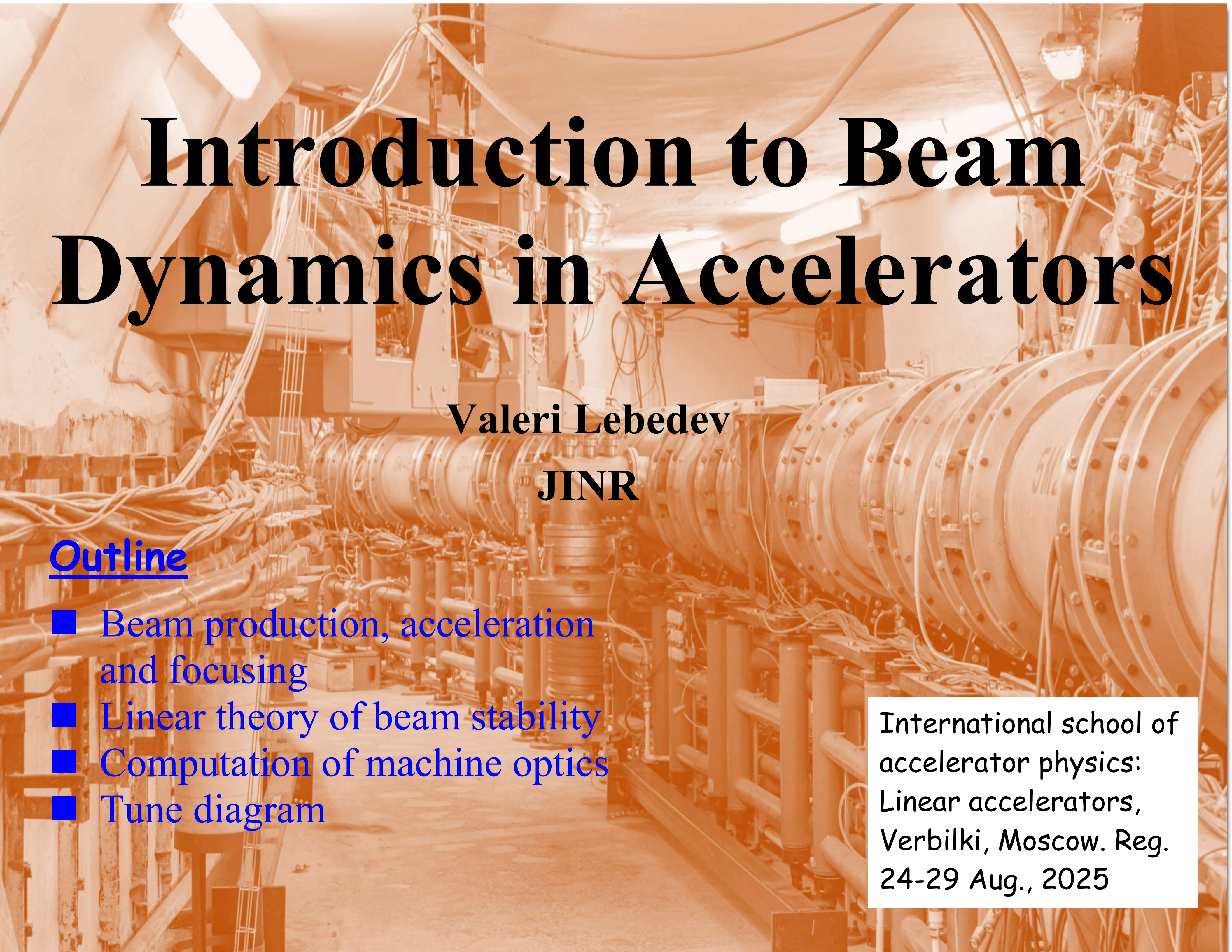

Introduction to Beam Dynamics in Accelerators

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JINR

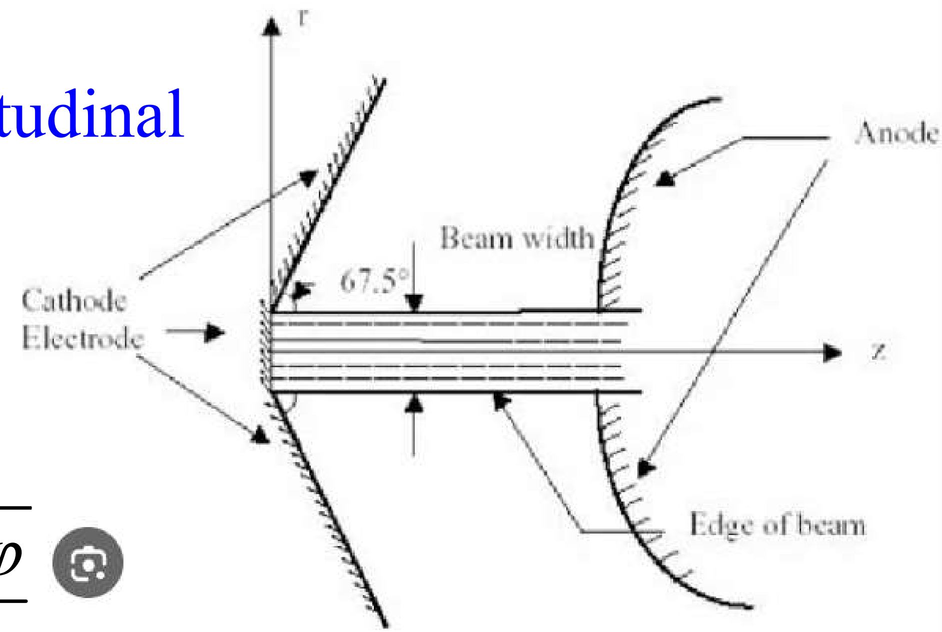
Outline

- Beam production, acceleration and focusing
- Linear theory of beam stability
- Computation of machine optics
- Tune diagram

International school of accelerator physics: Linear accelerators, Verbilki, Moscow. Reg. 24-29 Aug., 2025

Beam Production, Acceleration and Focusing

Pearce Gun - Child–Langmuir Law (3/2 law)



- "Ideal" electron gun does not have transverse electric field – just longitudinal field for particle acceleration
- Poisson's equation

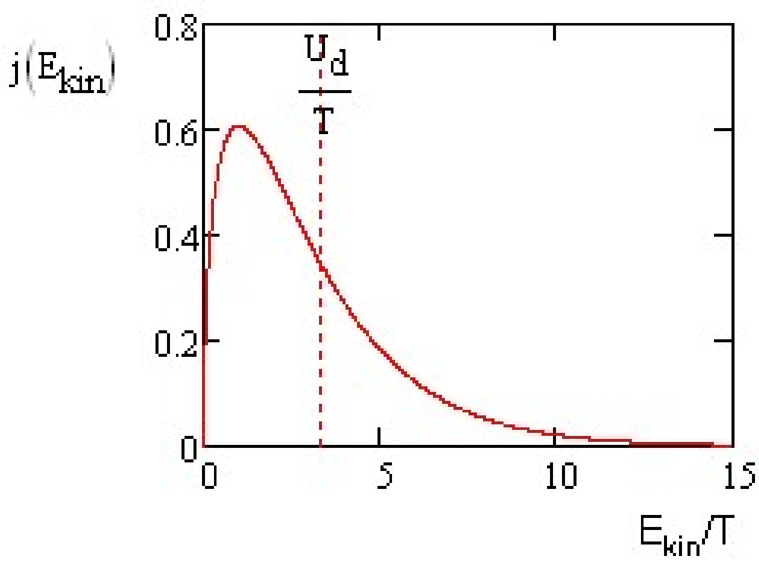
$$\Delta\varphi = 4\pi\rho \Rightarrow \frac{d^2\varphi}{dz^2} = 4\pi\rho$$

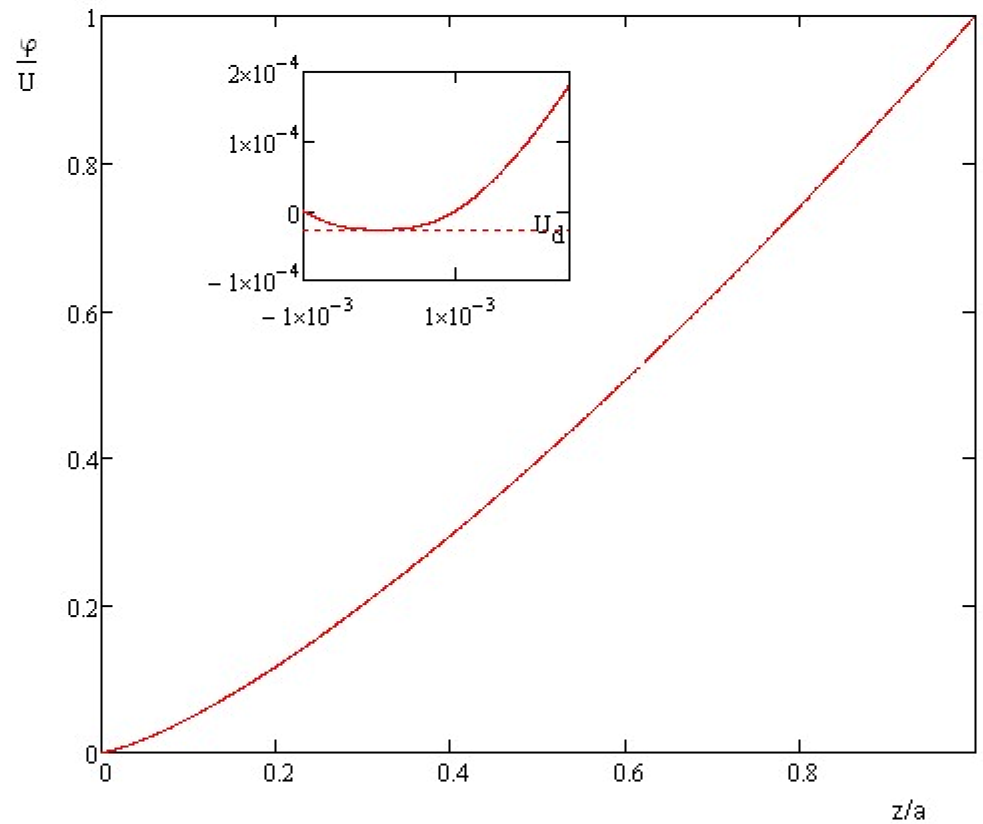
- Discontinuity equation: $j = \rho \mathrm{v} \Rightarrow \mathrm{j}{=}\rho\sqrt{\dfrac{2e\varphi}{m}}$
- Combining: $\dfrac{d^2\varphi}{dz^2} = 4\pi j\sqrt{\dfrac{m}{2e\varphi}}$
- Solution: $\varphi = Az^n$ => $n(n-1)Az^{n-2} = 4\pi j\sqrt{\dfrac{m}{2eAz^n}}$ => $n-2=-n/2$

$$n = \frac{4}{3} \Rightarrow \frac{4}{3}\frac{1}{3}A = 4\pi j\sqrt{\frac{m}{2eA}} \Rightarrow A^{3/2} = 9\pi j\sqrt{\frac{m}{2e}} \Rightarrow \varphi(z) = (9\pi j)^{2/3}\sqrt[3]{\frac{m}{2e}}z^{4/3}$$

- Child-Langmuir (CL) law: $j \propto U^{3/2}$

Virtual Cathode





- Excessive cathode current is reflected by virtual cathode
- The same space charge limitation is for the beam current of ion sources
- Space charge limitation of the beam current reduces shot noise

Electrostatic Axial-symmetric Focusing

- Laplace's equation

$$\Delta\varphi = 0 \Rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right) + \frac{\partial^2\varphi}{\partial z^2} = 0 \Rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(rE_r\right) + \frac{\partial E_z}{\partial z} = 0$$

- For small r we can assume dE_z/dz=const and integrate

$$rE_r \approx -\frac{r^2}{2}\frac{\partial E_z}{\partial z} \Rightarrow rE_r \approx -\frac{r^2}{2}\frac{\partial E_z}{\partial z} \Rightarrow \boxed{E_r \approx -\frac{r}{2}\frac{\partial E_z}{\partial z}}$$

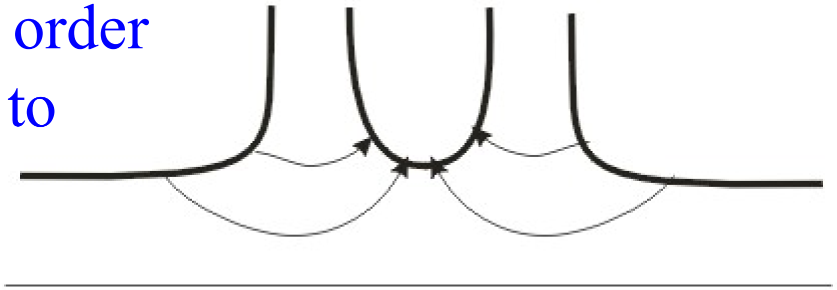
- Defocusing at the Pearce gun exit

$$\Delta p = e\int E_r dt \approx \frac{e}{\mathrm{v}_0}\int E_r dz \approx -\frac{e}{\mathrm{v}_0}\int \frac{r}{2}\frac{\partial E_z}{\partial z}dz \approx -\frac{er}{2\mathrm{v}_0}E_0$$

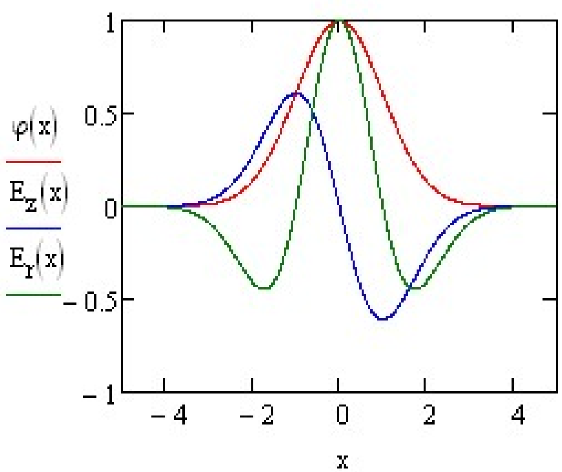
$$\theta = -\frac{r}{F} = \frac{\Delta p}{p} \approx \frac{1}{p}\frac{er}{2\mathrm{v}_0}E_0 \xrightarrow{eU = m\mathrm{v}_0^2/2} \frac{rE_0}{4U} \Rightarrow \boxed{\frac{1}{F} = -\frac{E_0}{4U}}$$

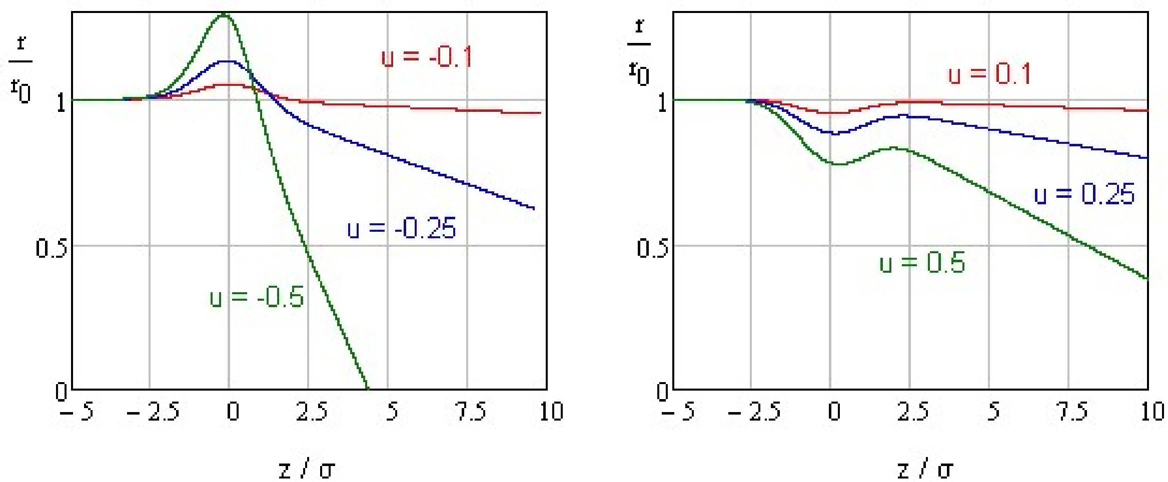
Electrostatic Axial-symmetric Focusing (2)

- Axial symmetric electrostatic lens
  - $\int E_r dz = 0$ => no focusing in the 1st order
  - Two contributions: Imbalances due to
    - change of longitudinal velocity
    - change of radial position



- Introduce: *U*/*U*$_0$=u(z) and assume $u = u_0 \exp(-z^2 / 2\sigma^2)$





Numerical solution in paraxial approximation

Magnetic Axial Symmetric Focusing

- Motion equations in cylindrical frame in paraxial approximation

$$\begin{cases} \dfrac{dp_r}{dt} = \dfrac{m\gamma \mathrm{v}_\theta^{\;2}}{r} + F_r \\ \dfrac{dM}{dt} \equiv \dfrac{d\left(rp_\theta\right)}{dt} = rF_\theta \end{cases} \Rightarrow \begin{cases} \dfrac{d^2r}{dt^2} = \dfrac{1}{r}\left(r\dfrac{d\theta}{dt}\right)^2 + \dfrac{e}{mc\gamma}\left(\left(r\dfrac{d\theta}{dt}\right)B_z - \mathrm{v}_z B_\theta\right) \\ \dfrac{1}{r}\dfrac{d}{dt}\left(r^2\dfrac{d\theta}{dt}\right) = \dfrac{e}{mc\gamma}\left(\mathrm{v}_z B_r - \mathrm{v}_r B_z\right) \end{cases}$$

$$\mathrm{v}_\theta = r\frac{d\theta}{dt}, \quad p_\theta = m\gamma r\frac{d\theta}{dt}, \quad B_\theta = 0, \quad B_r = -\frac{1}{2}\frac{dB_z}{dz}$$

- Transiting from t to z and substituting B_r

$$\begin{cases} \dfrac{d^2r}{dz^2} = r\left(\dfrac{d\theta}{dz}\right)^2 + \dfrac{eB_z}{mc^2\beta\gamma} r\dfrac{d\theta}{dz} \\ \dfrac{1}{r}\dfrac{d}{dz}\left(r^2\dfrac{d\theta}{dz}\right) = -\dfrac{e}{mc^2\beta\gamma}\left(\dfrac{r}{2}\dfrac{dB_z}{dz} + \dfrac{dr}{dz}B_z\right) \end{cases}$$

The last equation is easily integrated $r^2\dfrac{d\theta}{dz} + \dfrac{er^2B_z}{2mc^2\beta\gamma} = const$ (Busch's theorem)

For zero initial transverse velocity one obtains: $\dfrac{d\theta}{dz} = -\dfrac{eB_z}{2mc^2\beta\gamma}$

$$\frac{d^2r}{dz^2} = r\left(\frac{eB_z}{2mc^2\beta\gamma}\right)^2 - \frac{1}{2}\left(\frac{eB_z}{mc^2\beta\gamma}\right)^2 r = -\left(\frac{eB_z}{2mc^2\beta\gamma}\right)^2 r \quad \Rightarrow \quad \boxed{\frac{1}{F} \equiv -\frac{1}{r}\frac{dr}{dz}\bigg|_{fin} = \int\left(\frac{eB_z}{2mc^2\beta\gamma}\right)^2 dz}$$

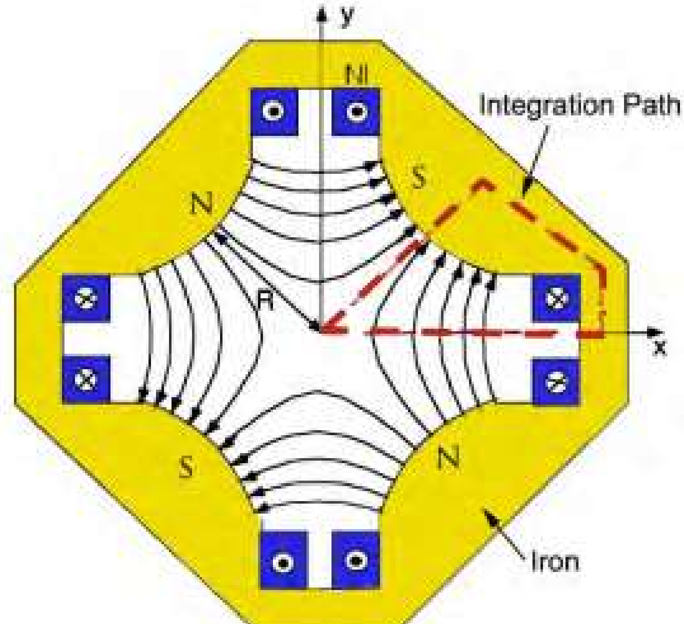
Focusing is proportional to B^2

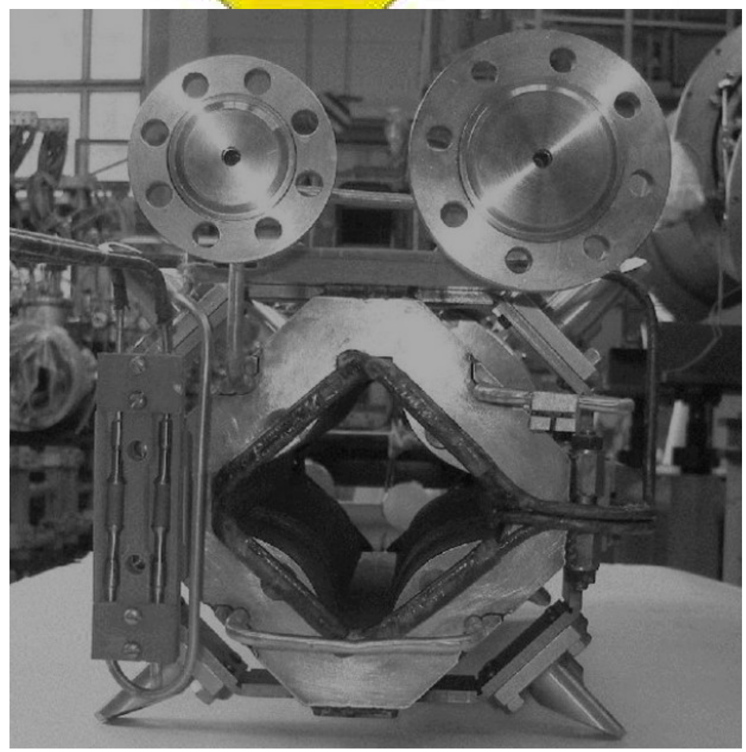
Quadrupole Focusing

Scalar potential: $\varphi \propto x^2 - y^2$

Magnetic field: $\begin{bmatrix} B_x \\ B_y \end{bmatrix} = G \begin{bmatrix} y \\ -x \end{bmatrix}$

Force: $\dfrac{d}{dt}\begin{bmatrix} p_x \\ p_y \end{bmatrix} = e\beta G \begin{bmatrix} -x \\ y \end{bmatrix}$





Angle change in the thin-lens approximation

$$\begin{bmatrix} \delta\theta_x \\ \delta\theta_y \end{bmatrix} = \frac{e\beta G}{p_0}\frac{L}{\beta c}\begin{bmatrix} -x \\ y \end{bmatrix}$$

$$\Rightarrow \quad \Phi_{x,y} \equiv 1/F_{x,y} = \frac{e(GL)}{p_0 c}$$

Equation of motion in paraxial approximation

$$\frac{d^2}{ds^2}\begin{bmatrix} x \\ y \end{bmatrix} = \frac{eG}{p_0 c}\begin{bmatrix} -x \\ y \end{bmatrix}$$

Focusing is proportional to G, i.e. B, but defocusing is present

Quadrupole Focusing (2)

Solution of motion equations

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} A\cos(kx) + B\sin(kx) \\ C\,\mathrm{ch}(kx) + D\,\mathrm{sh}(kx) \end{bmatrix}, \quad k^2 = \frac{eG}{p_0 c}$$

Transfer matrix

$$\begin{bmatrix} x \\ \theta_x \end{bmatrix}_2 = \begin{bmatrix} \cos(kx) & k^{-1}\sin(kx) \\ -k\sin(kx) & \cos(kx) \end{bmatrix} \begin{bmatrix} x \\ \theta_x \end{bmatrix}_1, \quad \begin{bmatrix} y \\ \theta_y \end{bmatrix}_2 = \begin{bmatrix} \mathrm{ch}(kx) & k^{-1}\mathrm{sh}(kx) \\ k\,\mathrm{sh}(kx) & \mathrm{ch}(kx) \end{bmatrix} \begin{bmatrix} y \\ \theta_y \end{bmatrix}_1$$

In agreement with the Liouville theorem $\det \mathbf{M} = 1$

Transfer matrix of empty space

$$M_O = \begin{bmatrix} 1 & L \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} x \\ \theta_x \end{bmatrix}_2 = \begin{bmatrix} x + L\theta_x \\ \theta_x \end{bmatrix}$$

Transfer matrix of triplet (thin lens approximation)

$$\begin{pmatrix} 1 & 0 \\ -\Phi & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 2\cdot\Phi & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -\Phi & 1 \end{pmatrix} = \begin{bmatrix} 1 - 2\cdot L^2\cdot\Phi^2 & 2\cdot L\cdot(L\cdot\Phi + 1) \\ -2\cdot L\cdot\Phi^2\cdot(1 - L\cdot\Phi) & 1 - 2\cdot L^2\cdot\Phi^2 \end{bmatrix}$$

Overall focusing still goes as B^2

Linear Theory of Beam Stability

Equations for Uncoupled Motion

- Linearized equation of motion

$$x'' + \left(K_x^{\,2} + k\right)x = 0$$

where: $K_x(s) \equiv K_x = eB_y(s)/Pc$, $k(s) \equiv k = eG(s)/Pc$

- In Hamiltonian form

$$\begin{cases} \dfrac{dx}{ds} = \dfrac{\partial H}{dp} \\ \dfrac{dp}{ds} = -\dfrac{\partial H}{dx} \end{cases} \quad \text{with} \quad H = \frac{p^2}{2} + \left(K_x^{\,2} + k\right)\frac{x^2}{2}$$

- General solution of 2-nd order linear equation

$$x(s) = C(s)x(0) + S(s)\theta(0)\,, \quad \theta(s) \equiv dx/ds$$

where $C(s)$ and $S(s)$ two linear independent solutions

We can rewrite it in matrix form

$$\begin{bmatrix} x(s) \\ \theta(s) \end{bmatrix} = \begin{bmatrix} M_{11}(s) & M_{12}(s) \\ M_{21}(s) & M_{22}(s) \end{bmatrix} \begin{bmatrix} x(0) \\ \theta(0) \end{bmatrix} \quad \text{or} \quad \boxed{\mathbf{x}(s) = \mathbf{M}(s)\mathbf{x}(0)}$$

Conservation of the Phase Space Volume

Jacobian does not depend on time

$$\frac{d}{ds}\left(\frac{\partial(p,q)}{\partial(p_0,q_0)}\right)=\frac{d}{ds}\left(\begin{vmatrix}\frac{\partial}{\partial p_0}\left(p_0+\frac{dp}{ds}ds\right) & \frac{\partial}{\partial p_0}\left(x_0+\frac{dx}{ds}ds\right)\\ \frac{\partial}{\partial x_0}\left(p_0+\frac{dp}{ds}ds\right) & \frac{\partial}{\partial x_0}\left(x_0+\frac{dx}{ds}ds\right)\end{vmatrix}\right)=\frac{d}{ds}\left(\begin{vmatrix}1-\frac{\partial^2 H}{\partial s\partial p}ds & \frac{\partial^2 H}{\partial p^2}ds\\ \frac{\partial^2 H}{\partial^2 x}ds & 1+\frac{\partial^2 H}{\partial x\partial p}ds\end{vmatrix}\right)=0$$

where we used $\begin{cases}\frac{dx}{ds}=\frac{\partial H}{dp}\\ \frac{dp}{ds}=-\frac{\partial H}{dx}\end{cases}$

⇨ The phase space volume is conserved in the course of motion and, consequently, $|\mathbf{M}|=1$

- The conservation of the phase space volume is also justified for multidimensional motion.
  It is called **Liouville theorem**

Betatron Motion in a Ring

- Arbitrary turn-by-turn betatron motion at a given place may be presented through eigen-vectors

$$\mathbf{x}_n = \mathrm{Re}\left(\Lambda_1^{\,n}\left(A_1\mathbf{v}_1\right) + \Lambda_2^{\,n}\left(A_2\mathbf{v}_2\right)\right) \quad \text{where} \quad \mathbf{M}\mathbf{v}_k = \Lambda_k\mathbf{v}_k\ , \quad k = 1, 2$$

  - Stable betatron motion requires $|\Lambda_k| = 1$ => $\Lambda_2 = \Lambda_1^{\,*}$ (since real $\mathbf{M}$)
  - Introduce betatron frequencies so that $\Lambda_{1,2} = e^{\pm i\mu}$

    Corresponding betatron tune (fractional part): $Q = \mu / 2\pi$

- Description of betatron motion for the entire ring
  - The eigen-vector $\mathbf{v}(s) = \mathbf{M}(0,s)\mathbf{v}$ is the eigen-vector for the total ring transfer matrix for coordinate s.
  - Then we normalize the eigen-vectors so that

    $\mathbf{v}(s) = \mathbf{M}(0,s)\mathbf{v}(0)e^{-i\mu(s)}$

    and require $\mathrm{Im}(\mathrm{v}_{1'}(s)) = 0$ and $\mathbf{v}^+(s)\mathbf{S}\mathbf{v}(s) = -2i$, where $\mathbf{S} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

    Then we can describe the entire ring betatron motion

$$\mathbf{x}(s) = \sqrt{2I}\,\mathrm{Re}\left(e^{i(\psi - \mu(s))}\mathbf{v}\right)$$

where the action I and the betatron phase ψ determine initial part. pos.

The Eigen-vector Parameterization

- Parametrize the eigen-vector

$$\mathbf{v} \equiv \begin{bmatrix} \mathrm{v}_1 \\ \mathrm{v}_2 \end{bmatrix} \equiv \mathbf{v}(s) = \begin{bmatrix} \sqrt{\beta(s)} \\ -\dfrac{i+\alpha(s)}{\sqrt{\beta(s)}} \end{bmatrix} \quad , \quad \begin{cases} \mathbf{v}_1 = \mathbf{v} \\ \mathbf{v}_2 = \mathbf{v}^* \end{cases}$$

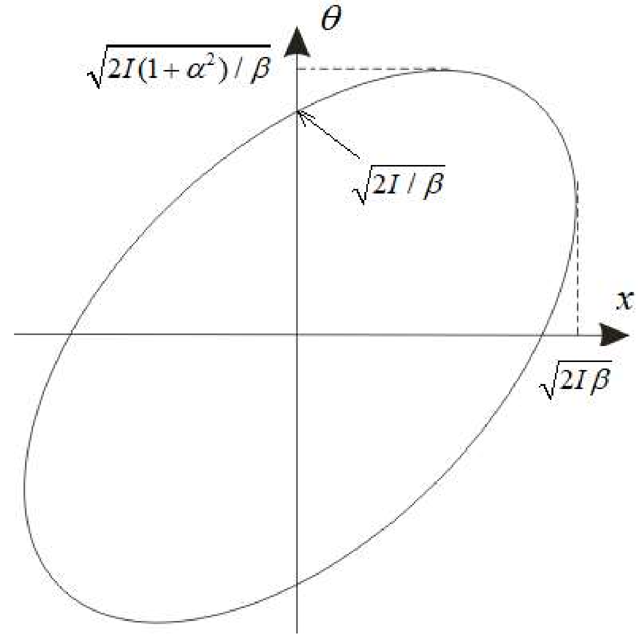
  - we define that $\mathrm{Im}(\mathrm{v}_1(s)) = 0$
  - The eigen-vectors are orthogonal and correctly normalized

$$\begin{cases} \mathbf{v}^+\mathbf{S}\mathbf{v} = \begin{bmatrix} \sqrt{\beta(s)} & \dfrac{i-\alpha(s)}{\sqrt{\beta(s)}} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{\beta(s)} \\ -\dfrac{i+\alpha(s)}{\sqrt{\beta(s)}} \end{bmatrix} = -2i \\ \mathbf{v}^T\mathbf{S}\mathbf{v} = 0 \quad \text{or} \quad \mathbf{v}_2^+\mathbf{S}\mathbf{v}_1 = 0 \end{cases}$$

Courant-Snider Invariant

The betatron amplitude (maximum particle displacement) $=\sqrt{2I\beta}$

The maximum angle $=\sqrt{\frac{2I}{\beta}\left(1+\alpha^2\right)}$



The maximum angle for x=0 is achieved when

$$\sqrt{2I}\,\mathrm{Re}\left(\begin{bmatrix}\sqrt{\beta(s)}\\ -\dfrac{i+\alpha(s)}{\sqrt{\beta(s)}}\end{bmatrix}e^{i\pi/2}\right)=\sqrt{2I}\,\mathrm{Re}\left(\begin{bmatrix}i\\ \dfrac{1-i\alpha(s)}{\sqrt{\beta(s)}}\end{bmatrix}\right)$$

⇨ Local angular spread: $\theta_m=\sqrt{\dfrac{2I}{\beta}}$

- Finding action from the known x and θ

$$\mathbf{v}^+\mathbf{S}\left[\mathbf{x}=\sqrt{2I}\left(\frac{e^{i\psi}\mathbf{v}+CC}{2}\right)\right]\xrightarrow[\text{condition}]{\text{orthogonality}}\mathbf{v}^+\mathbf{S}\mathbf{x}=-i\sqrt{2I}\rightarrow I=\frac{1}{2}\left|\mathbf{v}^+\mathbf{S}\mathbf{x}\right|^2$$

- Courant-Snyder invariant

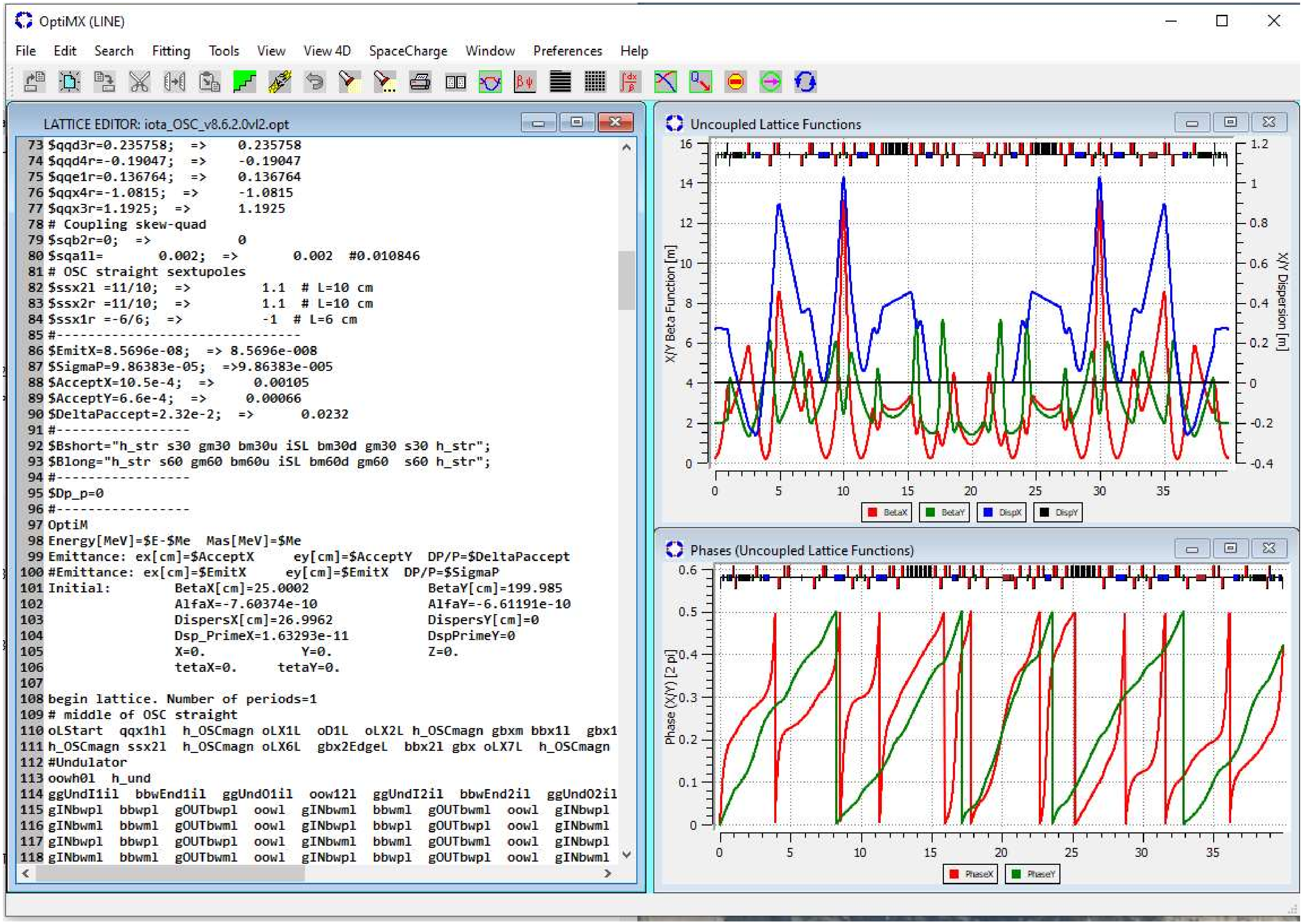
Remember that: $\mathbf{S}=\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$

$$2I=\left|\mathbf{v}^+\mathbf{S}\mathbf{x}\right|^2=\beta\theta^2+2\alpha x\theta+\frac{1+\alpha^2}{\beta}x^2$$

Computation of Machine Optics

Software for Computation of machine optics

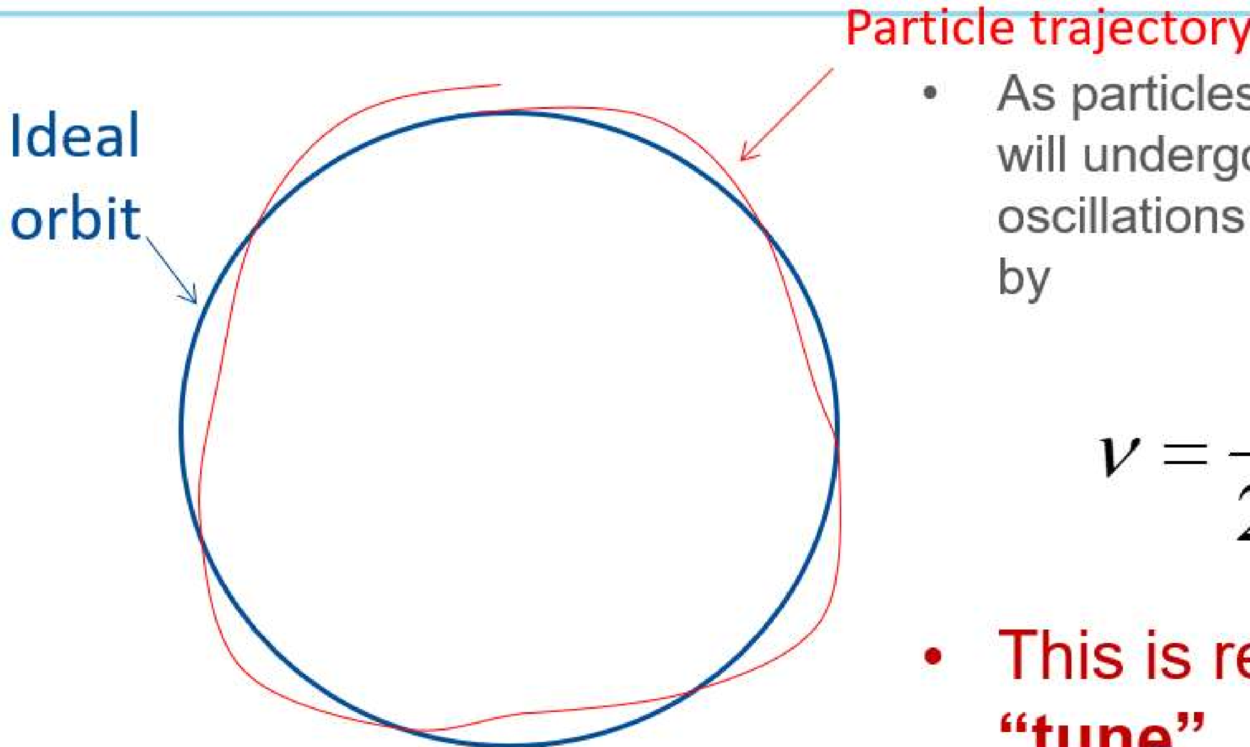
- There are many computer codes allowing one to compute beam optics
- I mention 3 of them
  1. MAD -> MAD-8 -> MADX - supported by CERN
     https://mad.web.cern.ch/mad/
  2. Elegant - supported by ANL
     https://www.aps.anl.gov/Accelerator-Operations-Physics/Software#elegant
  3. OptiMX - supported by Fermilab
     https://home.fnal.gov/~ostiguy/OptiM/ (temporary link because of Fermilab security:
     https://www.dropbox.com/s/56l4nctnwegf7w7/OptimX64-20210526-setup.exe?dl=0)
- In this course we will be using OptiM
  - Interactive, GUI driven, easy to learn
  - Operates on major computer platforms: Windows, Unix, MAC
  - Free installation, Easy to install
  - Online help (documentation)
- Input file consists of:
  - Math header
  - Main body starting from keyword OptiM. It includes: (1) beam parameters, (2) element sequence, (3) parameters of elements, (4) service blocks



Computations can be done in a ring and beam line modes

Tune Diagram

Betatron Oscillations and Tune



- As particles go around a ring, they will undergo a number of betatron oscillations ν (sometimes Q) given by

$$\nu = \frac{1}{2\pi} \oint \frac{ds}{\beta(s)}$$

- **This is referred to as the "tune"**

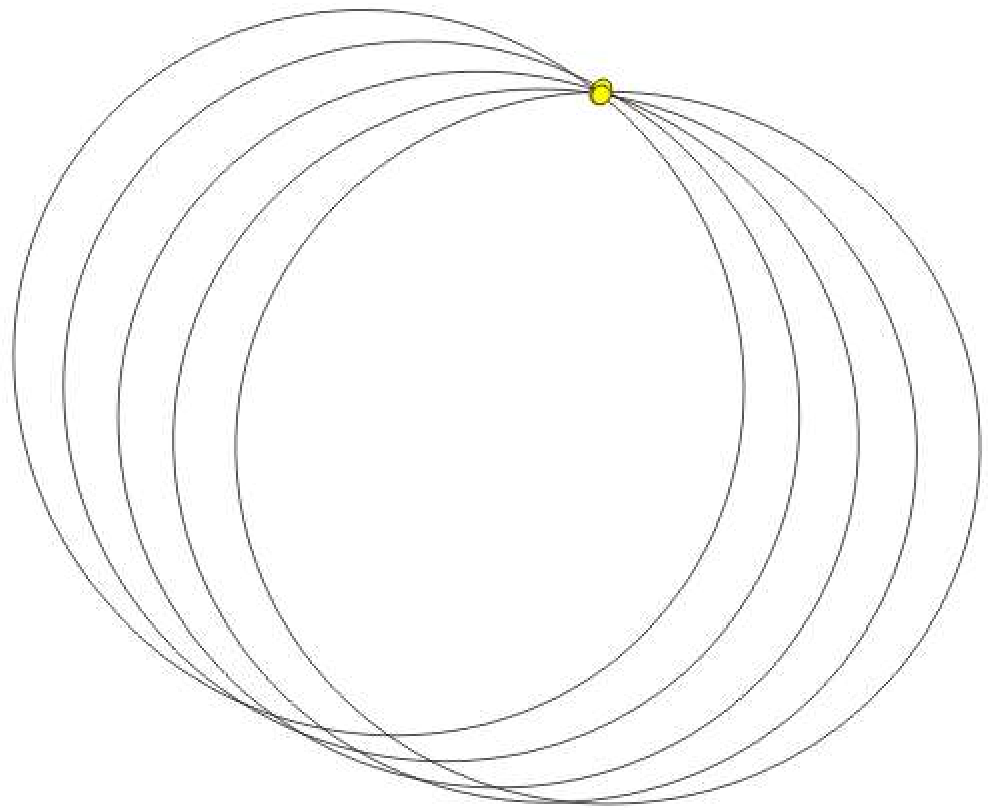
- We can generally think of the tune in two parts:

64.31

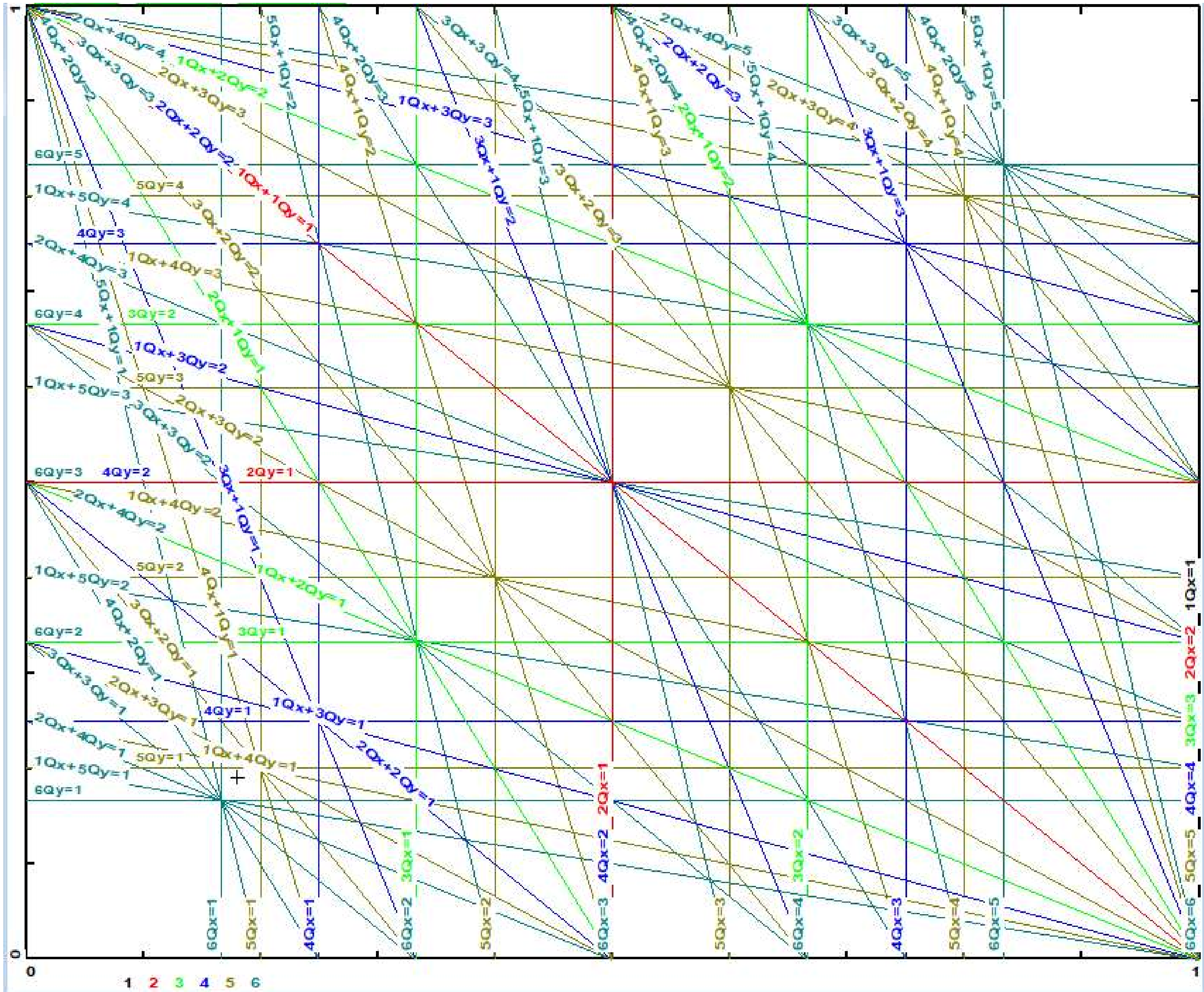
Integer : magnet/aperture optimization

Fraction: Beam Stability

Integer Resonance Example



Tune Diagram



■ Tevatron could see resonances up to 12th order

Problems

1. Find the beam current for Pierce electron gun and the focusing distance of its exit lens: Cathode radius – 1 cm, Anode-cathode distance 3 cm.
   Learn how transit from Gauss system to SI system
2. For uncoupled betatron motion prove that the normalization of eigen-vectors, $\hat{\mathbf{v}}_k^{+}\mathbf{S}\hat{\mathbf{v}}_k = -2i$, yields that $d\mu / ds = 1/\beta$.
$$\mathbf{S} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$
3. Prove that
   if **v** is the eigen-vector for matrix **M** corresponding to the one turn matrix starting at s=0 (point 1)
   then the vector $\mathbf{M}_{12}\,\mathbf{v}$ will be the eigen-vector of the transfer matrix corresponding to the point 2. Here $\mathbf{M}_{12}$ is the transfer matrix from point 1 to point 2.