Ideal relativistic spin fluid model

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Outline

Dynamics of spin and fundamental physics

- Study of spin dynamics for matter with dipole moments (anomalous magnetic and electric one) is important for search of new physics beyond Standard Model [Vergeles-Nikolaev-YNO-Silenko-Teryaev, *Phys. Usp.* **66** (2023) 109; Budker, Cong, et al, *Rev. Mod. Phys.* **97** (2025) 025005]
- Tests of foundations (Lorentz symmetry, equivalence principle, spacetime structure beyond Riemann geometry)
- Colella-Overhauser-Werner (1975) and Bonse-Wroblewski experiments - equivalence principle for quantum systems; Extend to spin [Silenko, Teryaev, *PRD* **71** (2005) 064016]
- Modern applications: heavy ion collisions physics, spin polarization and spin transport phenomena
- Action principle (Lagrange-Noether machinery) fundamental $\Longrightarrow$ Develop variational approach for relativistic spin fluid

From points to microstructured elements

- Physical objects: points, system of points, continuous media, fluids, rigid extended bodies
- In classical continuum theory material elements are structureless points characterized by positions only;
- their internal properties are disregarded.
- Generalization to media with microstructure: material body (fluid, etc) formed of particles whose internal properties contribute to macroscopic dynamics
- Early model of *Cosserat* brothers (1909): structureless material point replaced by point that carries a triad of axes (an oriented *frame*, in the modern geometrical language)
- $\Longrightarrow$ variety of models known as oriented or multipolar medium theory, asymmetric elasticity, micropolar elasticity
- Applications: ferromagnetic materials, cracked & granular media, liquid crystals, superfluids, heavy-ion collisions

THÉORIE

DES

CORPS DÉFORMABLES

PAR

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1909

- *Micromorphic continuum*: 3-frame attached to elements is deformable; hence 9 new degrees of freedom introduced in addition to classical ones (3 rotations, 1 volume expansion, 5 shear deformations)
- *Microstretch continuum*: 3-frame can rotate and change volume (4 degrees of freedom: 3 microrotations and 1 microvolume expansion)
- *Micropolar continuum*: structured material point is rigid, possessing 3 microrotation degrees of freedom
- Important example: media (point particles, extended bodies, fluids) with *spinning* matter elements
- *Microscopic origin*: can classical internal properties be derived from underlying fundamental quantum theory?
- Here: Discuss construction of relativistic variational theory of an ideal spin fluid

Example: Nematic liquid crystals

- Liquid crystal = fluid with elements characterized by average 4-velocity u^i, and *director* N^i (microstructure)
- These vectors satisfy normalization conditions

$$u^i u_i = 1, \qquad N^i N_i = -1, \qquad N_i U^i = 0$$

- Dynamics of director: $\omega^i := \epsilon^{ijk} N_j \dot{N}_k, \ \epsilon_{ijk} := \eta_{ijkl} \frac{U^l}{c}$
- Relativistic Lagrangian model

$$L = -\rho(\nu, s) - \tfrac{1}{2} J \nu \omega^i \omega_i$$
$$- \tfrac{1}{2} K_1 \left(\nabla_i N^i\right)^2 - \tfrac{1}{2} K_2 \left(\epsilon^{ijk} N_i \nabla_j N_k\right)^2 + \tfrac{1}{2} K_3 \left(\epsilon_{ijk} N^j \epsilon^{kln} \nabla_l N_n\right)^2$$

- ρ - internal energy density, ν - particle number density, J - element's moment of inertia, K_i - Frank's elastic moduli
- YNO, T. Ramos, G. Rubilar, *Phys. Rev.* **E86** (2012) 031703

Matter with spin: micropolar media

- History: spinning particle model Frenkel (1926) Thomas (1927) Phenomenological fluid Weyssenhoff-Raabe (1947)
- Weyssenhoff fluid's elements are characterized by u^i, energy-momentum $\mathcal{P}_i$ and *spin* density tensor $\mathcal{S}_{ij} = -\mathcal{S}_{ji}$
- Dust (noninteracting gas) has canonical currents

$$T_i{}^k = u^k \mathcal{P}_i, \quad S_{ij}{}^k = u^k \mathcal{S}_{ij} \qquad (\text{recall} \quad J^k = u^k \rho_{\rm e})$$

- Conservation of energy-mom. and angular momentum

$$\nabla_k T_i{}^k = 0, \qquad u_i \mathcal{P}_j - u_j \mathcal{P}_i = 2\nabla_k S_{ij}{}^k$$

- Inclusion of interactions of particles introduces pressure

$$T_i{}^k = u^k \mathcal{P}_i - p\left(\delta_i^k - u^k u_i/c^2\right)$$

- 4-momentum recovered explicitly ($u^i u_i = c^2$; $\rho := u^i \mathcal{P}_i$)

$$\mathcal{P}_i = \frac{1}{c^2}\left(\rho u_i - 2u^j \nabla_k S_{ij}{}^k\right)$$

Ideal fluid: variational approach

Ideal fluid of structureless elements (no internal degrees of freedom) is continuous medium characterized by 4-velocity u^i, *internal energy* density $\rho = \rho(\nu, s)$, *particle density* ν, *entropy* s, and *identity (Lin) coordinate* X. Assume

$$\nabla_i(\nu u^i) = 0, \qquad u^i \nabla_i s = 0, \qquad u^i \nabla_i X = 0$$

– number of particles, entropy and identity are conserved. Fluid dynamics governed by action $I = \frac{1}{c}\int \sqrt{-g}\, d^4x\, (-\rho)$. Write internal energy $\rho = \rho_{\rm m} c^2 + \mathcal{F}_0$ as a sum of "rest-mass" density and hydrodynamic energy density. In comoving frame $\sqrt{-g}\, d^4x = c\, dV_0\, d\tau$ with 3-volume dV_0 and proper time τ; rest mass of fluid's element $dm_0 = \rho_{\rm m} dV_0$ same in all frames and invariant volume $dV_0\, d\tau = dV\, dt$. In the nonrelativistic limit $dV_0\, d\tau = dV\, dt$ with $d\tau = \sqrt{1 - \boldsymbol{v}^2/c^2} dt \approx (1 - \boldsymbol{v}^2/2c^2) dt \Longrightarrow$

recover action $\frac{1}{c}\int \sqrt{-g} d^4x\, (-\rho) \approx \boxed{\int dt dV \left(\rho_{\rm m} \boldsymbol{v}^2/2 - \mathcal{F}_0\right)}$

Fluid with microstructure: Cosserat approach

Along with u^i fluid elements carry triad b^i_A, $A = 1, 2, 3$. Together

$$h^i_\alpha = \left\{u^i, b^i_{\hat{1}}, b^i_{\hat{2}}, b^i_{\hat{3}}\right\}$$

– *material frame* of fluid's elements. It's evolution encoded in

$$\Omega^\alpha{}_\beta := h^\alpha_i u^k \nabla_k h^i_\beta$$

generalized acceleration tensor $[\Omega^\alpha{}_\beta] = 1/$s. Here $h^\alpha_i h^i_\beta = \delta^\alpha_\beta$.
Standard orthogonality and normalization assumed:

$$g_{ij} u^i u^j = c^2, \qquad g_{ij} u^i b^j_A = 0$$

Explicitly components of inverse material frame

$$h^{\hat{0}}_i = \frac{1}{c^2} u_i, \qquad h^A_i = \left\{h^{\hat{1}}_i, h^{\hat{2}}_i, h^{\hat{3}}_i\right\},$$

where $h^A_i := g_{ij} g^{AB} b^j_B$. Obviously $u^i h^A_i = 0$ (or $u^i h^\alpha_i = \delta^\alpha_{\hat{0}}$), thus

$$h^i_A h^B_i = \delta^B_A, \qquad h^i_A h^A_j = \delta^i_j - \frac{1}{c^2} u^i u_j$$

Ideal fluid $\Longrightarrow$ spin fluid

Now, *specific spin density* ${\mu^\alpha}_\beta$ characterizes fluid's elements. Internal energy density $\rho(\nu, s, {\mu^\alpha}_\beta)$, and Gibbs law generalized

$$Tds = d\left(\frac{\rho}{\nu}\right) + p\,d\left(\frac{1}{\nu}\right) - \frac{1}{2}{\omega^\alpha}_\beta\,d{\mu^\beta}_\alpha,$$

where T – temperature, p – pressure, chemical potential ${\omega^\alpha}_\beta$ conjugate to ${\mu^\alpha}_\beta$. Total Lagrangian of spin fluid

$$L^{\rm m} = -\,\rho(\nu, s, {\mu^\alpha}_\beta) - \frac{1}{2}\nu{\mu^\alpha}_\beta{\Omega^\beta}_\alpha + L^{\rm c}$$

Constraints imposed by means of Lagrange multipliers:

$$L^{\rm c} = -\,\nu u^i\nabla_i\lambda_1 + \lambda_2 u^i\nabla_i X + \lambda_3 u^i\nabla_i s$$
$$+\lambda_0(g_{ij}u^iu^j - c^2) + \lambda^A g_{ij}u^ib^j_A + \lambda^{AB}(g_{ij}b^i_Ab^j_B - g_{AB})$$

Physical variables: $\{u^i, b^i_A, \nu, s, X, {\mu^\alpha}_\beta, \lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda^A, \lambda^{AB}\}$

Spin fluid dynamics: Euler-Lagrange equations

Variation of action wrt fluid variables yields equations of motion. Define $\dot\Phi = \nabla_i(u^i\Phi)$. Introduce spin and momentum density

$$\mathcal{S}^i{}_j = \frac{1}{2}\nu\mu^\alpha{}_\beta h^i_\alpha h^\beta_j\,, \qquad \mathcal{P}_k = \frac{1}{c^2}\left(\rho u_k - 2u^j\dot{\mathcal{S}}_{kj}\right)$$

Euler-Lagrange equations encompass

$$\begin{aligned} \dot{\mathcal{S}}^{ij} - \frac{u^i u_k}{c^2}\dot{\mathcal{S}}^{kj} - \frac{u^j u_k}{c^2}\dot{\mathcal{S}}^{ik} &= 0, \\ \omega_{\alpha\beta} + \Omega_{\alpha\beta} &= 0 \end{aligned}$$

Canonical tensors of spin and energy-mom. ($p = \nu\frac{\partial\rho}{\partial\nu} - \rho$):

$$S_{ij}{}^k = u^k\mathcal{S}_{ij}\,, \qquad T_i{}^k = u^k\mathcal{P}_i - p\left(\delta^k_i - \frac{u_i u^k}{c^2}\right)$$

Noether theorem for energy-mom. and angular momentum

$$\nabla_k T_i{}^k = 0, \qquad \nabla_k S_{ij}{}^k + T_{[ij]} = 0$$

Hyperfluid: micromorphic medium

- Early work: YNO, Tresguerres, *Phys. Lett.* **A184** (1993) 17
- Element described by 4-velocity u^i and attached *triad* b^i_A, $A = 1, 2, 3$. Legs *not* orthonormal: $g_{AB} = g_{ij} b^i_A b^j_B$ nontrivial
- Together (u^i, b^i_A) comprise *deformable* material frame h^i_α
- Internal properties: *particle density* ν, *specific entropy* s, *Lin (identity) variable* X, *specific hypermomentum* $\mu^\alpha{}_\beta$
- Total hypermomentum $\mathcal{J}^i{}_j = \frac{1}{2}\nu\mu^\alpha{}_\beta h^\beta_j h^i_\alpha$ comprises spin density $\mathcal{S}_{ij} = \mathcal{J}_{[ij]}$, dilation density $\mathcal{J} = \mathcal{J}^i{}_i$, and tensor of proper hypermomentum density $\not{\mathcal{J}}_{ij} = \mathcal{J}_{(ij)} - g_{ij}\mathcal{J}/4$.
- Improved hyperfluid model developed recently: YNO, Hehl, *Phys. Rev. D* **108** (2023) 104044
- Application: metric-affine gravity; geometric description of macroscopic dynamics of hadron matter on the basis of infinite $SL(4, R)$ representations (Ne'eman-Sijacki, 1988)

Conclusions and Outlook

- Why models of matter with microstructure are of interest?
- Spin is a fundamental property, playing a central role in the physical phenomena in high-energy and condensed matter
- Variational model developed for ideal relativistic spin fluid in framework of Cosserat approach to continuous media
- Direct applications to ferromagnetics, liquid crystals, defects, cracked and granular materials, heavy-ion collision physics
- History: phenomenologic (Weyssenhoff, 1947); variational (Halbwachs, 1960); Ray-Smalley, 1982; Kopczynski, 1986; YNO-Korotky, 1987; YNO-Tresguerres, 1993
- New: Euler (field-theoretic) formalism instead of Lagrange; solved important issue of spin *supplementary condition* (SC) – free choice of SC, not fixed as Frenkel or Tulczyjew etc

Thanks !