A method for identifying and calculating a singularity of type 1/(x-c) in a multidimensional divergent integral

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Problem

Evaluate an n-dimensional improper integral of the form:

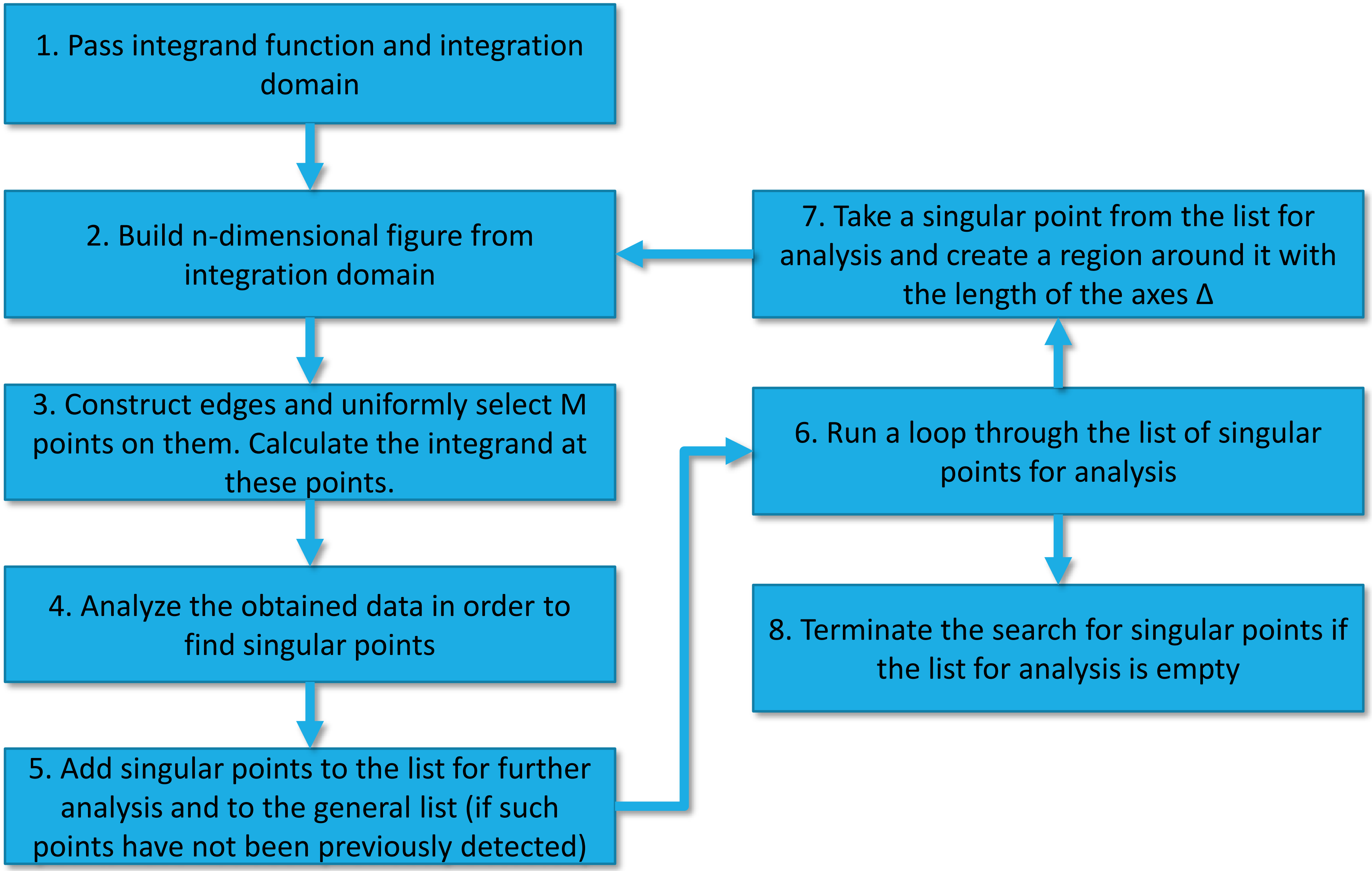
$$I = \int_b^a dx \frac{f(x)}{x - c}$$

where the integrand f(x) has a singularity at an unknown point $c \in [a, b]$.

Task

Develop an algorithm for computing multidimensional integrals with an unknown singularity of the type $\frac{1}{x-c}$

Search for singular points



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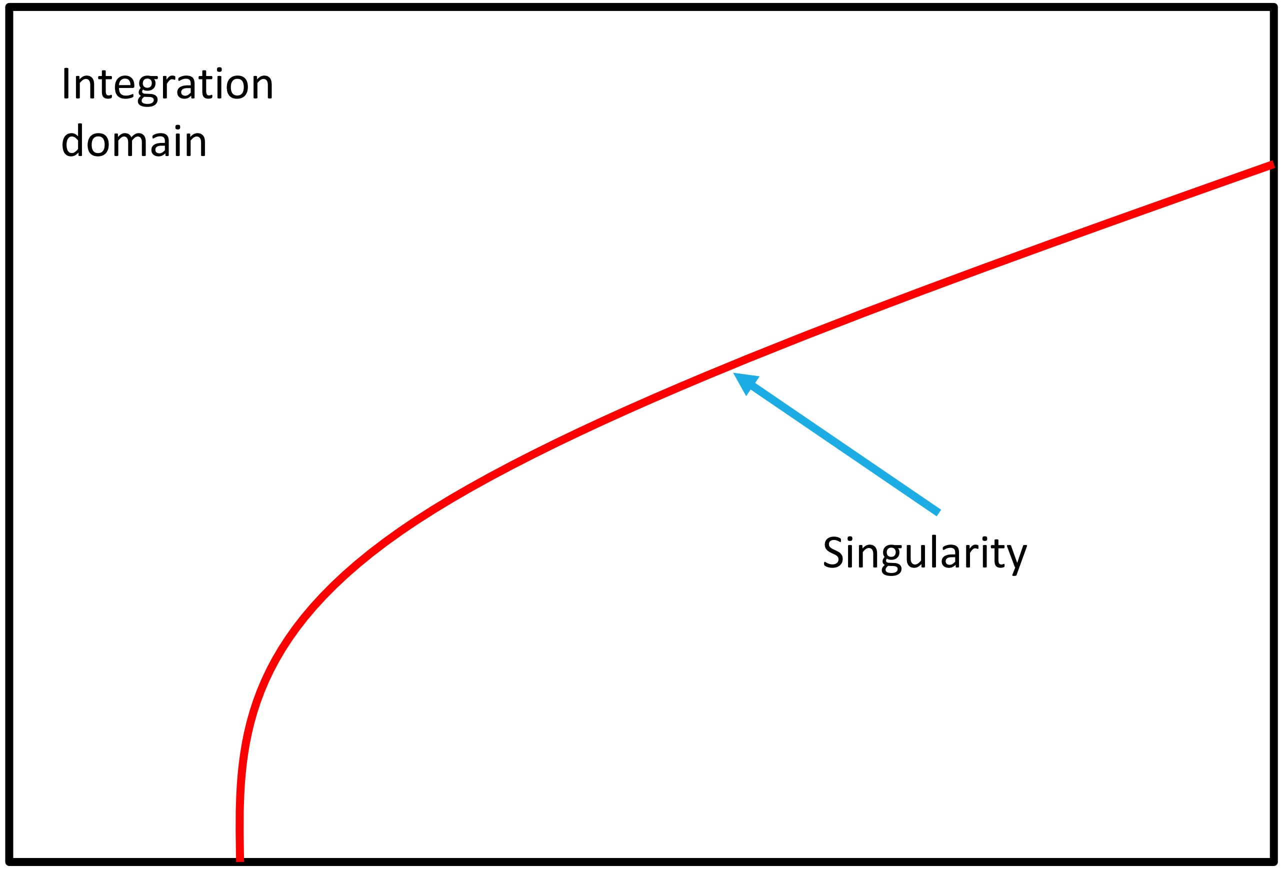

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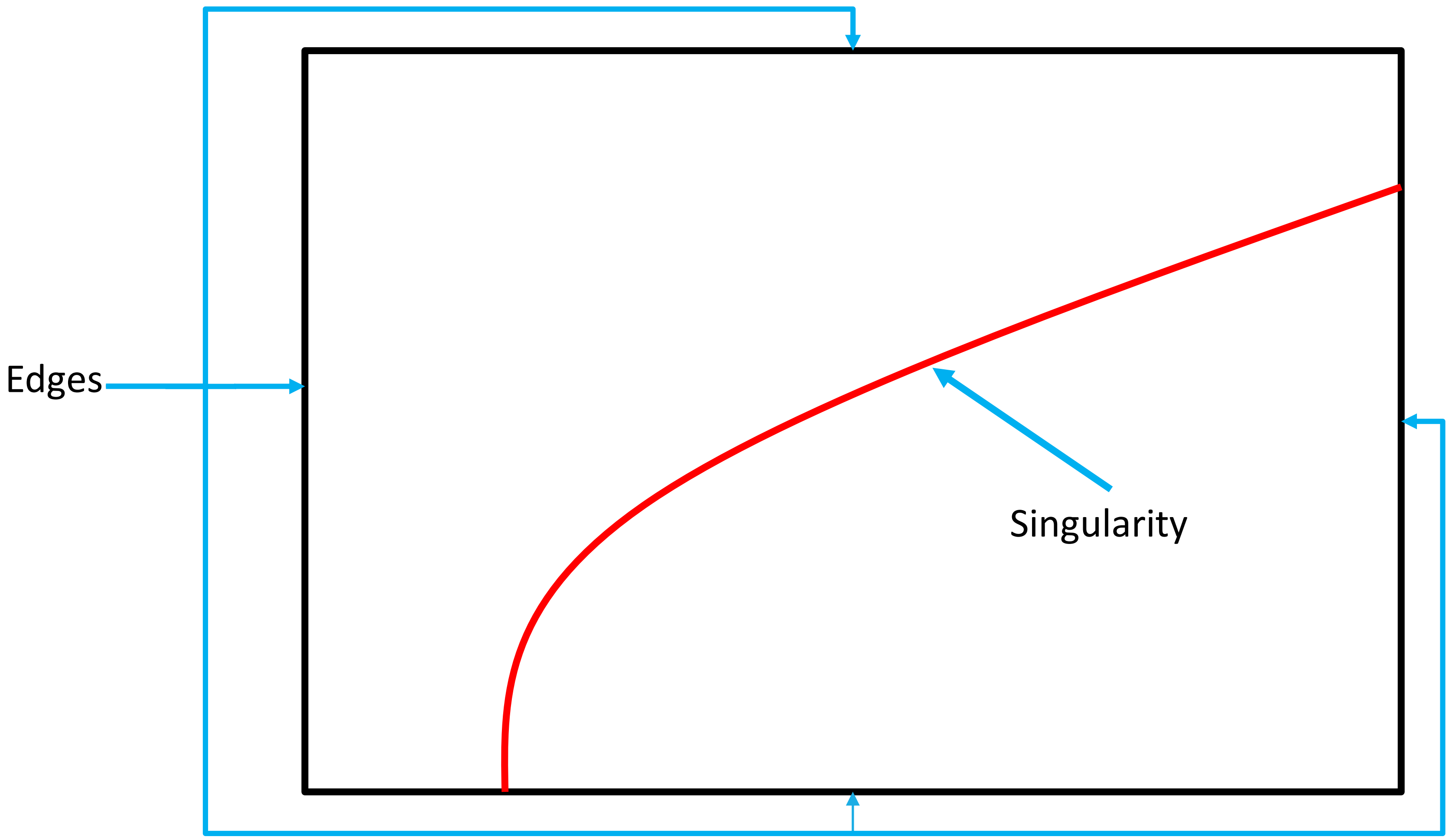

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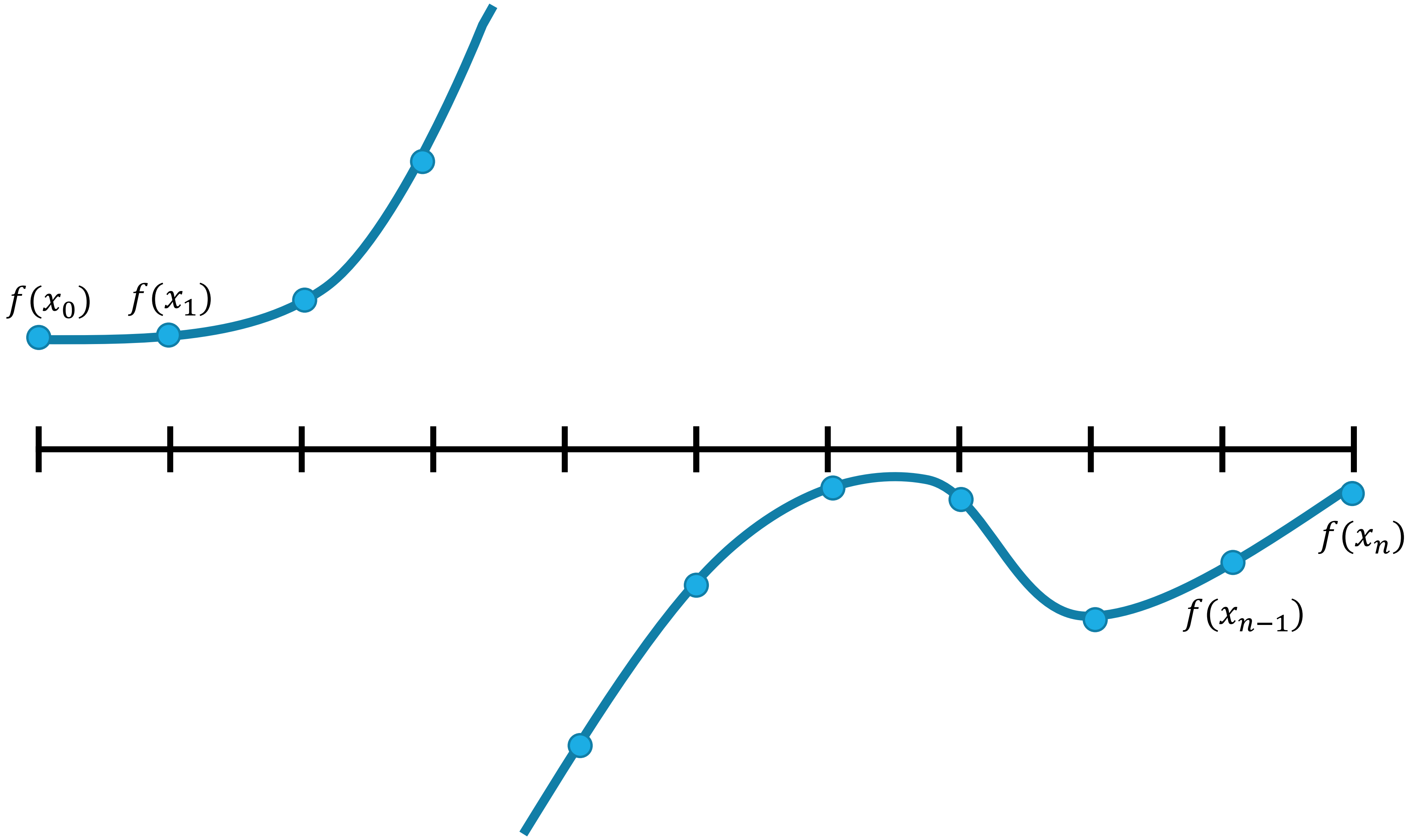

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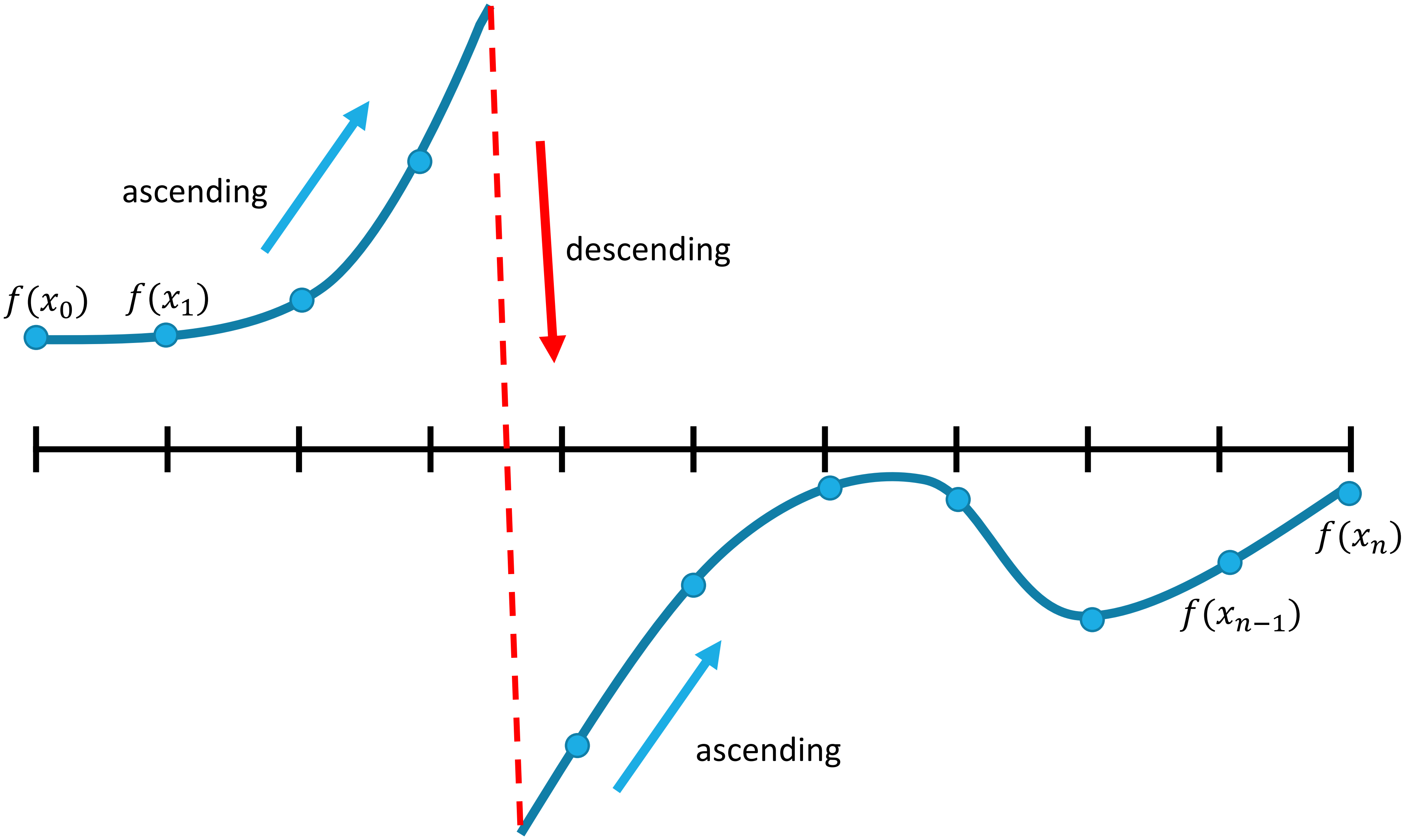

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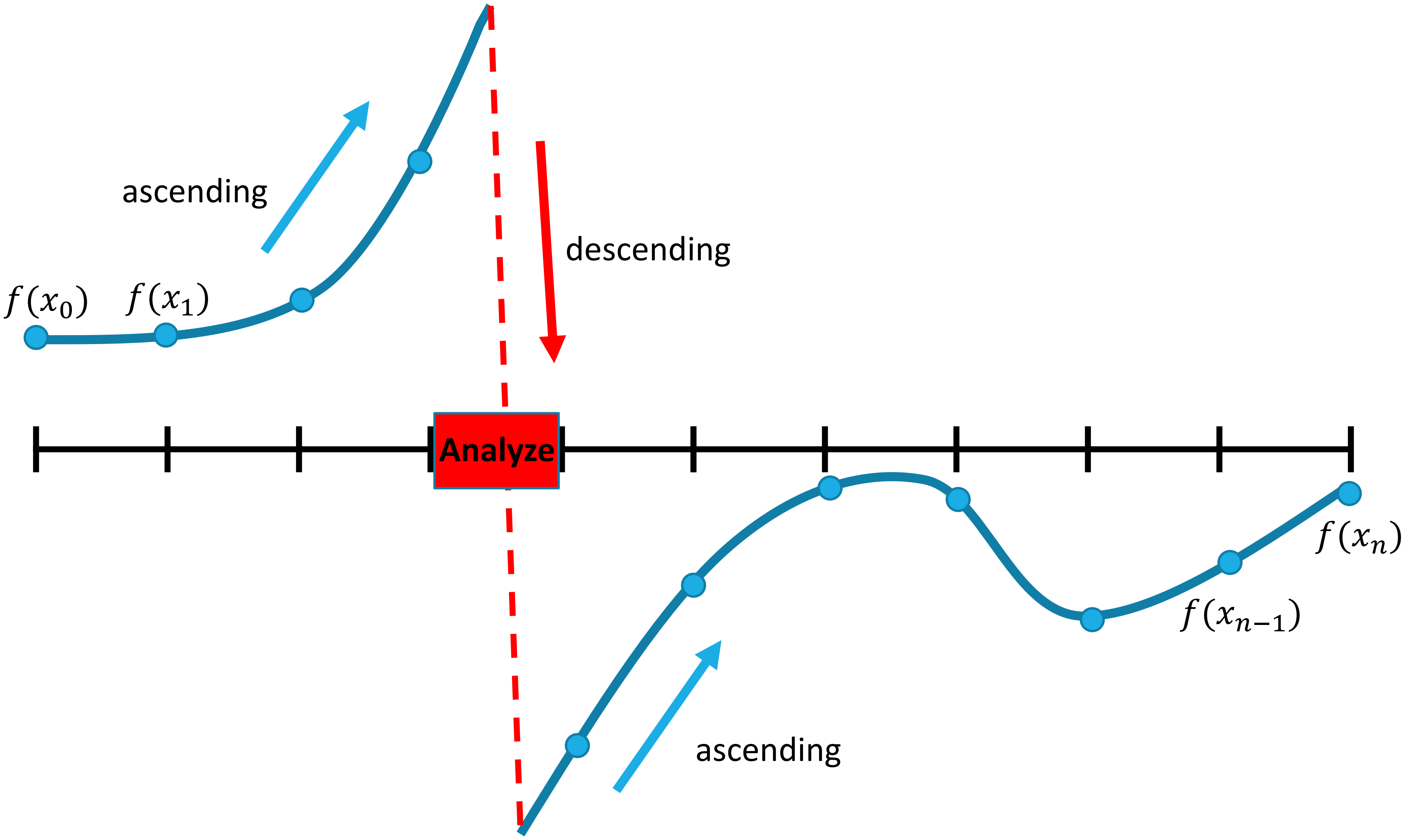
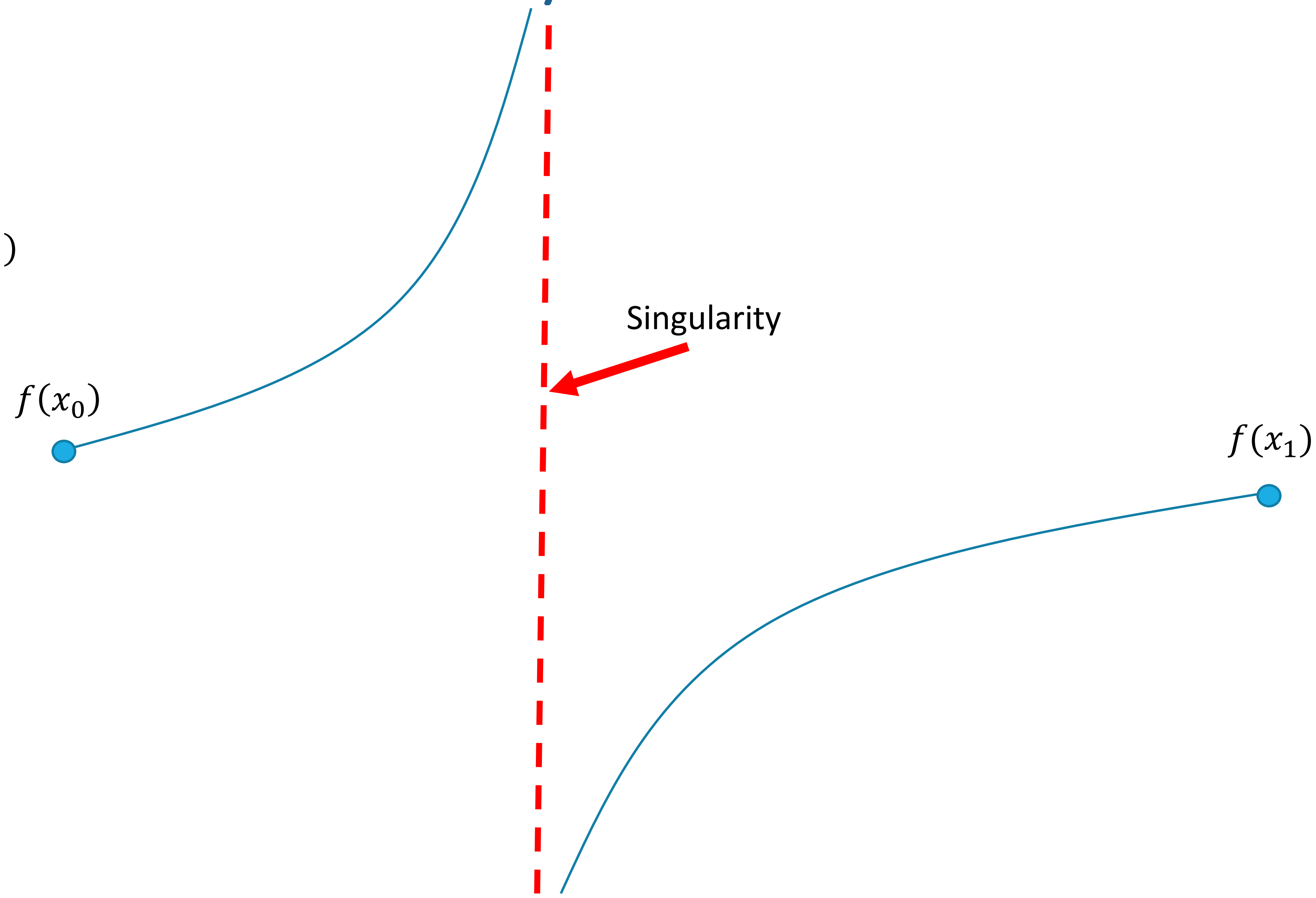


Binary search

$f(x_0) > f(x_1)$



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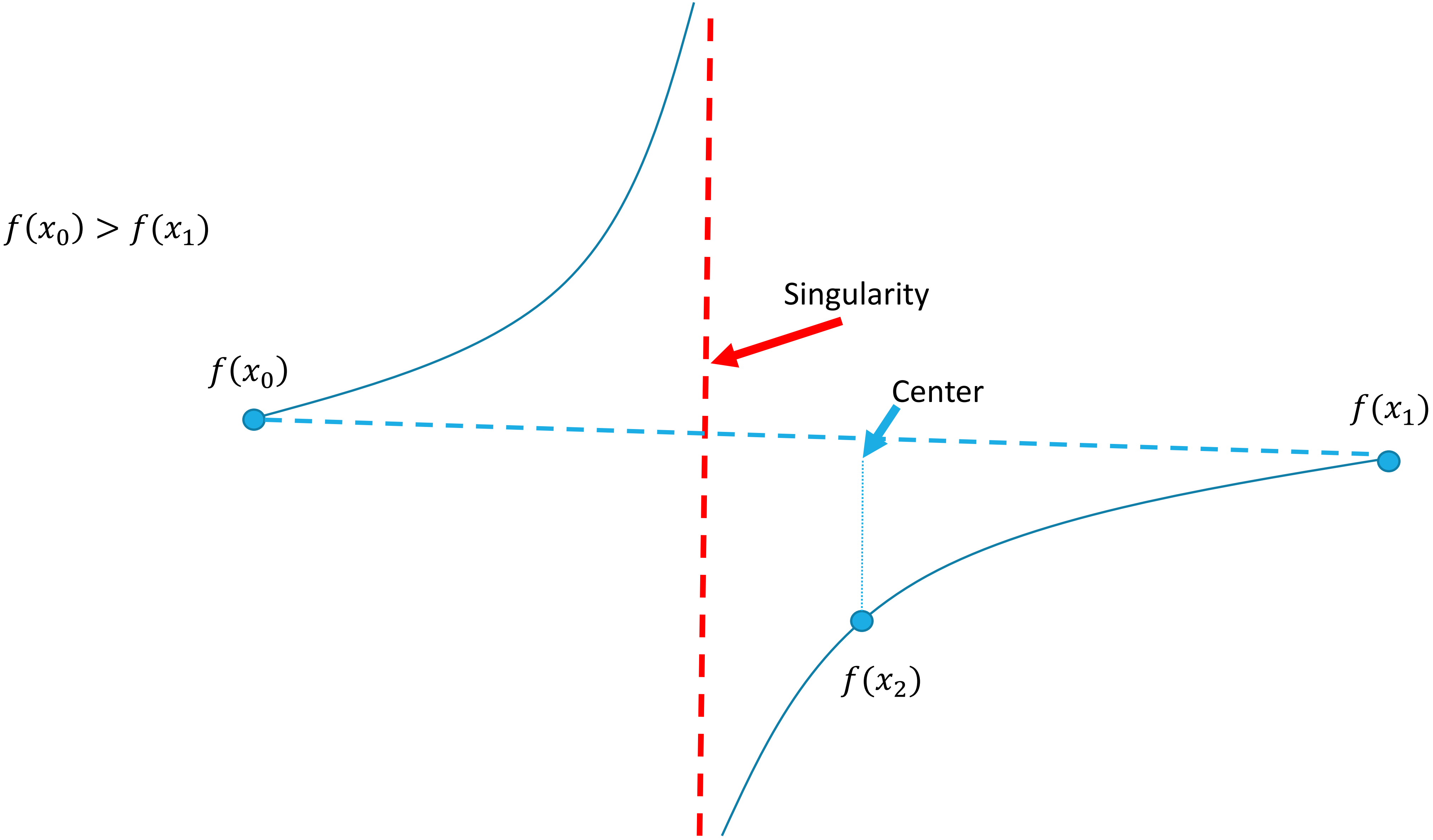

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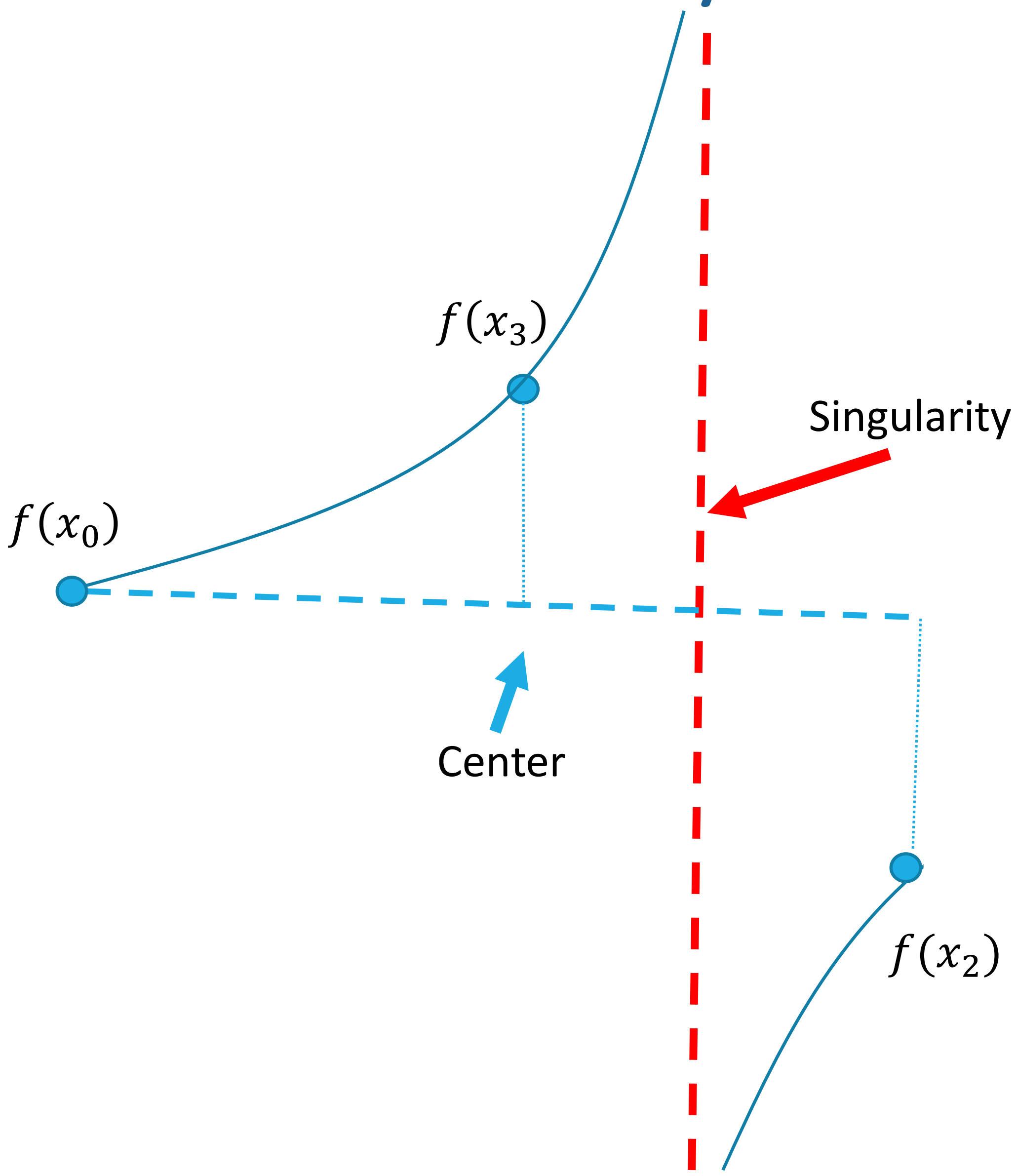

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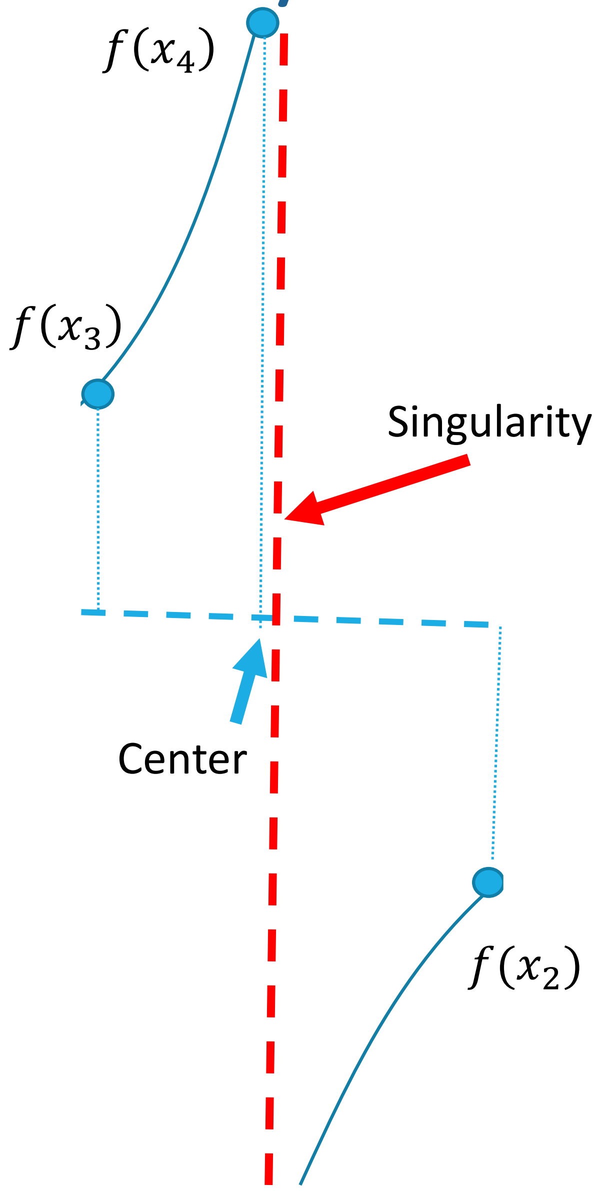
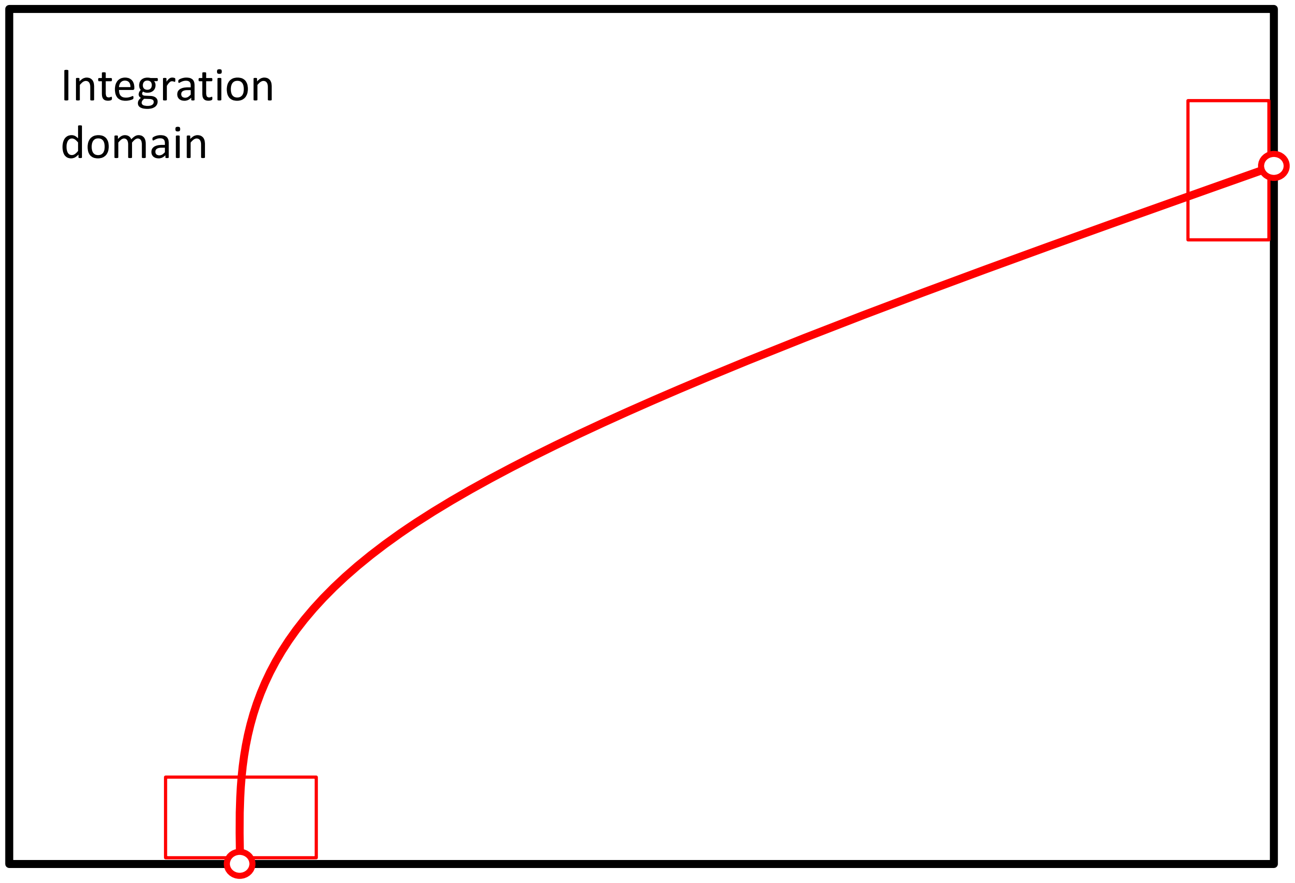


Constructing the list of singular points



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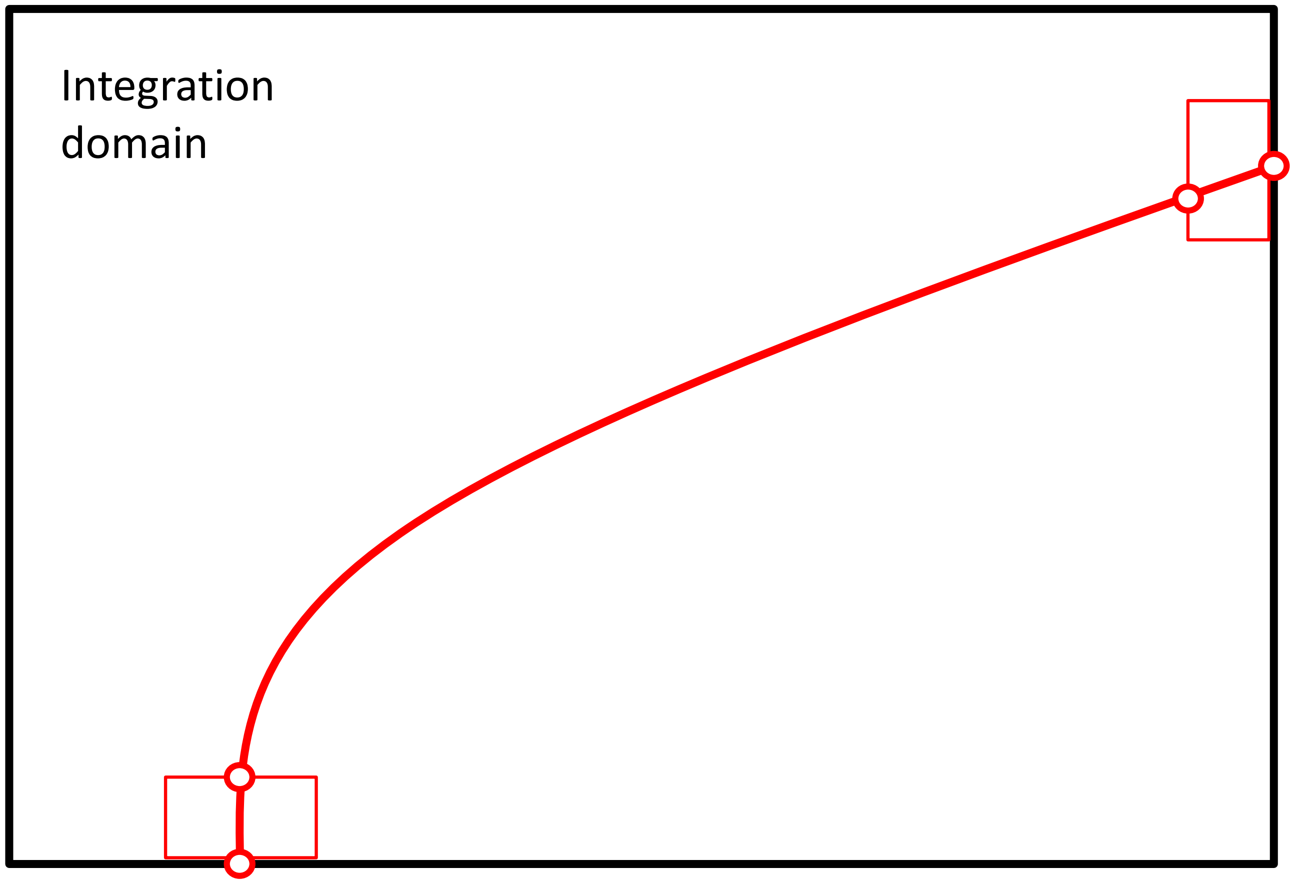

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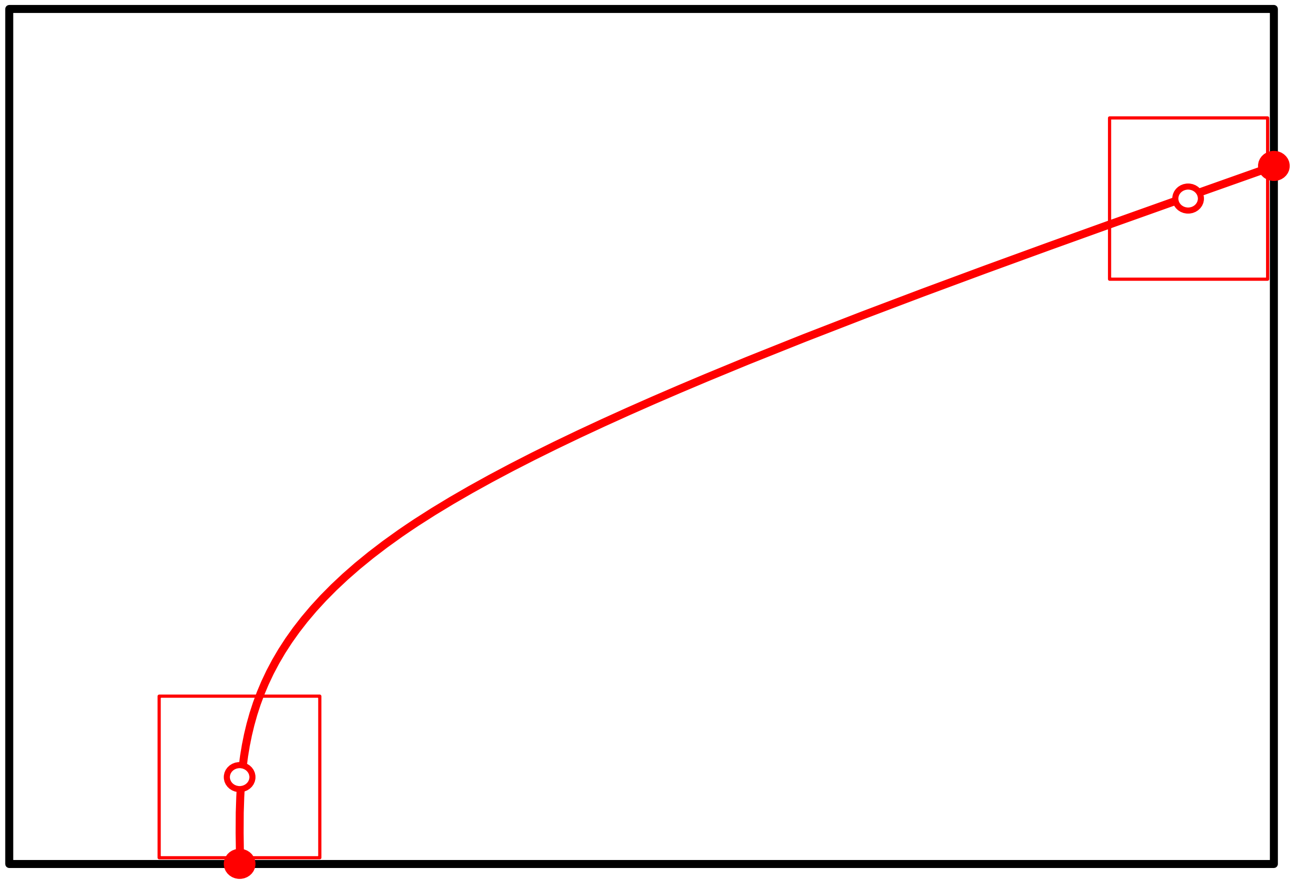

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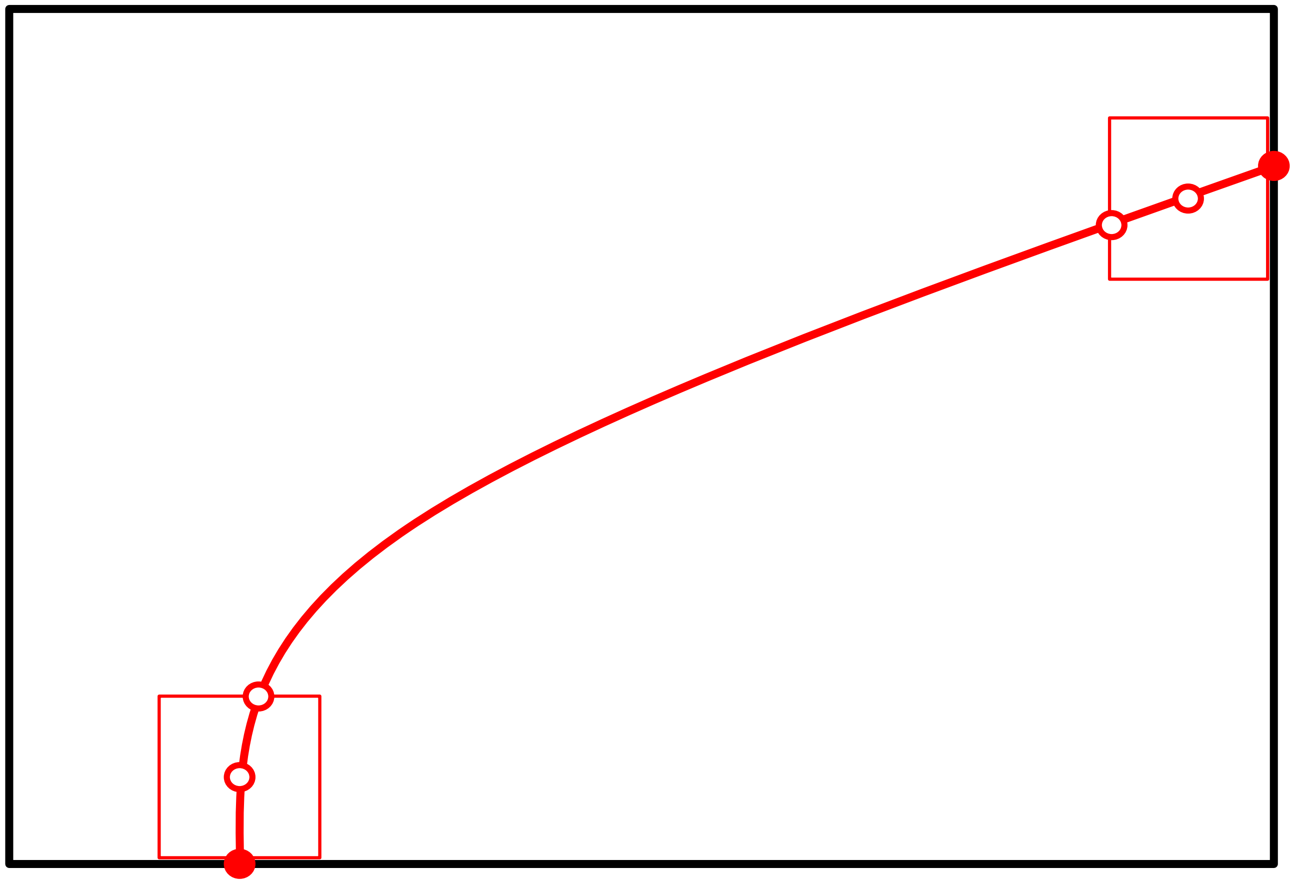

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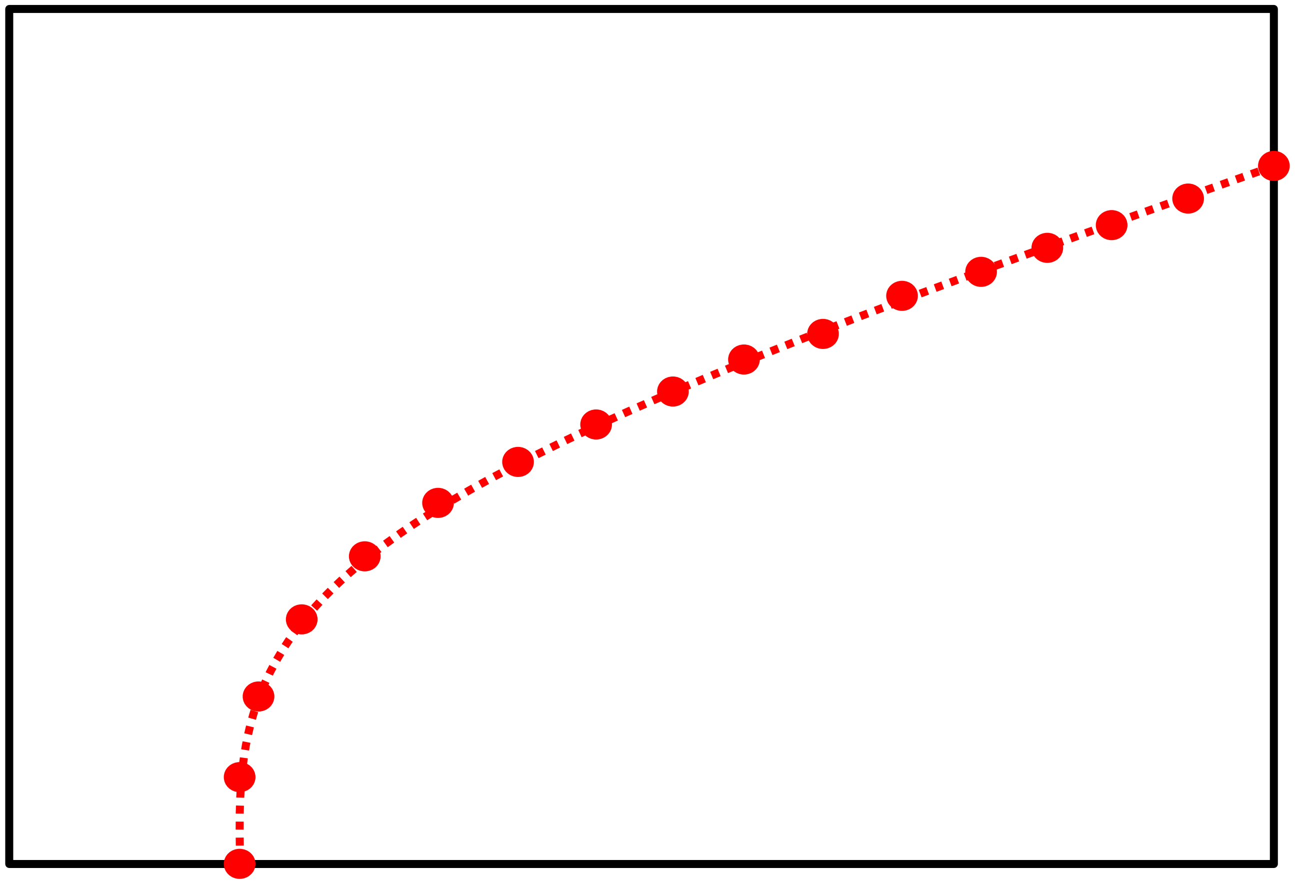
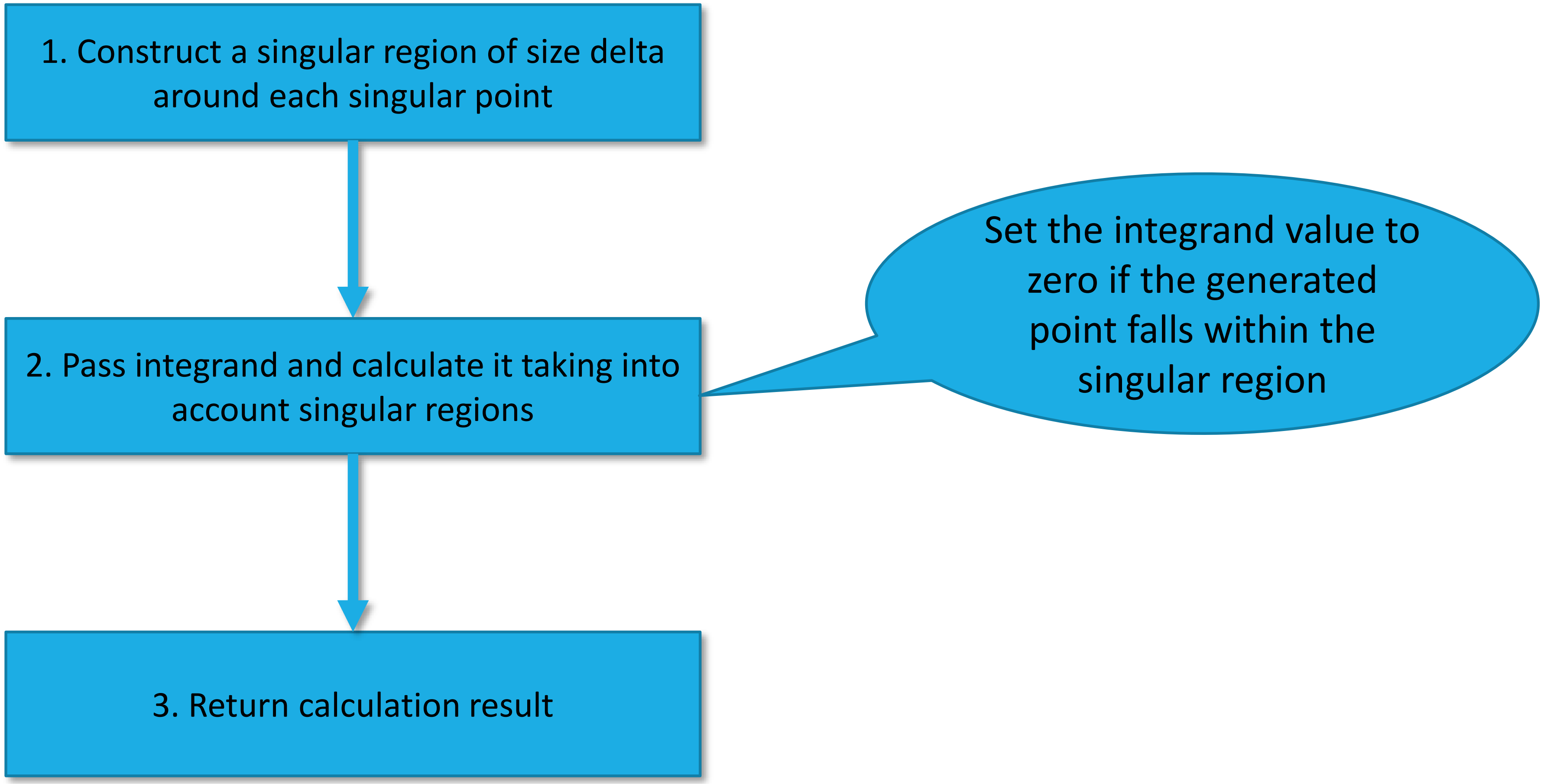
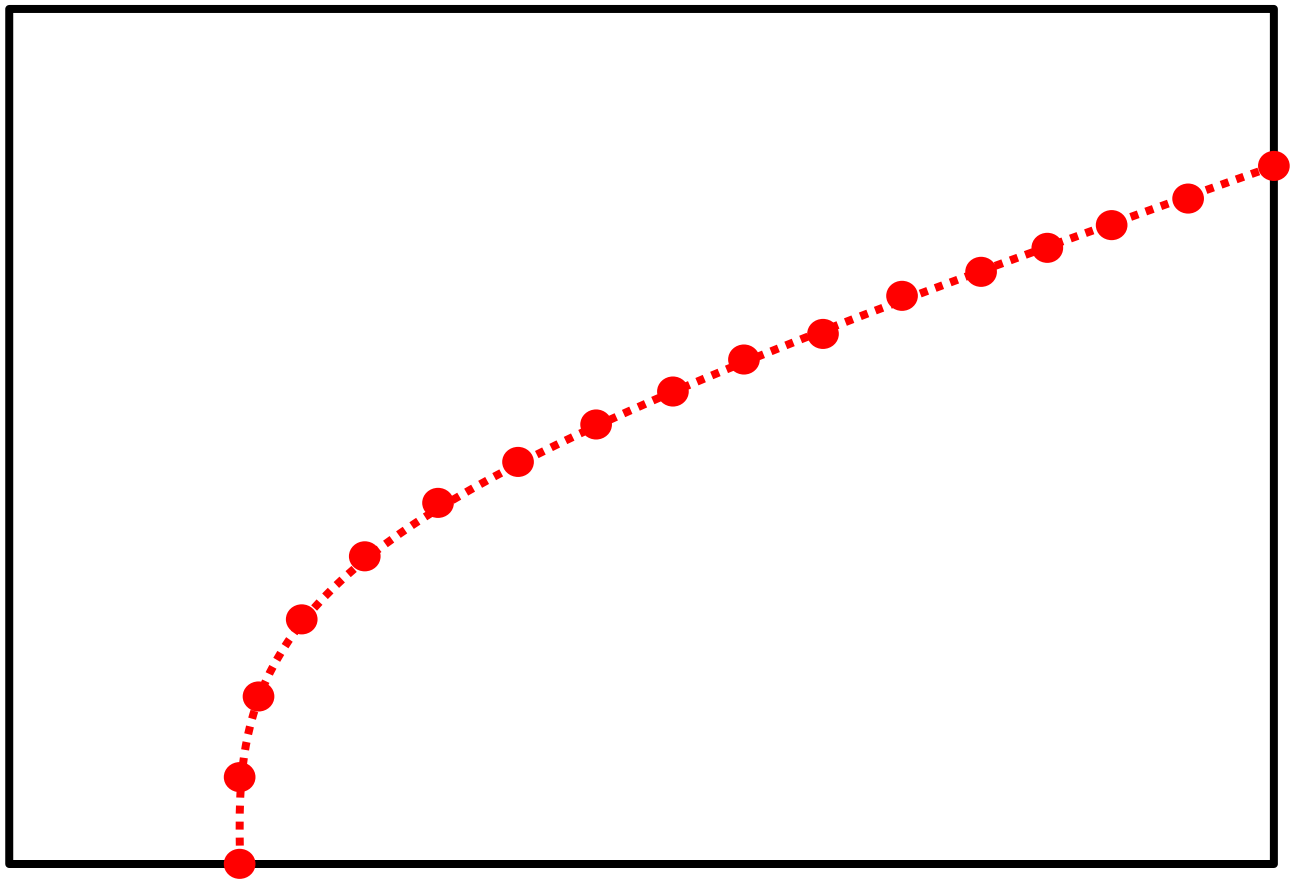


Calculation of the integral with GSL Monte Carlo VEGAS



Make singular areas



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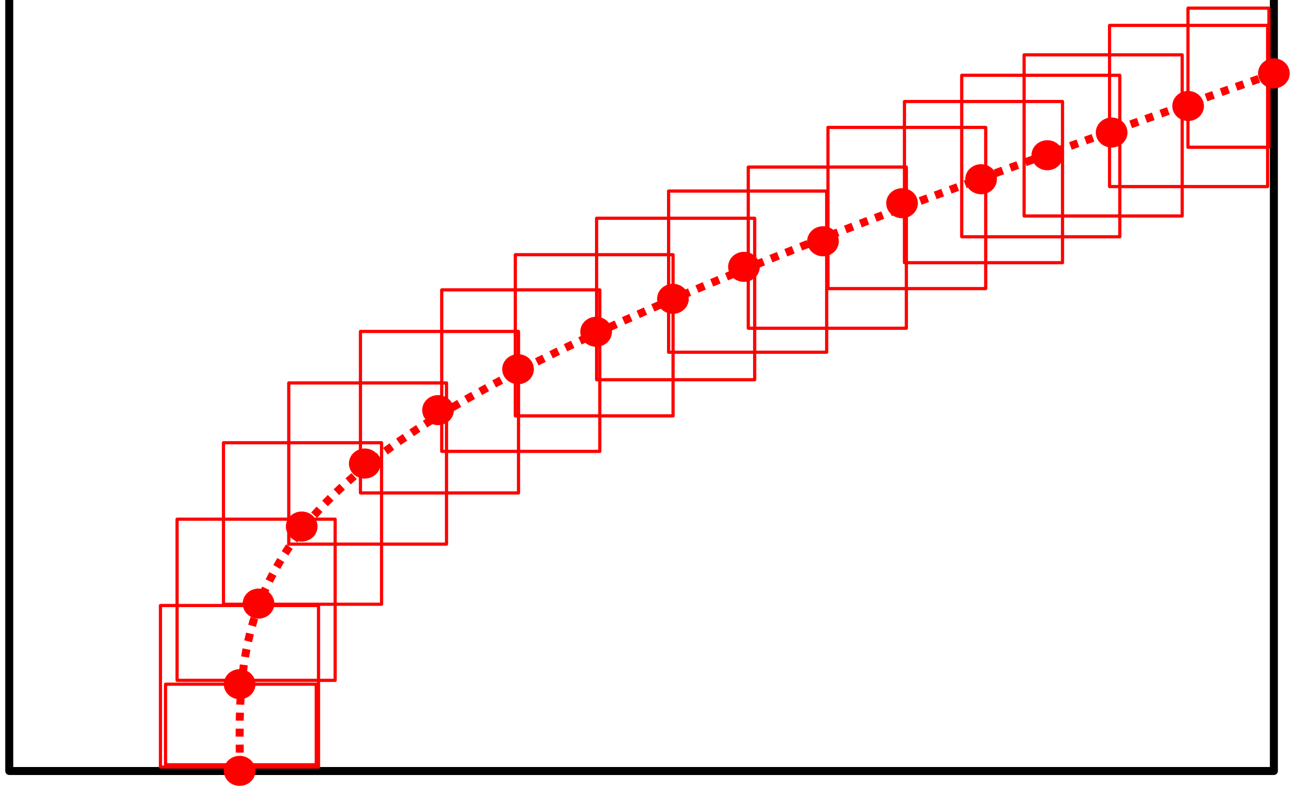

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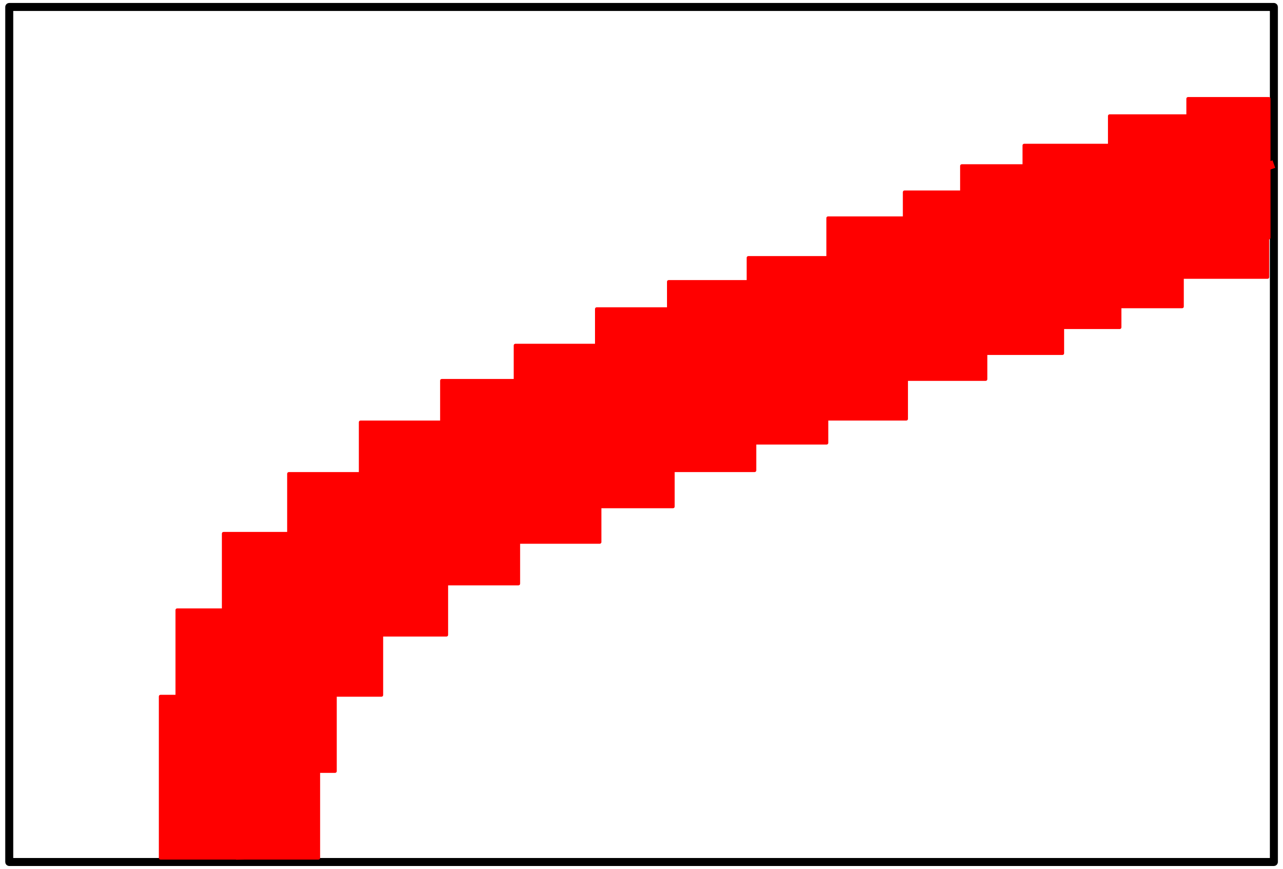
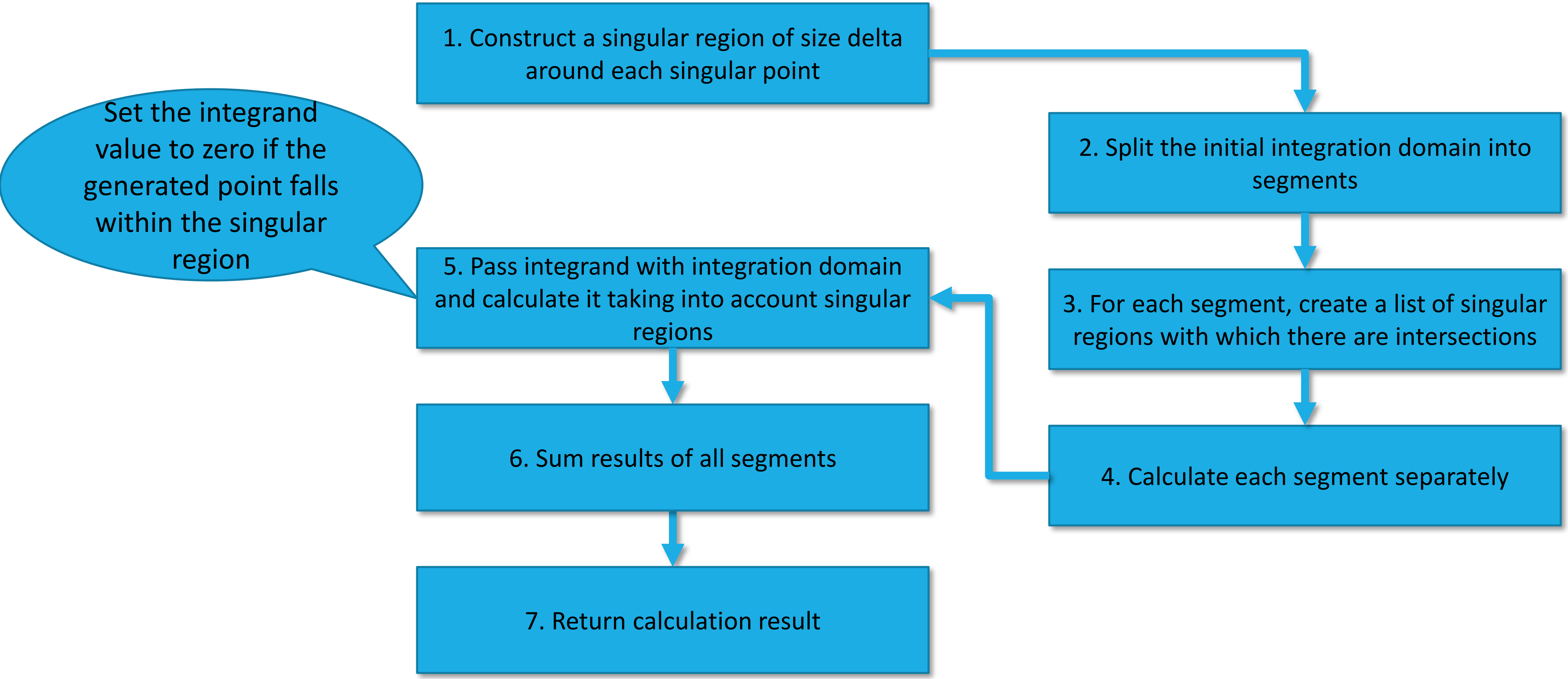
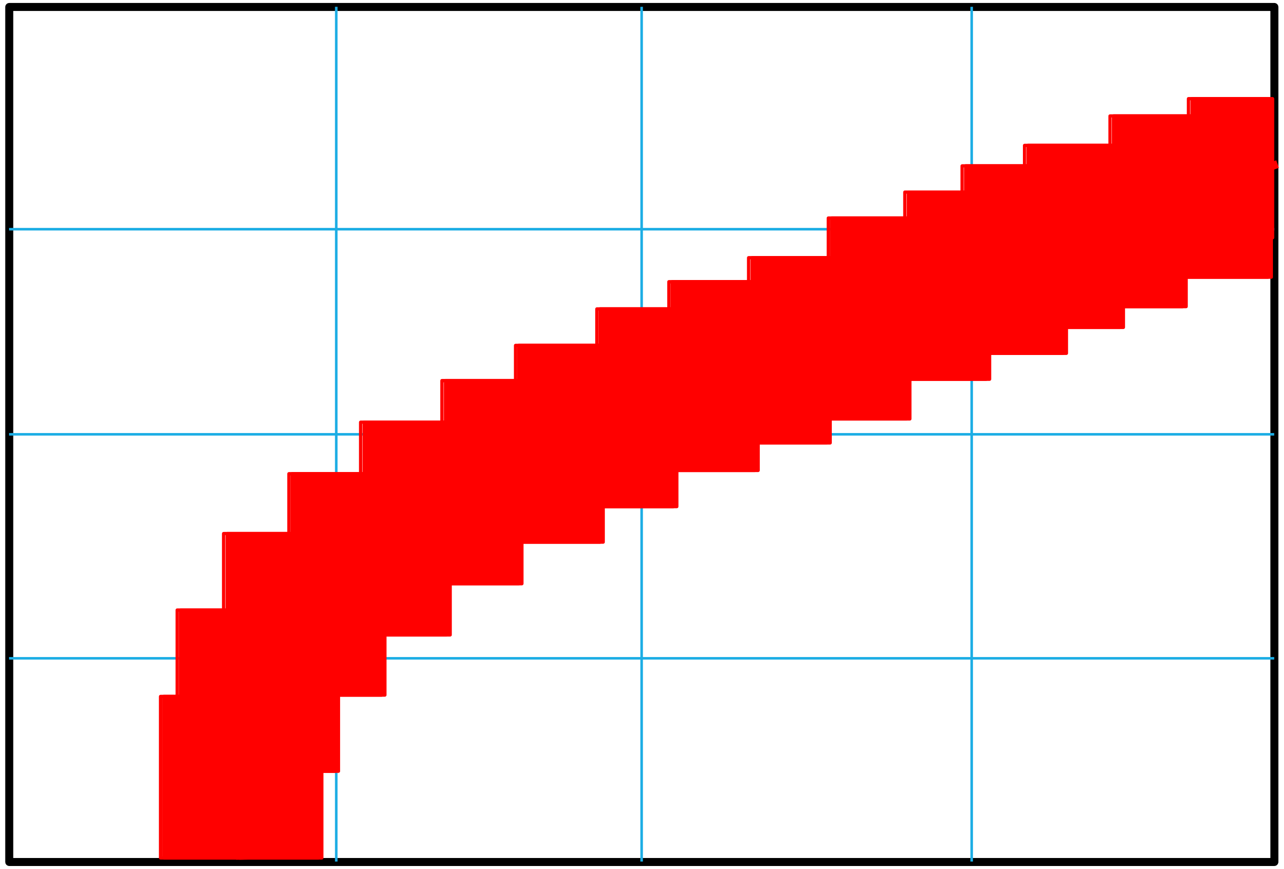


Add multithreading in calculation process



Calculate subsegments



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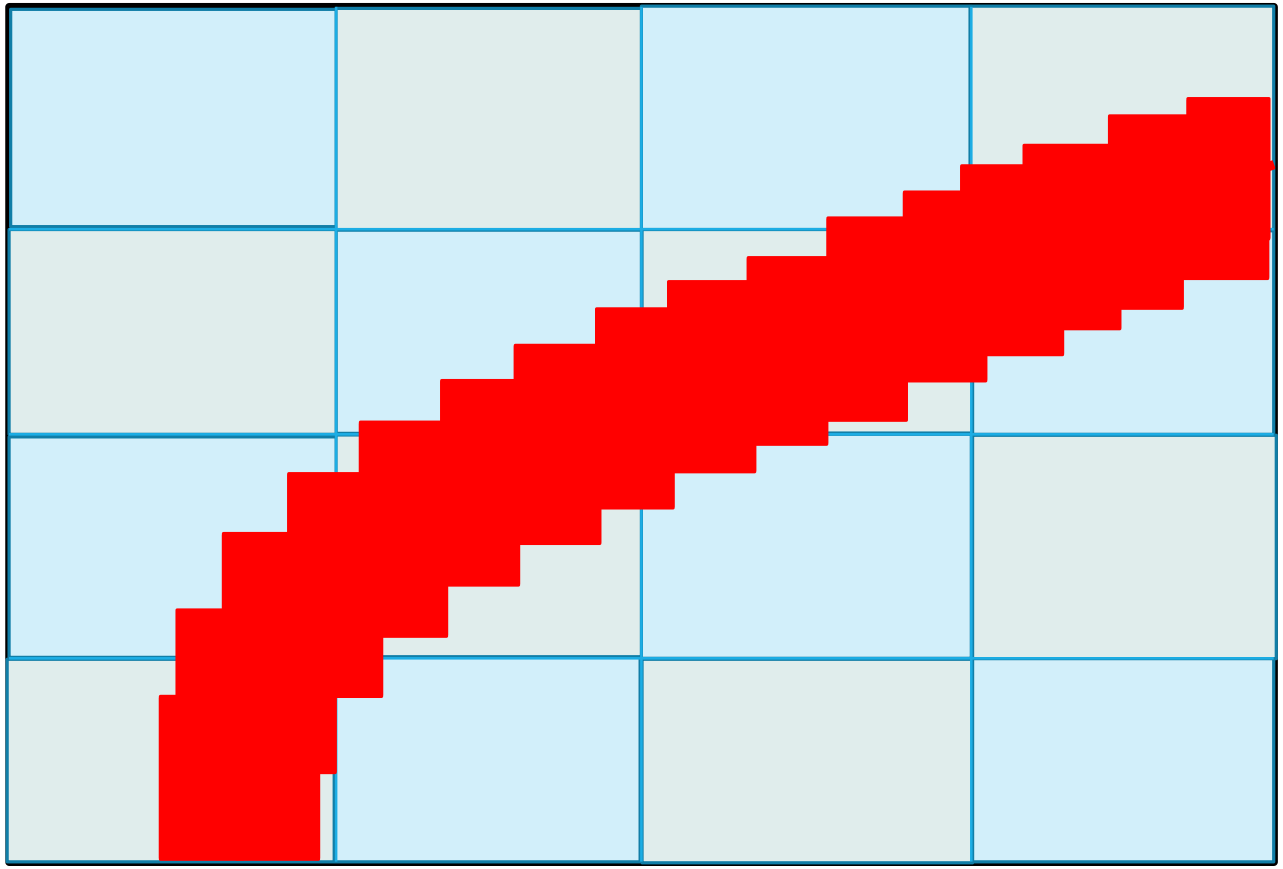

Test

$$\iint \frac{1}{y - x - 1}\,\mathrm{d}x\,\mathrm{d}y$$
$$x = [0{:}\,10]$$
$$y = [0{:}\,12]$$

Numerical answer: 2.0557374

Ξ - comparison to not duplicate singular points

Δ - exclusion area around singular points

β - areas size around singular point in searching algorithm

Intel Core i5-12500H
Cores: 4 + 8 x 2.5 GHz + 1.8 GHz

						1 thread. 1 segment	**16 threads. 36 segments**
GSL MC VEGAS error	Ξ	Δ	B	Points number	Result	Time	Time
0.05	0.001	0.1	0.01	1001	2.056268	25 sec	4 sec
0.5	0.001	0.1	0.01	1001	2.054786	19 sec	4 sec
0.05	0.001	0.1	0.1	101	2.056857	8 sec	< 1sec
0.5	0.001	0.1	0.1	101	2.05573	5 sec	< 1 sec

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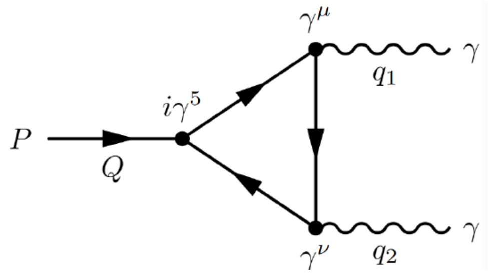
Calculation of the decay width $\pi^0 \to \gamma\gamma$

The calculation of the two-photon pion decay width was performed in the framework of the Nambu-Jona-Lasinio model.

The decay amplitude is calculated as

$$T^{(1)}(P^2; q_1^2, q_2^2) \quad = \quad Q_H \int \frac{dp}{(2\pi)^4} \mathrm{tr}\{\Gamma_H S_3 \gamma_\nu S_2 \gamma_\mu S_1\}$$



The Feynman diagram of the $\boldsymbol{\pi^0 \to \gamma\gamma}$ *decay*

with the pion vertex function $\Gamma_H \equiv \Gamma_\pi = (i\gamma_5) g_{\pi qq}$ and $Q_H = N_c(e_u^2 - e_d^2)e^2$ with e_u = 2/3 and e_d = −1/3 and after taking the trace, the amplitude can be written as:

$$T^{(1)}(P^2; q_1^2, q_2^2) \quad = \quad \epsilon_{\mu\nu\alpha\beta} \epsilon_1^\mu \epsilon_2^\nu q_1^\alpha q_2^\beta N_c \frac{4e^2}{3} g_{\pi qq} m I_3(P, q_1, q_2)$$

The width of the photoproduction is calculated by the simple relation:

$$\Gamma(\pi^0 \longrightarrow \gamma\gamma) = \frac{M_\pi^3}{64} |\, 32\alpha\pi g_{\pi qq} m I_3(0, 0, P^2)\, |^2$$

$$I_3(q_1^2, q_2^2, P^2) = \int \frac{dp}{(2\pi)^4} \frac{1}{(p_0^2 - E^2)((p_0 - q_{10})^2 - E_1^2)((p_0 + q_{20})^2 - E_2^2)}$$

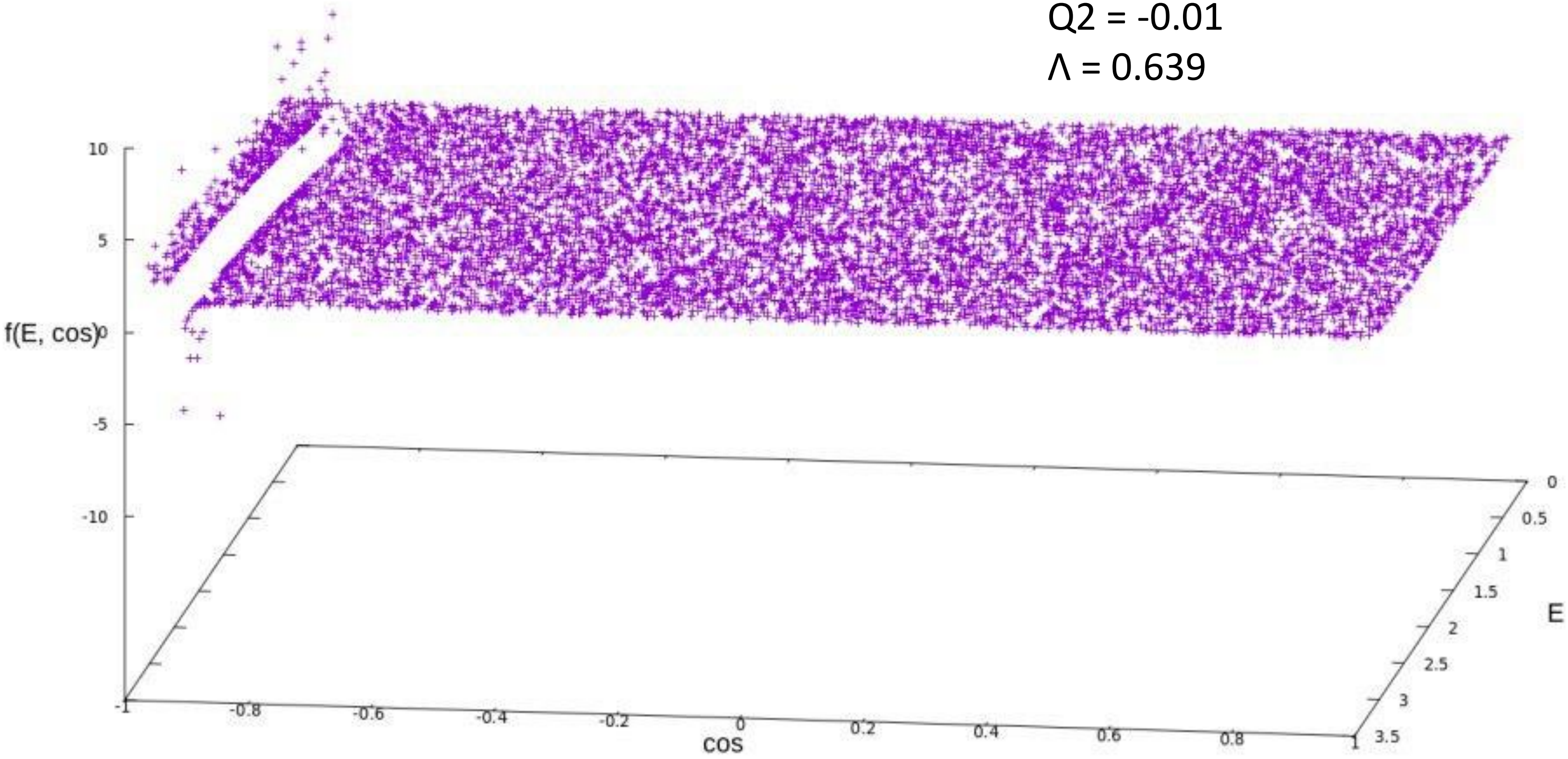
Integral I_3 at finite temperature

The main feature of the NJL model is the possibility to introduce finite temperature using the Matsubara summation. The integral I_3 at finite temperature splits into real and imaginary parts, and real part of the integral has a view:

$$ReI_3 = \frac{1}{8\pi^2}\int_m^{\sqrt{\Lambda^2+m^2}} \sqrt{E^2-m^2}\int_{-1}^{1} d\cos\theta \Big\{ f(E-\mu)\Big[-\frac{1}{2Eq_{10}-2\bar{p}q_{10}\cos\theta}\frac{1}{2EP_0+q_2^2} - \frac{1}{2Eq_{20}+2\bar{p}q_{10}\cos\theta}\frac{1}{2EP_0+q_2^2} + \frac{1}{2Eq_{10}-2\bar{p}q_{10}\cos\theta}\frac{1}{2EP_0+P_0^2}$$

Calculation of the integral $I_3(P^2, 0, 0; T)$

T = 0.005
Q2 = -0.01
Λ = 0.639



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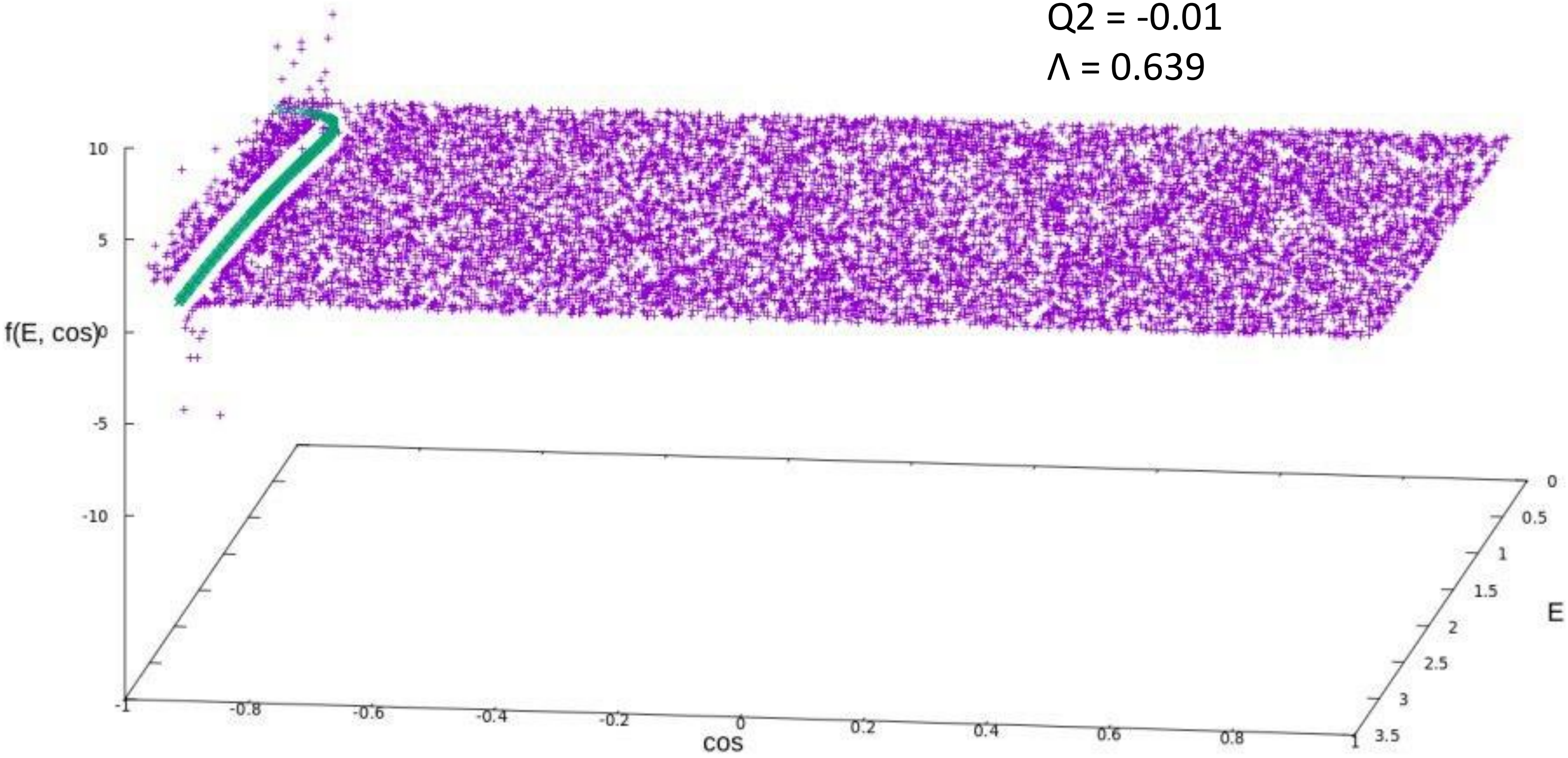
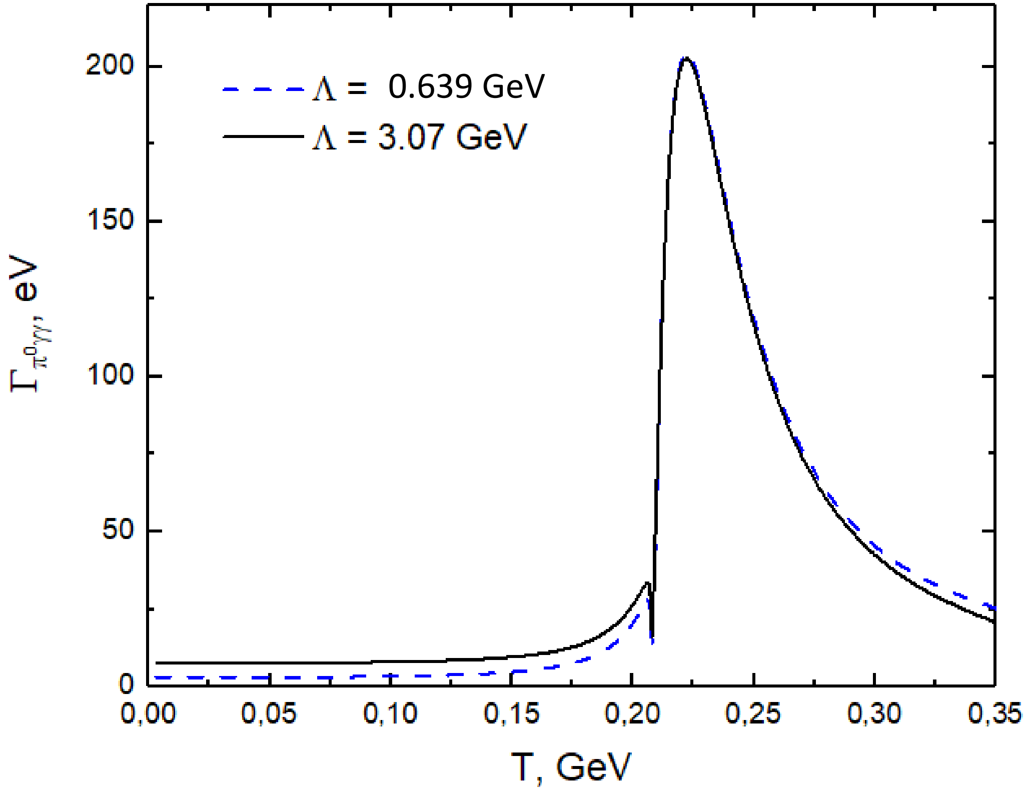


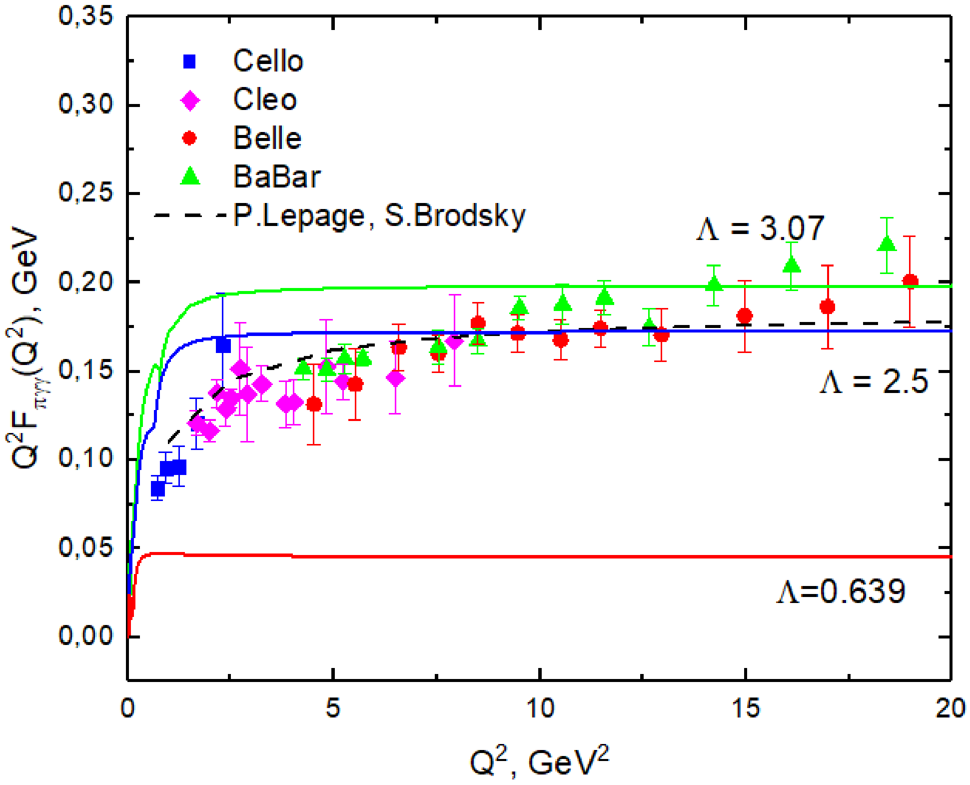
The decay width and transition form factor of $\pi^0 \to \gamma\gamma$

$$\Gamma(\pi^0 \longrightarrow \gamma\gamma) = \frac{M_\pi^3}{64} |\, 32\alpha\pi g_{\pi qq} m I_3(0,0,P^2)\,|^2$$

$$|F_{\pi\gamma}(0)|^2 = \frac{64\pi}{(4\pi\alpha)^2} \frac{\Gamma_{\pi^0\gamma\gamma}}{M_\pi^3}$$

$$F_{\pi\gamma}(Q^2) = 8 g_{\pi qq} m I_3(0, Q^2, P^2)$$





Summary

- Algorithm works and gives quite accurate results
- Parallelization allows to significantly reduce the calculation time
- The method was applied to calculate $\pi^0 \to \gamma\gamma$ problem
- The method will be applied for further calculations