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New analysis of the Bjorken sum rule based on the analytic perturbation theory

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BLTP

Outline

✓Motivation for a new analysis of the Bjorken sum rule

✓Overview of the standard perturbative (PT) and the analytic perturbation theory (APT) QCD descriptions

✓Higher twist (HT) analysis vs. low-energy data

✓New theoretical approach: behavior at small Q^2 near zero involving the Gerasimov-Drell-Hearn sum rule

✓Comparison with JLab data

✓Discussion and conclusions

THE POLARIZED BJORKEN SUM RULE

The **BSR** is defined by the integral of the difference of the spin-dependent structure functions $g_1(x, Q^2)$ of the proton g_1^p and neutron g_1^n over the possible values of the Bjorken variable x at a fixed square of the transferred momentum Q^2:

$$\Gamma_1^{p-n}(Q^2)\,|_{Q^2\to\infty} = \int_0^1 \left[g_1^p(x, Q^2) - g_1^n(x, Q^2)\right] dx = \frac{g_A}{6}$$

Axial charge: $g_A = 1.2724.$
Proton mass: $M = 0.938$ GeV

Feynman, Richard Phillips.
Photon-hadron interactions / Richard Feynman.
p. cm. — (Advanced book classics series)
Originally published : Reading, Mass. : W. A. Benjamin, Advanced Book Program, 1972. (Frontiers in physics)

Lecture 33: ((33.16) 18 Bjorken's relation) «...The first relation is remarkable since it simultaneously tests the quark picture. Its verification, or failure, would have a decisive effect on the future direction of high-energy theoretical physics.»

It is really so in after 50 years!

BSR + OPE

OPE (*Operator Product Expansion*):

$$\Gamma_1^{p-n}(Q^2) = \frac{g_A}{6}\left[1 - D_{\rm BS}(Q^2)\right] + \sum_{i=2}^{\infty} \frac{\mu_{2i}(Q^2)}{Q^{2i-2}}$$

$$\Gamma_1^{p-n}(Q^2 \to \infty) = \frac{g_A}{6}$$

perturbative QCD correction (power series in α_s leading twist)

high twist terms: quark and gluon correlations, $\sim 1/Q^{2i-2}$

Important in very low Q^2

$$\Gamma_1^{p-n}(Q^2) = \int_0^1 \left[g_1^p(x,Q^2) - g_1^n(x,Q^2)\right]dx = \frac{g_A}{6}C_{Bj}(Q^2) + \frac{\mu_4}{Q^2} + \frac{\mu_6}{Q^2} + \dots$$

extracted from data

$$C_{Bj}^{PT}(Q^2) = 1 - D_{Bj}^{PT}(Q^2) = 1 - \sum_{k\geq 1}^{4} c_k \alpha_s^k(Q^2)$$

twist $\geq$ 4
twist = dimension - spin
first non-zero term - twist-4

Twist-2: parton moves freely inside nucleon.

Twist-4: two-quark correlations + gluon exchange (many-body dynamics).

Twist-6 and beyond: even more complex many-body interactions.

$$\sum_{i=2}^{\infty} \frac{\mu_{2i}(Q^2)}{Q^{2i-2}} \to \frac{\hat{\mu}_4 + m_{\rm HT}^2}{Q^2 + m_{\rm HT}^2}$$

QCD running coupling α_s

- $\alpha_s(Q^2)$ — the strong coupling constant, the analogue oh the «charge» for quarks and gluons.
- Depends on the momentum scale Q^2 → «running coupling».

The running coupling is defined as a solution of the renormalization group (RG) equation. In the analysis we use the exact numerical solutions of the RG equation in different orders (NLO, N2LO, N3LO).

- At **large** Q^2 : $\alpha_s \to 0$ → quarks are almost free (*asymptotic freedom*)
- At **small** Q^2 : α_s grows → strong interactions, PT breaks down.
- In standard PT there appears an **unphysical Landau pole** at $Q^2 = \Lambda^2$.

In **APT**

- Instead of powers $\alpha_s^k(Q^2)$, one uses analytic functions $\mathscr{A}^{(k)}(Q^2)$.
- No Landau pole, $\alpha_s^{APT}(Q^2)$ remains finite as $Q^2 \to 0$.

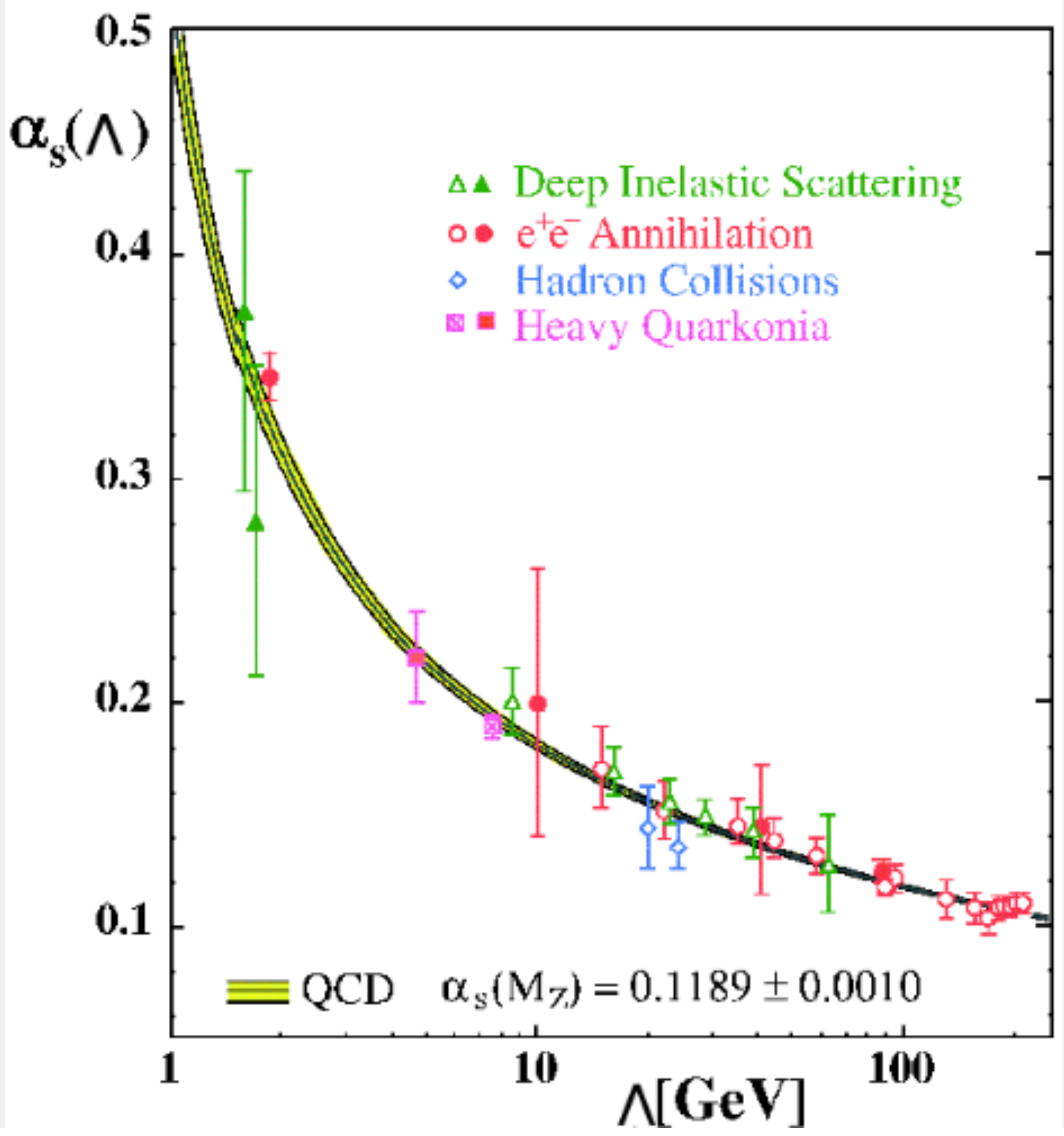
RG Equation

$$Q^2 \frac{d\alpha_s}{dQ^2} = -\beta(\alpha_s)$$

$$\beta(\alpha_s) = \beta_0 \alpha_s^2 + \beta_1 \alpha_s^3 + \ldots$$

At 1-loop:

$$\alpha_s(Q^2) = \frac{4\pi}{\beta_0 \ln(Q^2/\Lambda_{QCD}^2)}$$



ANALYTIC PERTURBATION THEORY

To describe the corrections in the region of large momentum Q^2 used the modification — Analytic perturbation theory (APT).

APT = PT + RG + Q^2-analiticity

[D.V. Shirkov, I.L. Solovtsov, Phys. Rev. Lett. 79 (1997) 1209]

APT solves the problem of eliminating non-physical singularities of the effective charge in QCD - «Landau pole» (1950)

In 1959, a solution to the Landau pole problem was proposed in QCD (see *Ref. Bogolyubov, Logunov, Shirkov, JETP, 1959*).

The analytical approach in the perturbation theory of QCD was initiated by the works of Solovtsov, Shirkov (see *Ref. (Shirkov, Solovtsov: 1996), (Shirkov, Solovtsov: 1997), (Shirkov, Solovtsov: 1998)*), where the main objects in this approach are spectral densities, by means of which the analytic effective charge and its integer degrees are defined in the Euclidean domain in the form of dispersion integrals.

BSR + APT

Analytic perturbation theory (APT)

Perturbation theory (PT)

$^*C_{Bj}(Q^2)$ encodes all short-distance QCD radiative corrections, while g_A carries the long-distance physics of the nucleon

$$D_{\rm BS}^{\rm APT} = \sum_{k\le 4} c_k \mathscr{A}_k \quad \Rightarrow \quad D_{\rm BS}^{\rm PT} = \sum_{k\le 4} c_k \alpha_s^k$$

$$C_{Bj}^{APT}(Q^2) = 1 - D_{Bj}^{APT}(Q^2) = 1 - \sum_{k\ge1} c_k \mathscr{A}^{(k)}(Q^2)$$

$$D_{\rm BS}^{APT}(Q^2) = c_1\mathscr{A}^{(1)}(Q^2) + c_2\mathscr{A}^{(2)}(Q^2) + c_3\mathscr{A}^{(3)}(Q^2) + c_4\mathscr{A}^{(4)}(Q^2)$$

$$\mathscr{A}^{(k)}(Q^2) = \frac{1}{\pi}\int_0^\infty d\sigma\, \frac{\varrho_k(\sigma)}{\sigma + Q^2}\cdot$$

$\mathscr{A}^{(k)}(Q^2)$ — analytical functions, eliminate the Landau pole

$$C_{Bj}^{PT}(Q^2) = 1 - D_{Bj}^{PT}(Q^2) = 1 - \sum_{k\ge1}^{4} c_k \alpha_s^k(Q^2)$$

$$D_{BS}^{PT}(Q^2) = c_1\alpha_s(Q^2) + c_2\alpha_s^2(Q^2) + c_3\alpha_s^3(Q^2) + c_4\alpha_s^4(Q^2) + \ldots$$

for N^3LO (4-loop):

$$D_{\rm BS}(\alpha_s) = 0{,}318\,\alpha_s + 0{,}363\,\alpha_s^2 + 0{,}652\,\alpha_s^3 + 1{,}804\,\alpha_s^4$$

[Baikov, Chetyrkin, Kühn (2010)]

Meaning of OPE for BSR:

- **Factorization by scales:** short distances $\rightarrow$ $C_{Bj}(Q^2)(PT)$, long distances $\rightarrow$ g_A and HT parameters.
- **Twist ordering:** leading (twist-2) dominates at large Q^2; higher twists are power-suppressed $(1/Q^{2i})$.
- **Practice:** gives a clear form for $\Gamma_1^{p-n}(Q^2)$, where C_{Bj} is calculate and $\mu_4, \mu_6, \ldots$ can be fitted from data

$$D_{BS}^{APT}(Q^2) = c_1\mathscr{A}^{(1)}(Q^2) + c_2\mathscr{A}^{(2)}(Q^2) + c_3\mathscr{A}^{(3)}(Q^2) + c_4\mathscr{A}^{(4)}(Q^2) \quad | \quad D_{BS}^{PT}(Q^2) = c_1\alpha_s(Q^2) + c_2\alpha_s^2(Q^2) + c_3\alpha_s^3(Q^2) + c_4\alpha_s^4(Q^2) + \ldots$$

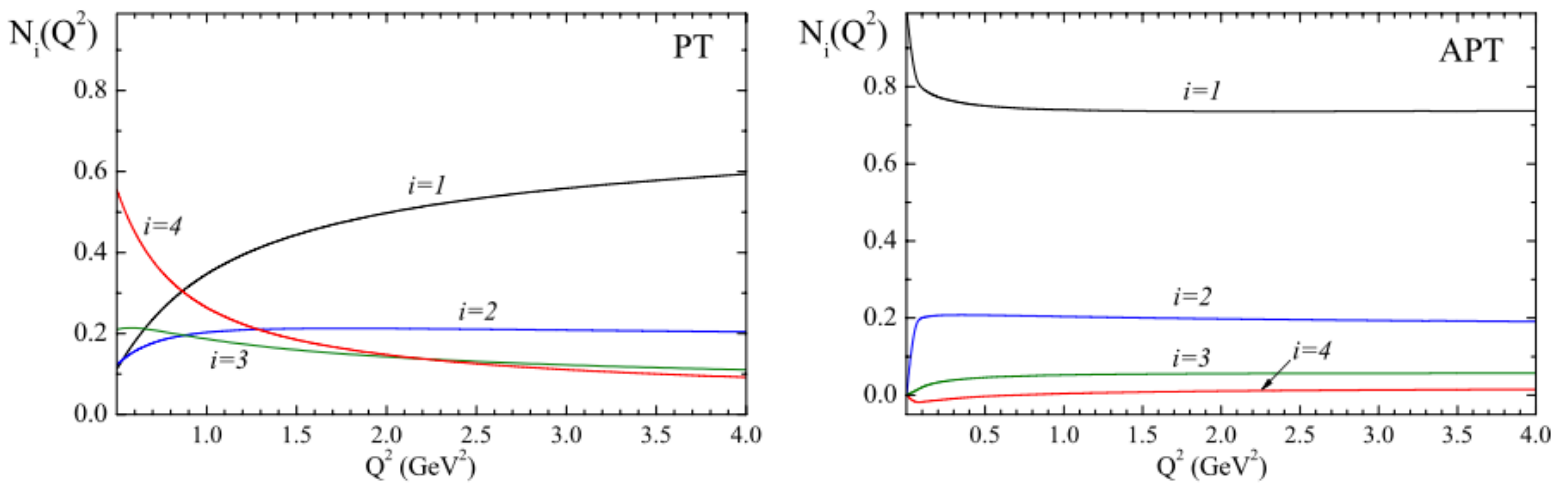


Fig. 1. The Q^2-dependence of relative contributions of perturbative terms in PT (the left panel) and APT (the right panel) at the four-loop level (N^3LO).

The difference in the convergence properties of PT and APT expansions, respectively, is demonstrated in Fig. 1. The value of the relative contribution of the term, $N_i(Q^2)$, is determined as:

$$N_i^{PT}(Q^2) = c_i\alpha_s^i(Q^2)/D_{BS}^{PT}Q^2 \qquad \text{and} \qquad N_i^{APT}(Q^2) = c_i\mathscr{A}_s^i(Q^2)/D_{BS}^{APT}Q^2$$

Small Q^2 matching the Gerasimov-Drell-Hearn sum rule

For the purpose of a smooth continuation of $\Gamma_1^{p-n}(Q^2)$ to the non-perturbative region, we follow the approach proposed in [*Soffer, Teryaev, Phys. Rev. D* ***70*** *(2004)*].

$$I_1(Q^2) \equiv \frac{2M^2}{Q^2}\Gamma_1(Q^2) = \frac{2M^2}{Q^2}\int_0^1 g_1(x,Q^2)dx$$ ← the Q^2-dependent integral that is defined for all Q^2.

where M is the mass of the nucleon.

$$I_{\text{GDH}} = \frac{2M^2}{Q^2}\Gamma_1(Q^2) = \lim_{Q^2\to 0}\left[I_1^p(Q^2) - I_1^n(Q^2)\right] = -\frac{1}{4}\left(\kappa_p^2 - \kappa_n^2\right) \simeq 0.112$$ ← the value of this integral at $Q^2 = 0$ is related to by the **GDH** sum rule

where κ_N is the nucleon anomalous magnetic moment:

$\kappa_p \approx 1.793, \;\; \kappa_n \approx -1.913.$

$$\lim_{Q^2\to\infty} I_1^{p-n}(Q^2) = \frac{M^2}{Q^2}\frac{g_A}{3}$$

GDH defines the slope of $\Gamma_1(Q^2)$ at low Q^2 and $I_1(Q^2)$

$$I_1^{p-n}(Q^2) = \theta(Q^2 > Q_0^2)I_1(Q^2) + \theta(Q^2 < Q_0^2)\sum_{l=0}^{k}\frac{1}{l!}\frac{\partial^l I_1(Q^2)}{(\partial Q^2)^l}\bigg|_{Q=Q_0}(Q^2 - Q_0^2)^l$$

$I_1^{p-n}(Q^2)$ allows to introduce a new qualitative characteristic: the position and the height of the maximum in transition region.

The function $I_1^{p-n}(Q^2)$ provides a smooth interpolation between the GDH sum rule at $Q^2 = 0$ and BSR sum rule at large Q^2.

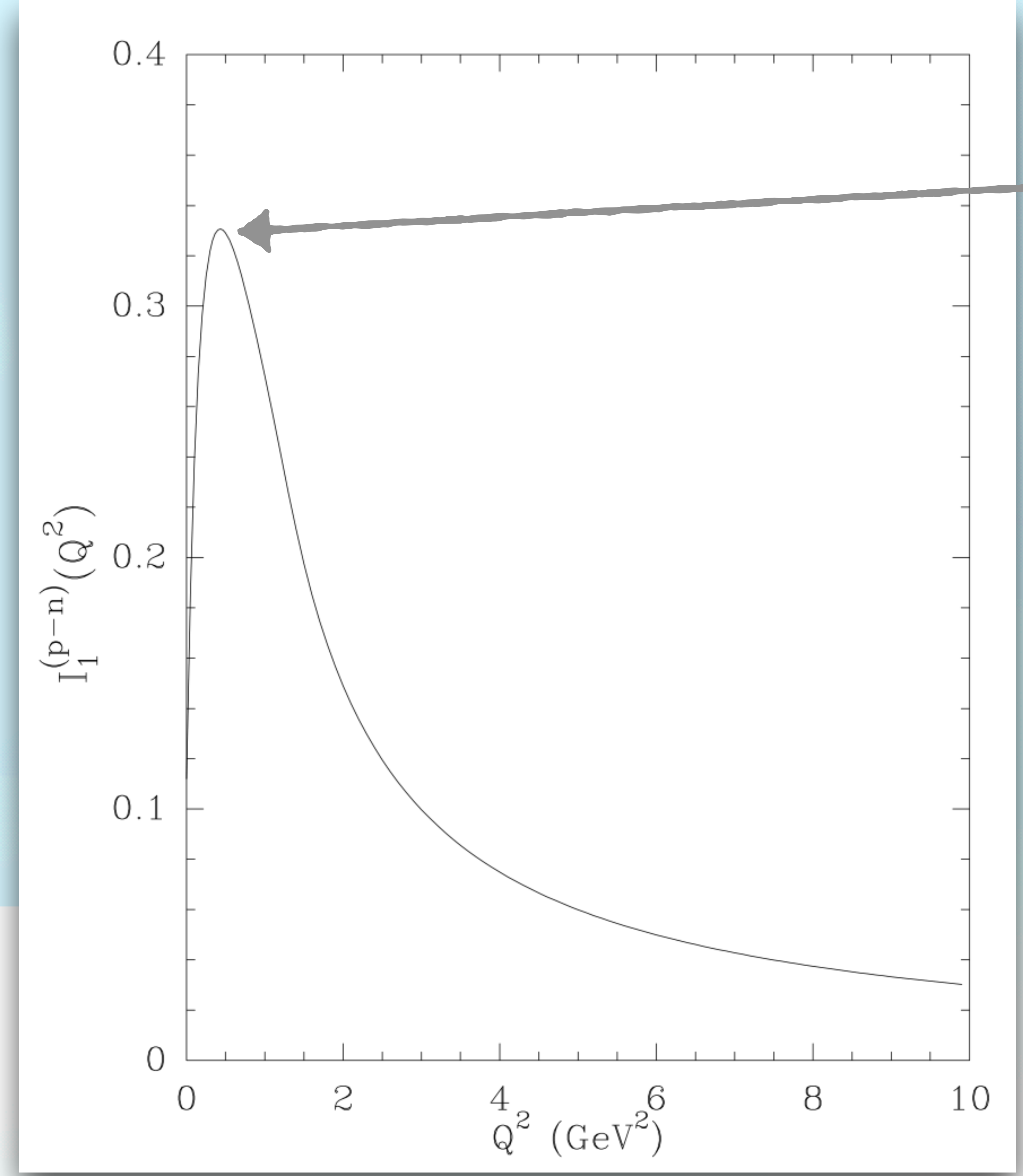


Fig 2. The prediction for $I_1^{p-n}(Q^2)$, directly related to the Bjorken sum rule.

[J. Soffer, O. V. Teryaev, Phys. Lett. B545 (2002)].

B.L. Ioffe, V.A. Khoze and L.N. Lipatov. *Hard Processes: Volume 1 Phenomenology, Quark-Parton Model*. Amsterdam: North-Holand, 1984.

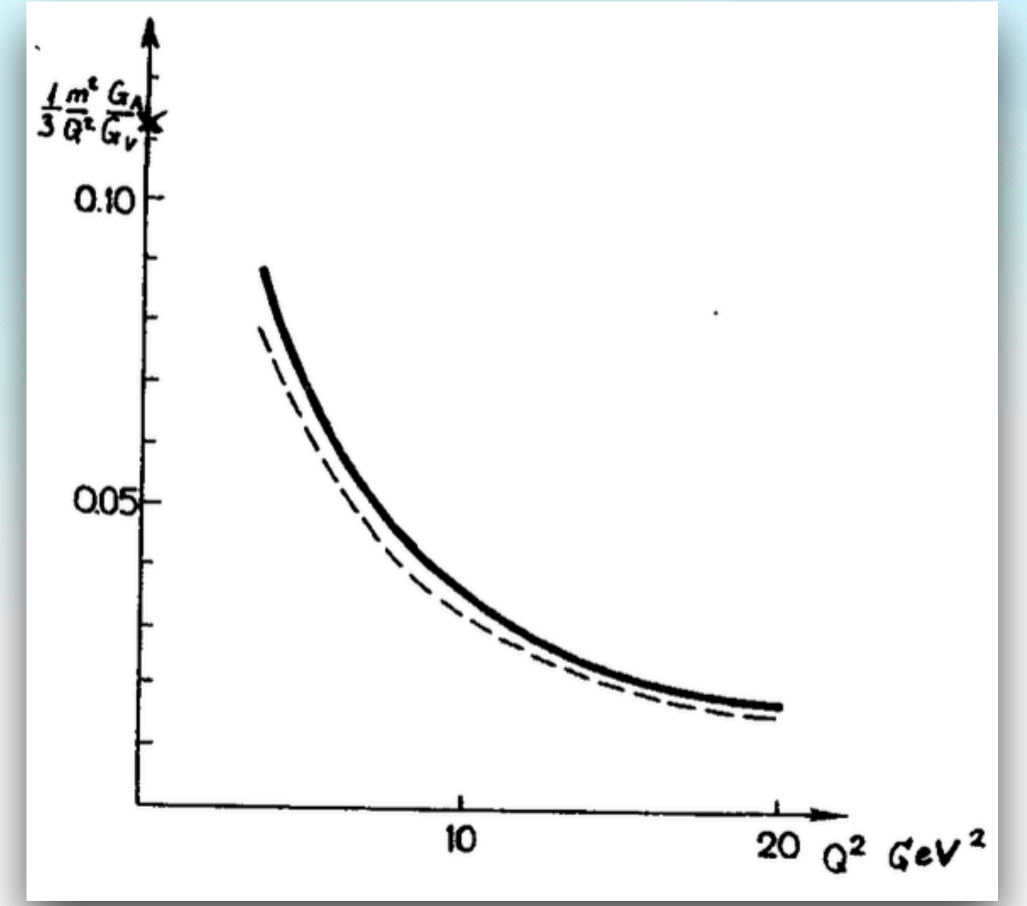


Fig 3. A function of Q^2 (solid curve). Dashed curve: the same, taking into account corrections $\sim 1/\ln Q^2$ in QCD.

Now the sum rule (4.22) written for the difference of the proton and neutron scattering cross sections,

$$\int_0^\infty \frac{d\nu}{\nu}[G_1^p(\nu, 0) - G_1^n(\nu, 0)] = -\tfrac{1}{4}(\varkappa_p^2 - \varkappa_n^2) = 0.112, \qquad (4.22')$$

is to be compared with the sum rule (4.17), rewritten in the form

$$\int_{Q^2/2}^\infty \frac{d\nu}{\nu}[G_1^p(\nu, Q^2) - G_1^n(\nu, Q^2)] = \frac{1}{3}\frac{m^2}{Q^2}\frac{G_A}{G_V}. \qquad (4.24)$$

The Q^2 dependence of the r.h.s. of (4.24) and the point at $Q^2 = 0$, given by (4.22) are presented in fig. 3. It is clear that the value shown at $Q^2 = 0$ is in reasonable agreement with the curve extrapolated from the region of large Q^2. Taking into account the corrections $\sim 1/\ln Q^2$, obtained in the framework of QCD, diminishes the r.h.s. of (4.24) in the region of intermediate Q^2, making the agreement even better (the dashed line in fig. 3). At the same time, it is evident that one should not expect the Bjorken sum rule to be accurate in the region of $Q^2 < 5$ GeV^2.

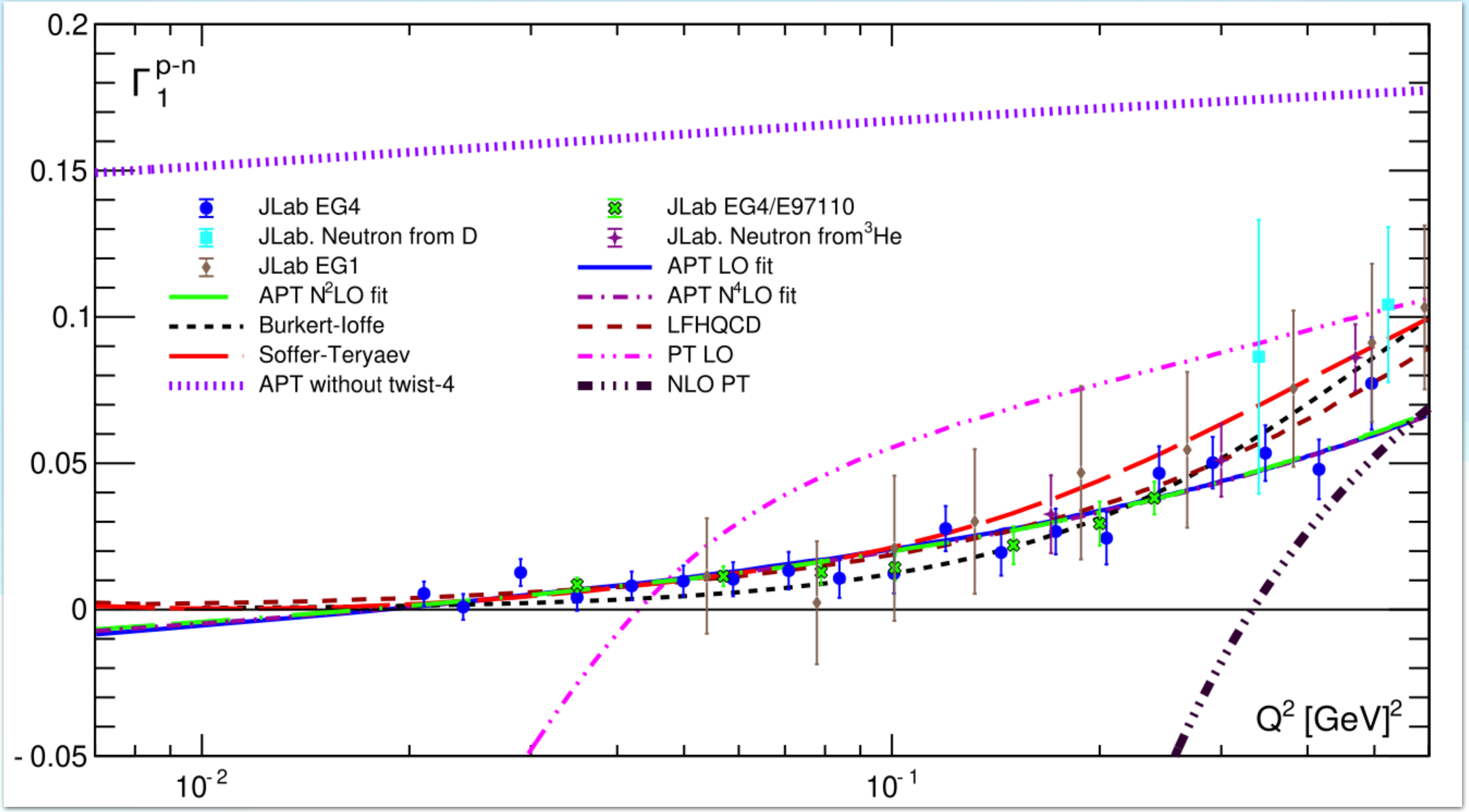


Fig. 4. The results for $\Gamma_1^{p-n}(Q^2)$ using the first four orders of APT, from fits of experimental data with $Q^2 < 0.6$ GeV^2.

[I.R. Gabdrakhmanov, N.A. Gramotkov, D.A. Volkova, I.A. Zemlyakov, A.V. Kotikov, O.V. Teryaev. Particles 2025, 8(1), 29.]

The four-loop approximation provides good description of the JLab data at intermediate values Q^2 > 1-2 GeV^2. Consider now small values Q^2 < 1 GeV^2 and how move to Q^2 = 0. For this purpose we use the technique of matching proposed by Soffer-Teryaev and the behavior at small Q^2 near zero by involving the Gerasimov-Drell-Hearn sum rule.

$$I_1^{p-n}(Q^2) = \theta(Q^2 > Q_0^2) I_1(Q^2) + \theta(Q^2 < Q_0^2) \sum_{l=0}^{k} \frac{1}{l!} \frac{\partial^l I_1(Q^2)}{(\partial Q^2)^l}\Bigg|_{Q=Q_0} (Q^2 - Q_0^2)^l \qquad (1)$$

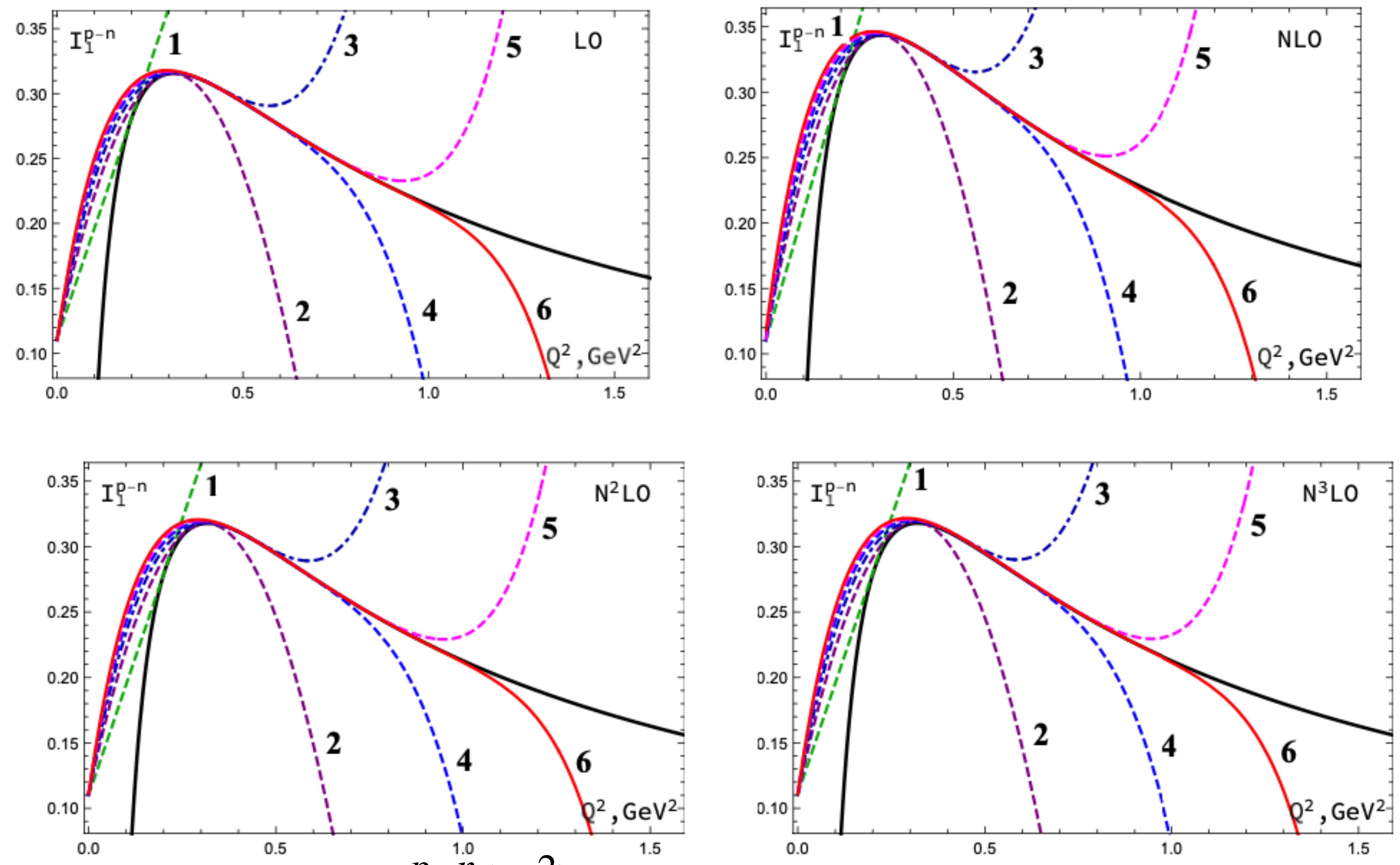


Fig. 5. *Behavior of $I_1^{p-n}(Q^2)$, Eq. (1), depending on the perturbative order (LO, NLO, N^{2}LO and N^{3}LO) and on the number of terms in the sum (numbers near the curves).*

	LO	NLO	N^{2}LO	N^{3}LO
k = 1	0.199	0.196	0.203	0.205
k = 2	0.284	0.280	0.292	0.292
k = 3	0.369	0.363	0.378	0.379
k = 4	0.456	0.448	0.476	0.467
k = 5	0.542	0.543	0.554	0.555

Table 1. Matching points Q_0^2 (in GeV^2) for different orders of APT (LO, NLO, N^2LO and N^3LO) and the number k terms in the sum (1).

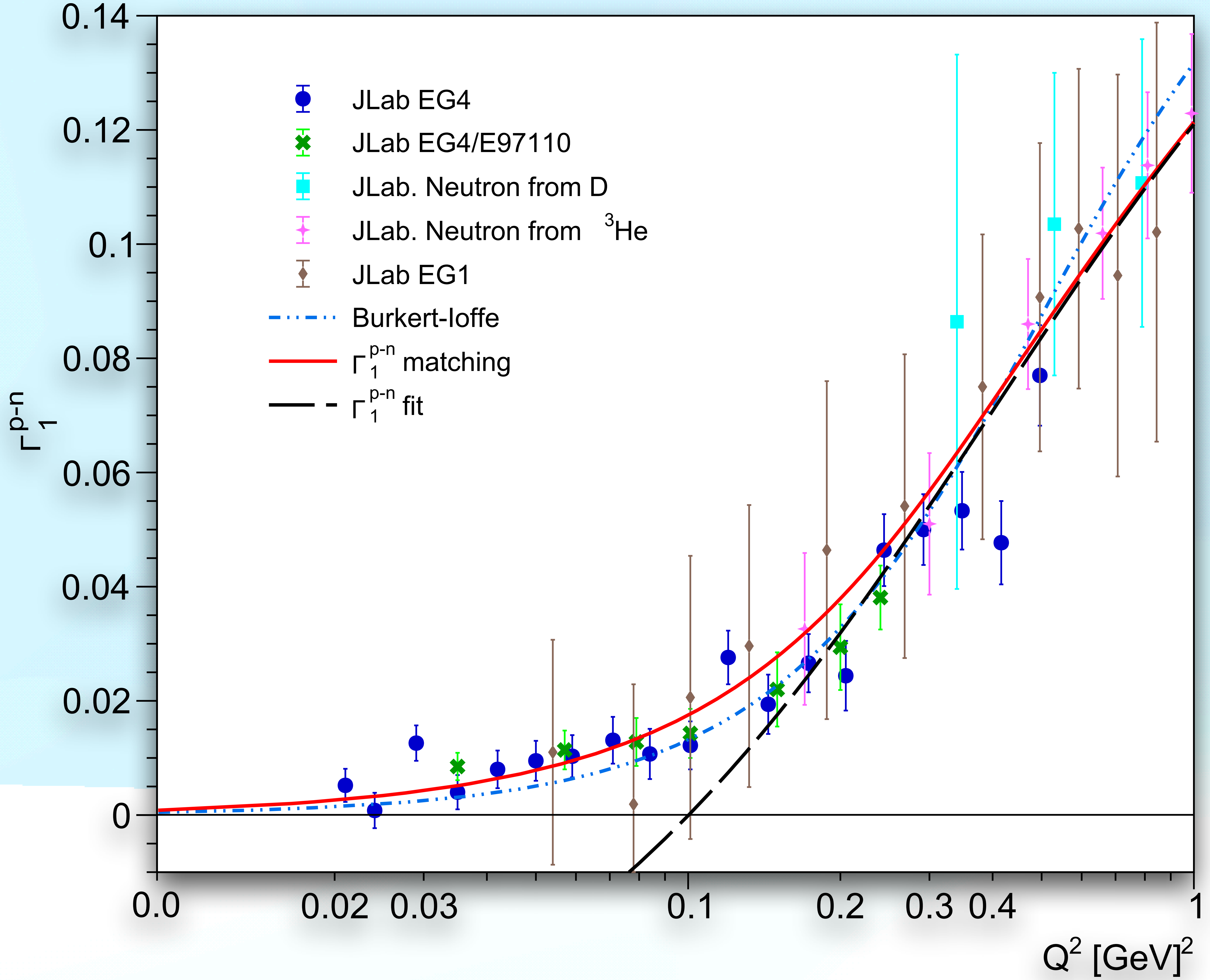


Fig. 6: *The behavior $\Gamma_1^{p-n}(Q^2)$ in various approach in the region of small transferred momenta.*

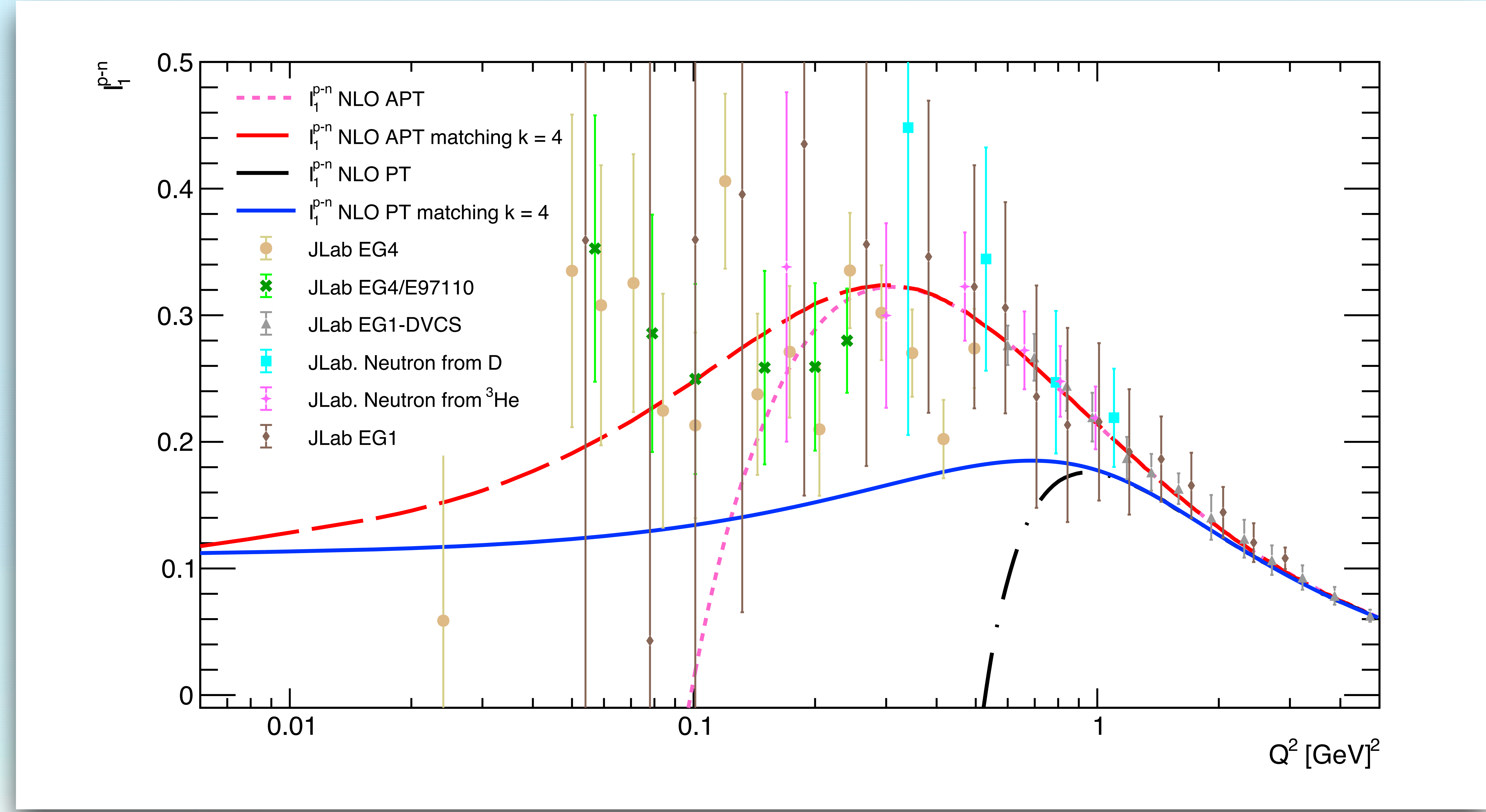


Fig. 7: *The behavior $I_1^{p-n}(Q^2)$ in various approach in the region of small transferred momenta. The solid line corresponds to the PT with matching, the dot-dashed line the PT without matching, the dashed line to the APT approach with matching and the short dashed line to the APT without matching.*

There is still a room for improvements in theoretical description of DIS sum rules that stimulates further studies along line associated with both perturbative and with nonperturbative effects.

OUTLOOK

- APT + new HT approximation → stable at low Q^2 .
- $I_1(Q^2)$: must have a maximum (small at 0 → rise → fall as $1/Q^2$). Maximum characterized by position and height.
- APT + matching: smooth, monotonic behavior; PT fails.
- The approach of «matching» the function is applied, where the sum rule is applied to the region of large values of Q^2 (Bjorken sum rule) and then continues to the region of small values of Q^2 → information on the behavior of structural functions in the area where experimental data are not available.
- A QCD method for analyzing experimental data on the Bjorken sum rule in the infrared region was proposed, based on the use of the function $I_1(Q^2)$, the GDH sum rule, and a matching procedure. The results obtained within APT at different perturbative orders are stable and close to each other, although due to large uncertainties in the data no definite conclusion on agreement with experiment can be made. Even standard PT (with its unphysical singularities) gives a formally reasonable description, which underlines the strong sensitivity to the perturbative contribution.

Perspective

→ This approach can be applied to other problems in the study of the low-energy region of QCD. Also, approach in different loop level of the perturbative part gives a stable good agreement with the experimental data in the whole region up to zero momentum transfer.

Thanks for your attention!