Relativistic quantum-mechanical description of acceleration of charged vortex particles by a uniform electric field

Релятивистское квантово-механическое описание ускорения заряженных закрученных частиц однородным электрическим полем

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Стационарные собственные волновые функции заряженного пучка Лагерра-Гаусса, ускоренного в однородном электрическом поле, строго рассчитаны в рамках релятивистской квантовой механики. Детально определена эволюция параметров пучка в процессе ускорения. Предложена простая классическая модель этой эволюции. Ускорение в чрезвычайной степени подавляет поперечное расплывание пучка. Наши результаты показывают, что пучки закрученных частиц можно ускорять, не нарушая их внутренних вихревых свойств.

Stationary wave eigenfunctions of a charged Laguerre-Gauss beam accelerated in a uniform electric field are rigorously derived in the framework of relativistic quantum mechanics. The evolution of beam parameters during acceleration is calculated in detail. The simple classical model of this evolution is proposed. The acceleration extraordinary suppresses transverse spreading of the beam. Our results show that vortex particle beams can be accelerated without destroying their intrinsic vortex properties.

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Vortex (twisted) particles possess the intrinsic orbital angular momentum (OAM). Vortex electrons are widely used in fundamental and applied physics (see Refs. [1–10] and references therein). Since an applicability of vortex electrons and other vortex particles significantly depends on their energy ranges, a successful acceleration of vortex beams would be an important achievement. The simplest possibility is the use of a uniform electric field [11]. For this purpose, an axisymmetric electromagnetic lens can also be used [12]. The injection and acceleration of electrons can also be carried out in a twisted laser beam [13, 14] and a RF field [15]. Production of high-energy vortex particle beams is now one of the most actual problems of physics of vortex particles [16–21]. However, the acceleration of *relativistic* particles is mostly studied with the Schrödinger equation [11,12,15] and the Dirac equation has been used only in Ref. [14]. Unfortunately, the last investigation is based on the non-normalizable Volkov-Bessel wave function. In contrast, we use the well-known normalizable Laguerre-Gauss (LG) wave function and fulfill a high-precision relativistic quantum-mechanical description of the acceleration of charged vortex particles needed for designing and implementing present and future [13–15, 22] high-energy physics experiments.

We use the system of units $\hbar = 1,\ c = 1$ with $\hbar$ and c explicitly included when this inclusion clarifies the problem.

To perform a rigorous quantum-mechanical analysis, we use the initial Dirac-Pauli Hamiltonian for a charged particle in a uniform electric field collinear to the z axis:

$$\begin{aligned} \mathcal{H} = \beta m + \mathcal{E} + \mathcal{O}, \qquad & \mathcal{E} = e\Phi, \\ \mathcal{O} = c\boldsymbol{\alpha}\cdot\boldsymbol{p} + i\mu'\boldsymbol{\gamma}\cdot\boldsymbol{E}, \qquad & \Phi = \Phi_0 - E_z z. \end{aligned} \tag{1}$$

Here m is the particle rest mass, μ' is an anomalous magnetic moment and standard denotations of the Dirac matrices (see Ref. [23]) are applied. A particle is accelerated along the z axis when $eE_z = |eE_z| > 0$.

Our analysis is applicable for an infinite space with a uniform electric field and also covers the transition of a charged particle beam from the free space or a solenoid to the electric field region. In the latter case, we use the "hard-edge" approximation based on the sharp boundary between two media at $z = 0$. As follows from quantum mechanics, the wave function and its first derivative should be continuous on the boundary. In particular, the beam width and its first derivative, $w(0)$ and $w'(0)$, should be continuous.

The Schrödinger picture of relativistic quantum mechanics can be obtained by the relativistic Foldy-Wouthuysen (FW) transformation [24–28]. The weak-field approximation is applicable when all terms in the FW Hamiltonian containing commutators of $\mathcal{E}$ are much less than mc^2. When $[\mathcal{O}, \mathcal{E}] = 0$, the FW transformation is exact [24]. Therefore, the term $\mathcal{E}$ is not covered by the weak-field approximation. When this approximation is applied, the relativistic FW transformation for a particle in a uniform electric field results in [24, 28]

$$\frac{\partial\psi_{FW}}{\partial t} = \mathcal{H}_{FW}\psi_{FW}, \quad \mathcal{H}_{FW} = \beta\epsilon + e\Phi + \left(\frac{\mu_0 m}{\epsilon+m} + \mu'\right)\frac{1}{\epsilon}\Big[\boldsymbol{\Sigma}\cdot(\boldsymbol{p}\times\boldsymbol{E})\Big], \tag{2}$$

where $\mu_0 = e\hbar/(2m)$ is the Dirac magnetic moment and $\epsilon = \sqrt{m^2 + \boldsymbol{p}^2}$.

We consider stationary states ($\mathcal{H}_{FW}\psi_{FW} = \mathbb{E}_0\psi_{FW}$). They can be realized only in stationary external fields. Any vortex beam is a continuum of coherent partial de Broglie waves with the same total energy. In stationary states, the total energy is constant, and the coherence of the beam is not violated. As a result, stationary LG beams remain coherent in any stationary electric, magnetic, and gravitational fields. We do not consider nonstationary LG beams studied in Refs. [29, 30].

As a rule, practically used vortex beams satisfy the paraxial approximation ($|\boldsymbol{p}_\perp| \ll |p_z|$). In this case, a transition to the second-order paraxial equation is rather convenient. It can be fulfilled with any required precision [28, 31, 32], and we obtain [28]

$$\left[\left(i\frac{\partial}{\partial t} - \mathfrak{E}\right)^2 - \boldsymbol{p}^2 - m^2\right]\psi = 0, \qquad \mathfrak{E} = e\Phi + \left(\tfrac{\mu_0 m}{\epsilon+m} + \mu'\right)\tfrac{1}{\epsilon}\Big[\boldsymbol{\Sigma}\cdot(\boldsymbol{p}\times\boldsymbol{E})\Big]. \tag{3}$$

For stationary solutions,

$$\left[(\mathbb{E}_0 - \mathfrak{E})^2 - \mathbf{p}^2 - m^2\right]\psi = 0, \qquad \mathbb{E}_0 = \epsilon_0 + e\Phi_0 = \sqrt{m^2 + p_0^2} + e\Phi_0, \tag{4}$$

where $\mathbb{E}_0$ is the conserved total energy. The spin-dependent term in the formula for $\mathfrak{E}$ is rather small compared with $e\Phi$ and can be neglected ($\mathfrak{E} = e\Phi$). One needs to introduce the wavenumbers $k \equiv k(z) = p/\hbar$ and $k_0 = p_0/\hbar$. Here

$$k(z) = k_0\sqrt{1 + 2K_1 z + K_2^2 z^2} \tag{5}$$

and

$$K_1 = \frac{\epsilon_0|eE_z|}{c^2p_0^2}, \qquad K_2 = \frac{|eE_z|}{cp_0}. \tag{6}$$

Next necessary derivations have been carried out in Ref. [33]. The paraxial equation takes the form

$$\left[\boldsymbol{\nabla}_\perp^2 + 2ik(z)\frac{\partial}{\partial z}\right]\Psi = 0. \tag{7}$$

In free space ($k = const$), the paraxial equation has the following solution for LG beams [34–36]:

$$\begin{gathered}\Psi = \mathbb{A}\exp(i\Phi), \qquad \int \Psi^\dagger\Psi r dr d\phi = 1,\\ \mathbb{A} = \tfrac{C_{n\ell}}{w(z)}\left(\tfrac{\sqrt{2}r}{w(z)}\right)^{|\ell|} L_n^{|\ell|}\left(\tfrac{2r^2}{w^2(z)}\right)\exp\left(-\tfrac{r^2}{w^2(z)}\right)\eta,\\ \Phi = l\phi + \tfrac{kr^2}{2R(z)} - \Phi_G(z), \qquad C_{n\ell} = \sqrt{\tfrac{2n!}{\pi(n+|\ell|)!}},\end{gathered} \tag{8}$$

where

$$\begin{gathered}w(z) = w_0\sqrt{1 + \tfrac{z^2}{z_R^2}}, \quad R(z) = z + \tfrac{z_R^2}{z}, \quad z_R = \tfrac{kw_0^2}{2},\\ \Phi_G(z) = N\arctan\left(\tfrac{z}{z_R}\right), \quad N = 2n + |\ell| + 1,\end{gathered} \tag{9}$$

the real functions $\mathbb{A}$ and Φ define the amplitude and phase, k is the beam wavenumber, w_0 is the beam waist (minimum beam width), $R(z)$ is the radius of curvature of the wavefront, $\Phi_G(z)$ is the Gouy phase, z_R is the Rayleigh diffraction length, $L_n^{|\ell|}$ is the generalized Laguerre polynomial, and $n=0,1,2,\dots$ is the radial quantum number. The spin function η is an eigenfunction of the Pauli operator σ_z: $\sigma_z\eta^\pm=\pm\eta^\pm$, $\eta^+=\begin{pmatrix}1\\0\end{pmatrix}$, $\eta^-=\begin{pmatrix}0\\1\end{pmatrix}$. Ψ is a spinor and, evidently, is not an eigenfunction of the operator p_z. Therefore, the free-space wave function (8), (9) characterizes a beam formed by partial waves with different p_z.

The derivation shows that the eigenfunction of the LG beam in the uniform electric field has the form (8) with replacing $k\rightarrow k_0$ in the phase Φ [33]. However, the beam parameters are not defined by Eq. (9) and the calculations result in [33]

$$\begin{gathered}
w(z)=w_0\sqrt{\left(1+\frac{2A(z)w_0'}{K_2w_0}\right)^2+\frac{16A(z)^2}{k_0^2K_2^2w_0^4}},\\
R(z)=\frac{2A(z)}{K_2}+\frac{k_0^2w_0^3\left[2A(z)w_0'+K_2w_0\right]}{k_0^2w_0^2w_0'\left[2A(z)w_0'+K_2w_0\right]+8A(z)},\\
\Phi_G(z)=\Phi_G(0)+N\operatorname{arccot}\left[\frac{k_0w_0w_0'}{2}+\frac{k_0K_2w_0^2}{4A(z)}\right].\\
A(z)=\operatorname{arctanh}\left[\frac{K_2}{2K_1+K_2^2z}\left(\frac{k(z)}{k_0}-1\right)\right].
\end{gathered}\tag{10}$$

Here N is defined by Eq. (9) and the boundary conditions are $w(0)=w_0$, $w'(0)=w_0'$. For $z>0$, the beam is accelerated and the argument of the arctanh function in Eq. (10) lies within the interval (0,1), ensuring that $A(z)$ remains real. The equation for the beam waist can be simplified for $w_0'=0$:

$$w(z)=w_0\sqrt{1+\frac{4}{k_0^2w_0^4K_2^2}\ln^2\left|\frac{K_1+K_2^2z+K_2\sqrt{1+2K_1z+K_2^2z^2}}{K_1+K_2}\right|}.\tag{11}$$

It has been proven [33] that the beam undergoing initial focusing for $w_0'<0$ reaches a minimum width of

$$w_{\min}=\frac{w_0}{\sqrt{1+\frac{1}{4}k_0^2w_0^2w_0'^2}}.\tag{12}$$

The corresponding focal position is given by

$$z_{\mathrm{f}}=\frac{K_1}{K_2^2}\left[\cosh\left(\frac{k_0^2K_2w_0^3w_0'}{4+k_0^2w_0^2w_0'^2}\right)-1\right]-\frac{1}{K_2}\sinh\left(\frac{k_0^2K_2w_0^3w_0'}{4+k_0^2w_0^2w_0'^2}\right).\tag{13}$$

Two focuses exist for $w_0'\neq0$. When $w_0'>0$, the beam has a real focus for its free-space part and an imaginary one for an electric-field part. Everything is the opposite for $w_0'<0$.

An analog of the Rayleigh range in the electric field, defined as the propagation length over which the beam's cross-sectional area doubles from the focus, is given by [33]

$$z_R^{(E)}=\frac{2K_1}{K_2^2}\sinh\left(\frac{1}{4}k_0K_2w_0^2\right)^2+\frac{\sinh\left(\frac{1}{2}k_0K_2w_0^2\right)}{K_2}.\tag{14}$$

This Rayleigh range is extended as compared to the free-space case.

Calculated beam behavior can be properly explained in the framework of classical particle physics [33]. A paraxial partial de Broglie wave with the transverse momentum $\boldsymbol{p}_\perp$ and the total momentum $p \approx p_z$ describes a particle with the same parameters. At the waist, a LG beam moving in free space can be modeled by a continuum of coherent particle states with identical p_ϕ and p and various ϕ. The distribution of probability of the particle position azimuth is uniform with respect to ϕ. Let r_0 be the distance between the particle and the axis of symmetry of the beam. In the transverse plane, any particle moves freely with constant velocity. At the waist, $v_r = 0$ and the direction x of its movement is orthogonal to $\boldsymbol{r}$: $dx = v_\phi dt$. This direction remains unchanged all the time. After time t, the azimuth of the particle position changes, the radial velocity becomes nonzero, and the distance to the axis of symmetry increases: $r(t) = \sqrt{r_0^2 + v_\phi^2 t^2}$. To check the compatibility with wave theory, we need to determine the connection between r and z. Since $dz = v_z dt \approx v dt$ and $v_\phi/v = p_\phi/p$, we obtain

$$r(z) = \sqrt{r_0^2 + \frac{p_\phi^2 z^2}{p^2}}. \tag{15}$$

Comparison with Eq. (9) shows the equivalence of two approaches at $r_0 = w_0$, $p_\phi = 2\hbar/w_0$.

If the beam waist coordinate is $z = 0$ and the beam is accelerated at $z \geq 0$, then $p = \hbar k(z)$ and p_ϕ remains unchanged. However, v_ϕ decreases in classical and quantum-mechanical pictures because $v_\phi = p_\phi/\epsilon(z)$. In this case, $dx = [p_\phi/\epsilon(z)]dt = [p_\phi/p(z)]dz$. After integration, we obtain

$$x = \frac{p_\phi}{p_0}\int_0^z \frac{dz}{\sqrt{1+2K_1z+K_2^2z^2}} = \frac{p_\phi}{p_0K_2}\ln\left|\frac{K_1+K_2^2z+K_2\sqrt{1+2K_1z+K_2^2z^2}}{K_1+K_2}\right|. \tag{16}$$

It is amazing that the same substitution as before, $r_0 = w_0$, $p_\phi = 2\hbar/w_0$, results in Eq. (11) for the beam width.

This simple model perfectly explains a fundamental suppression of transverse spreading of the beam in an electric field, but it is approximate. It does not take into account the real quantum-mechanical structure of the LG state. In particular, there is a radial motion even at the waist and the transverse and longitudinal momenta are not definite. Nevertheless, the model perfectly explaines Eq. (11) which describes an evolution of the beam width.

Vortex beams can enter an electric field not only from free space but also from a solenoid. In this case, $w_0' = 0$ for a Landau state, but w_0' can be positive and negative for a LG beam with a spatially oscillating width [9,10]. While an electric field extraordinarily suppresses transverse spreading of the beam, the beam focusing can be important. We suppose that it can be performed with magnetic lenses.

Our results can be directly applied to electrostatic linear accelerators (linacs) [37] accelerating electrons by a static and approximately uniform electric field. They also cover RF linacs with a time-varying RF field [15,22]. When the beam motion is synchronized with the RF oscillations, the beam is governed by an approximately static and uniform electric field acting on moving electrons. Further studies of this regime are necessary and can be considered as an outlook.

It is known that the helical motion of particles in a magnetic field is followed by the radiation of vortex photons and can lead to loosing OAMs [38–46]. Certainly, the decrease of the particle OAM is equal to the total OAM of radiated photons [38,44]. It can be proven [33] that the acceleration of particles in a uniform electric field does not decrease their OAMs. Although the radiation of non-vortex photons cannot change the particle OAM and destroy the beam coherence, but it can influence the wave function. This effect does not occur within the framework of the classical theory of radiation but can appear when the energies of the photon and particle are comparable.

In summary, the practically important problem of acceleration of vortex beams is studied. It is shown that charged particle beams keep their coherence and OAMs in a uniform electric field, despite the emission of photons. Stationary LG wave eigenfunctions of relativistic twisted Dirac fermions accelerated in such a field are rigorously derived in the "hard-edge" approximation. The evolution of the beam in the field is considered in detail. The important effect of extraordinary suppression of transverse spreading of the beam is discovered, carefully analyzed, and appropriately explained. Our results can be successfully used for the acceleration of vortex beams in linacs.

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