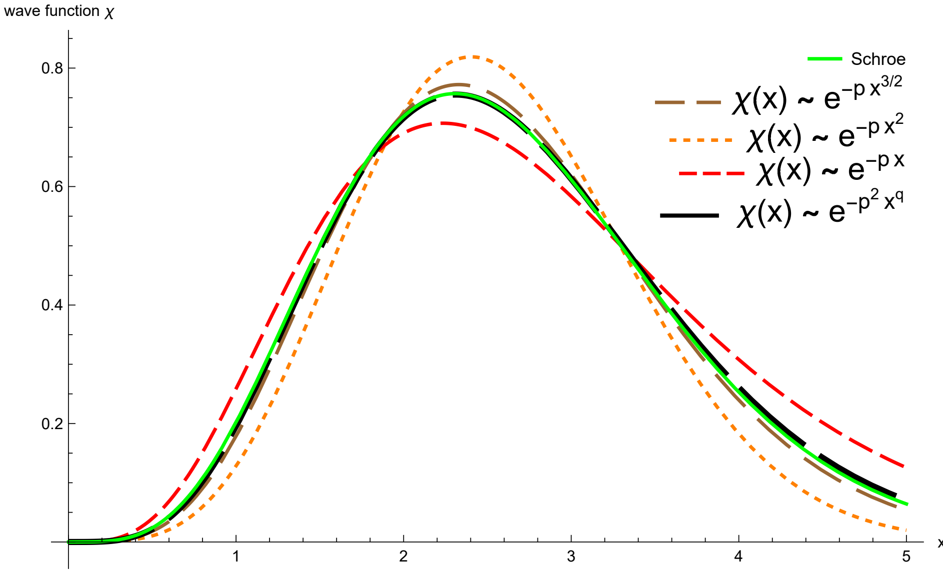
[6]Universe 7, 94 (2021)

[7]Phys. Rev. D, 97, 094015, (2018)

[8]EPJ C, 80, 1004, (2020)

[9]Eur. Phys. J. C (2023) 83:416

Thank You!


wave function χ
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