A method for identifying and calculating a singularity of type 1/(x-c) in a multidimensional divergent integral

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Problem

Evaluate an n-dimensional improper integral of the form:

$$I = \int_{b}^{a} dx \frac{f(x)}{x - c}$$

where the integrand f(x) has a singularity at an unknown point $c \in [a, b]$.

Task

Develop an algorithm for computing multidimensional integrals with an unknown singularity of the type $\frac{1}{x-c}$

1. Pass integrand function and integration domain

2. Build n-dimensional figure from integration domain

3. Construct edges and uniformly select M points on them. Calculate the integrand at these points.

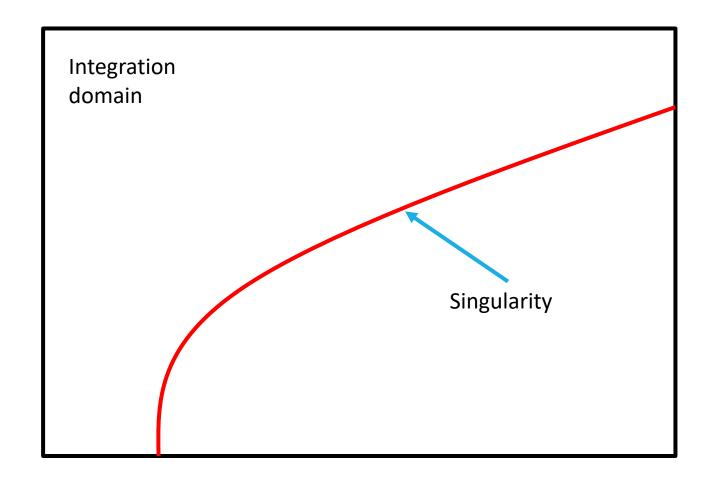
4. Analyze the obtained data in order to find singular points

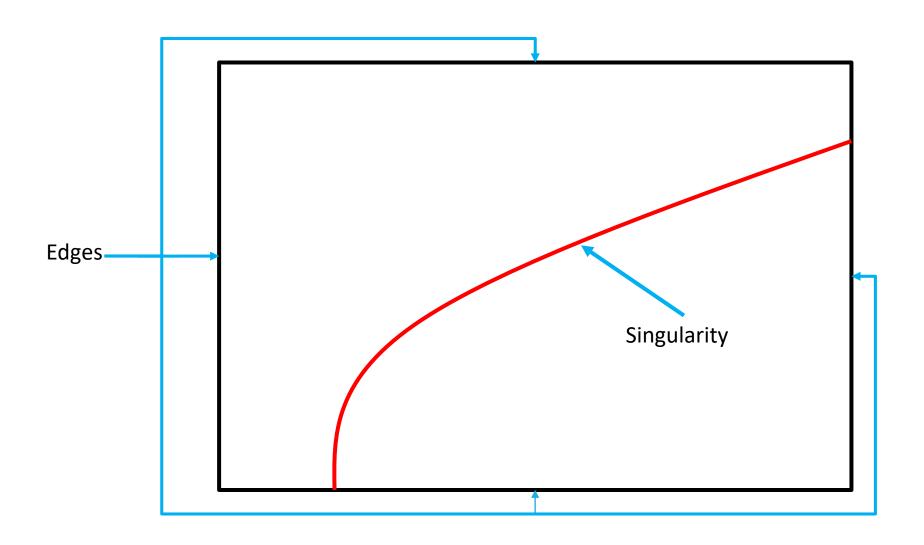
5. Add singular points to the list for further analysis and to the general list (if such points have not been previously detected)

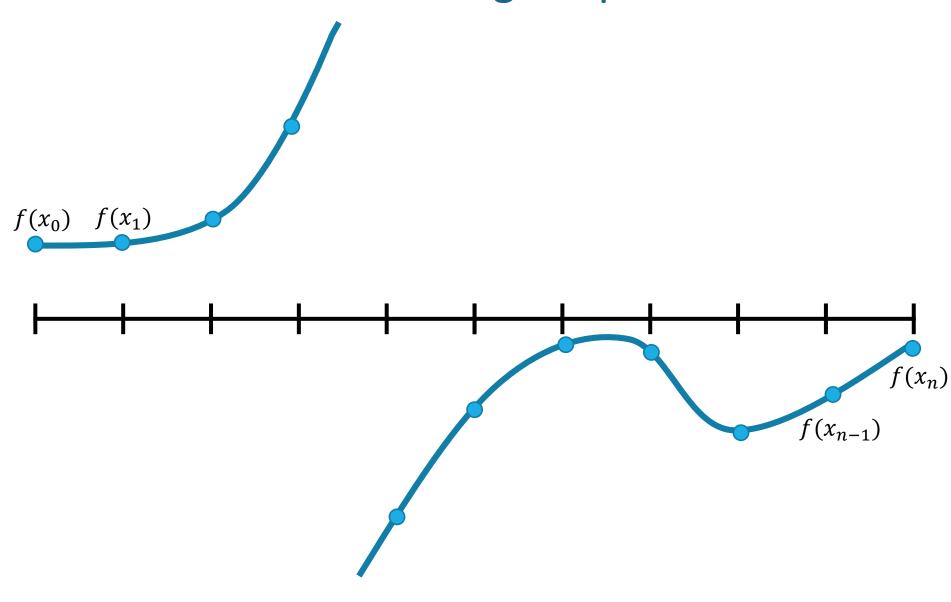
7. Take a singular point from the list for analysis and create a region around it with the length of the axes Δ

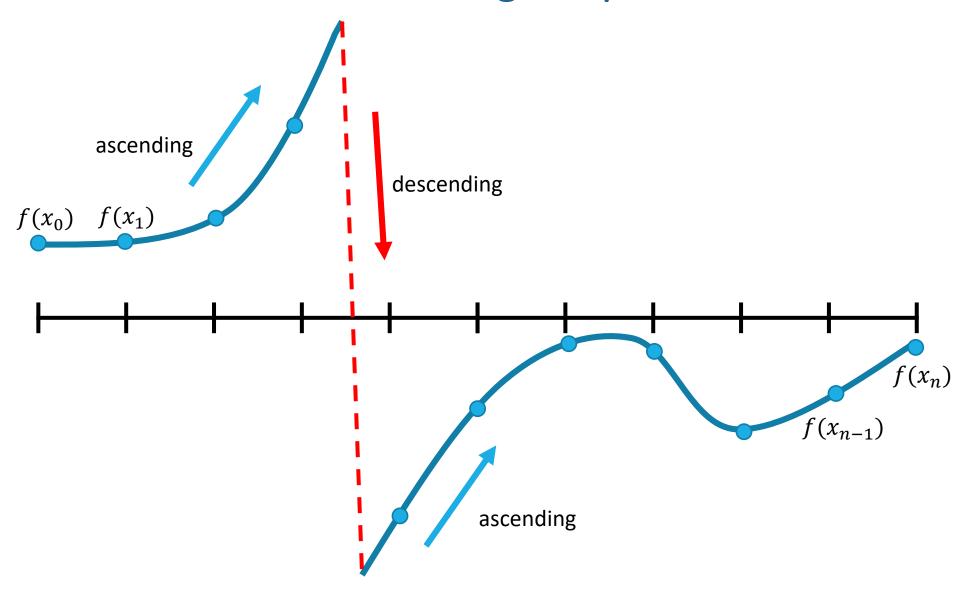
6. Run a loop through the list of singular points for analysis

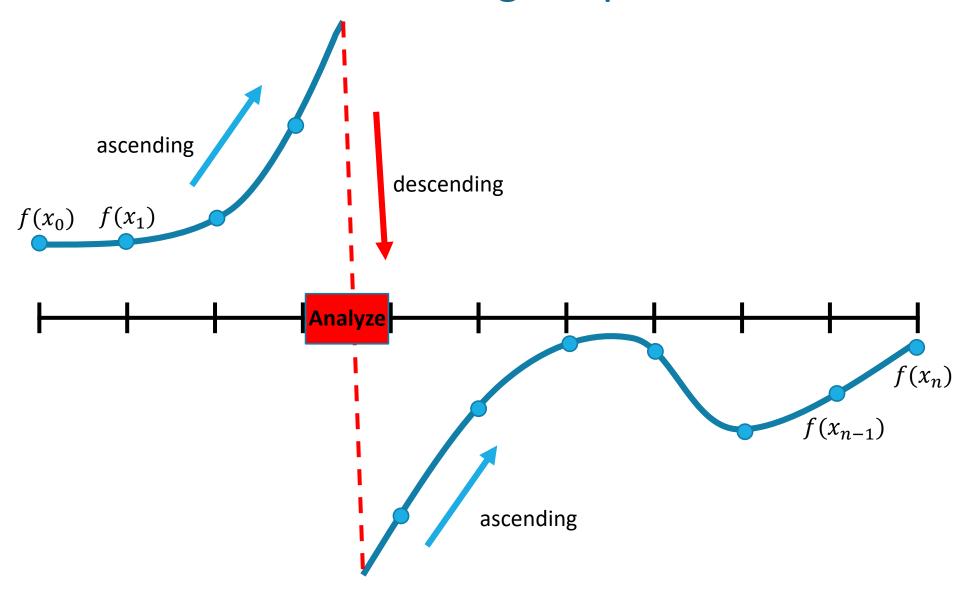
8. Terminate the search for singular points if the list for analysis is empty

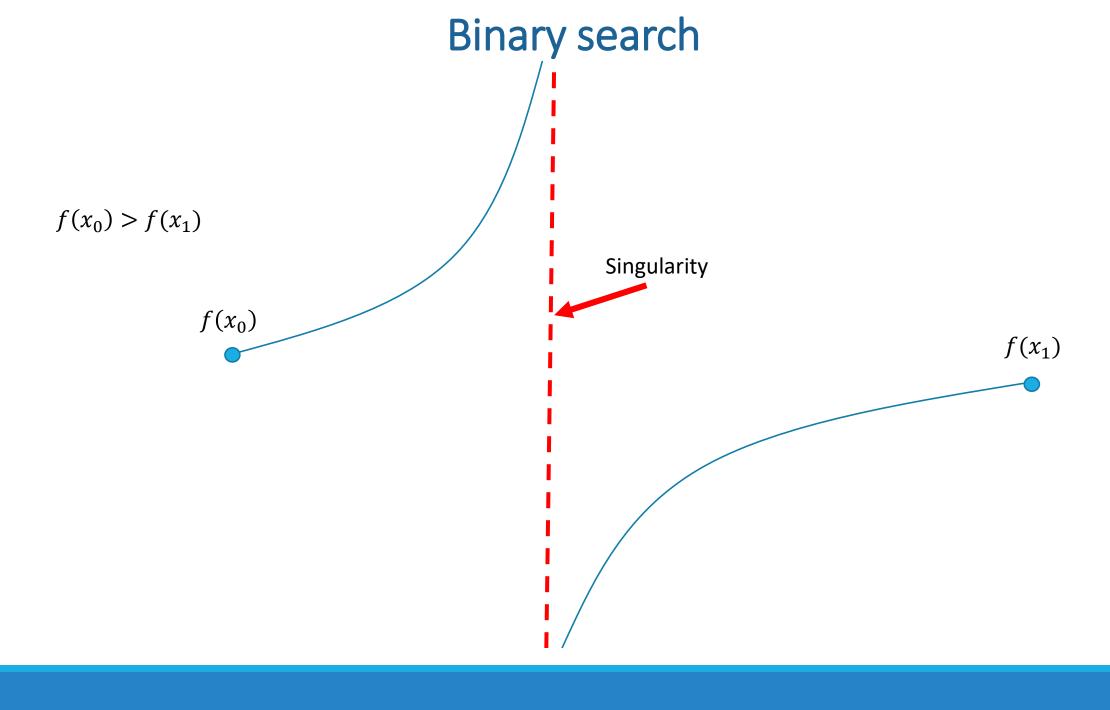


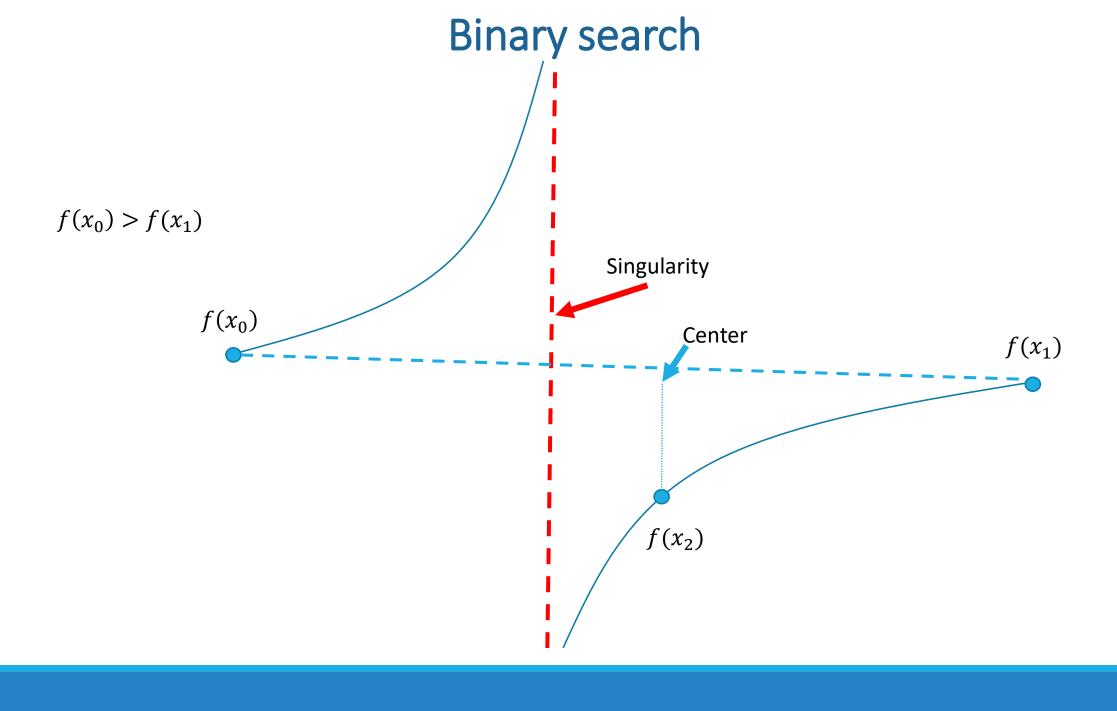


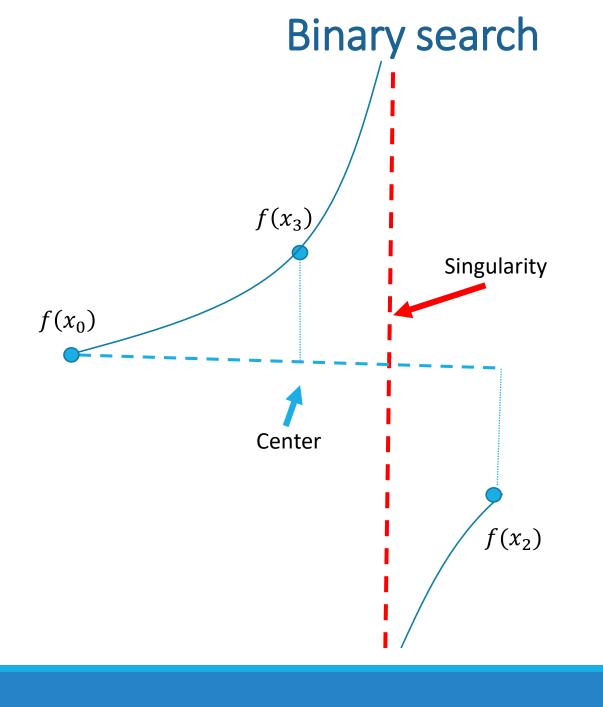


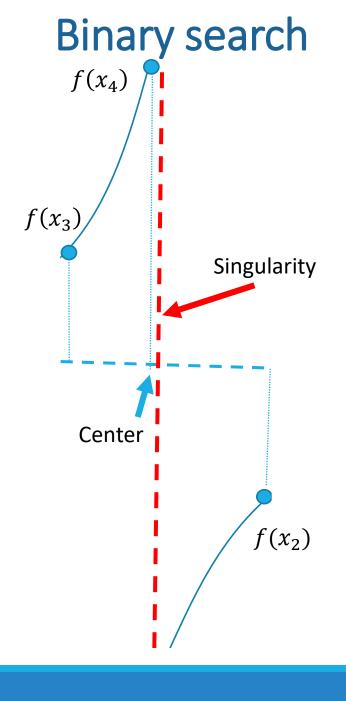


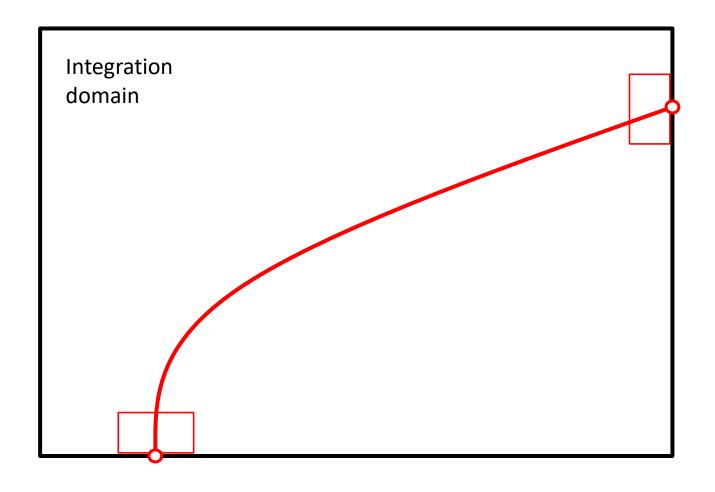


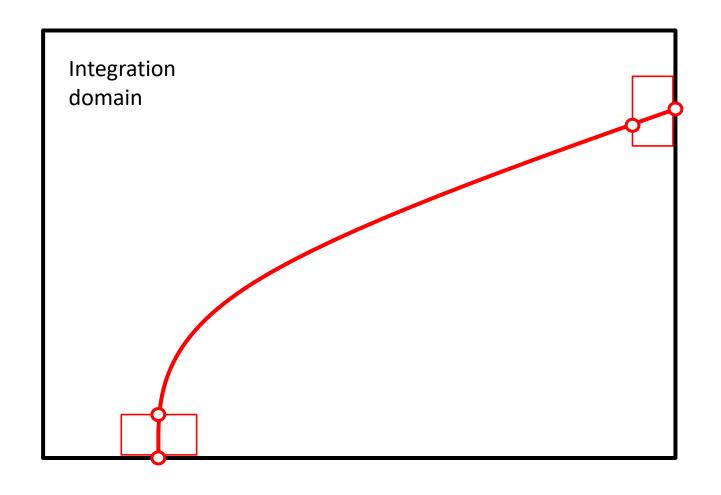


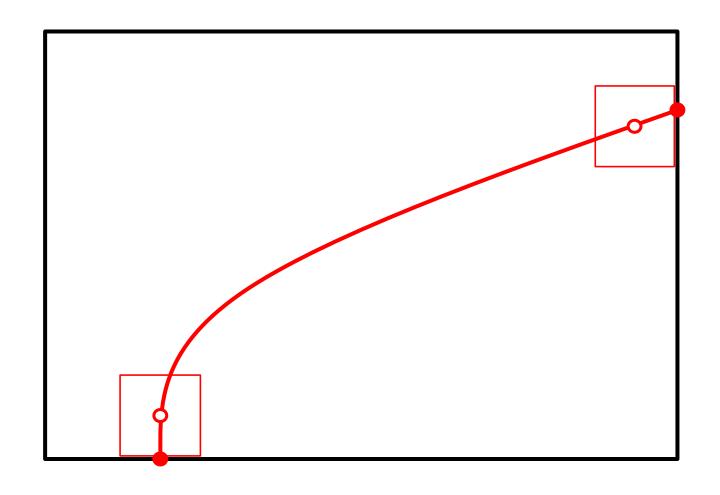


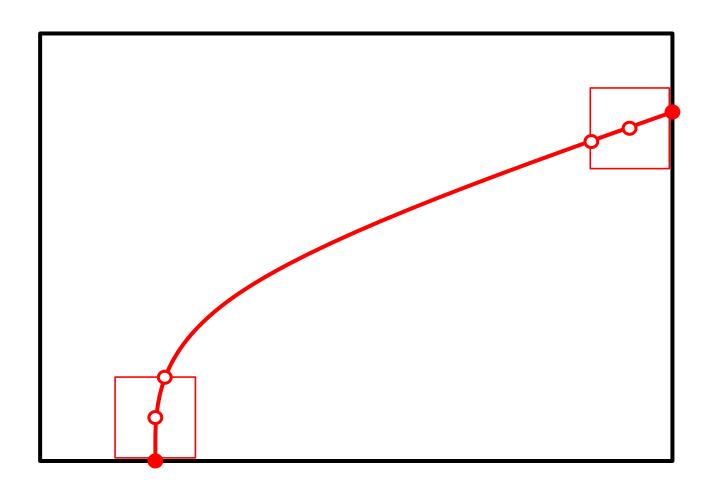


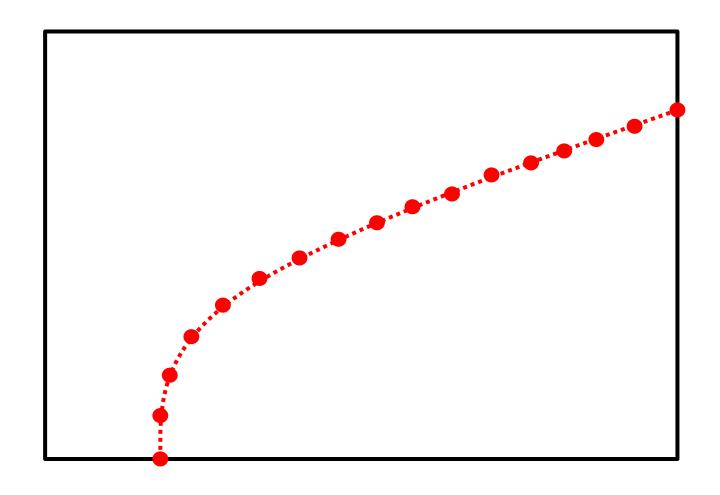




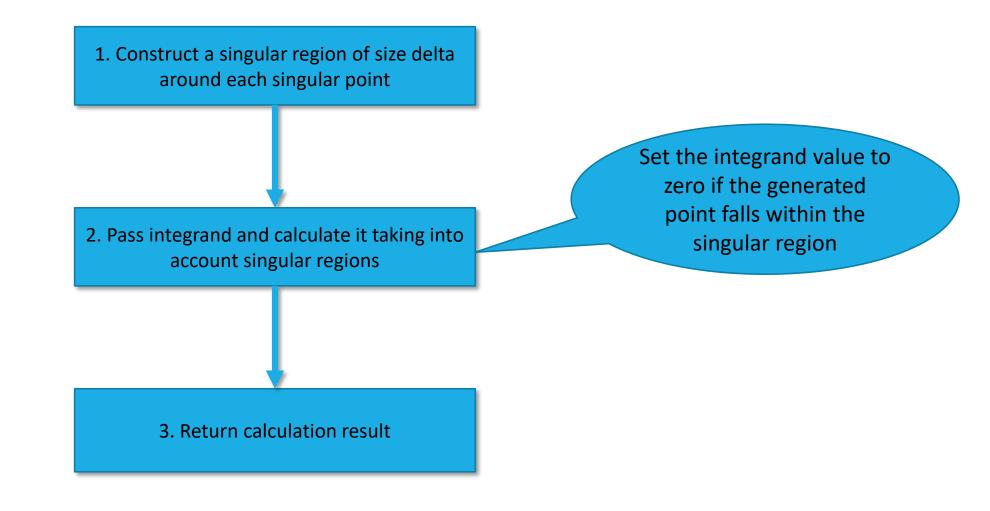




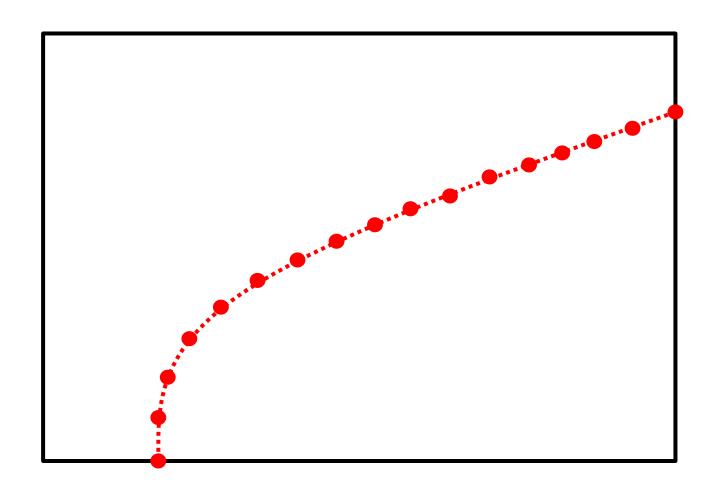




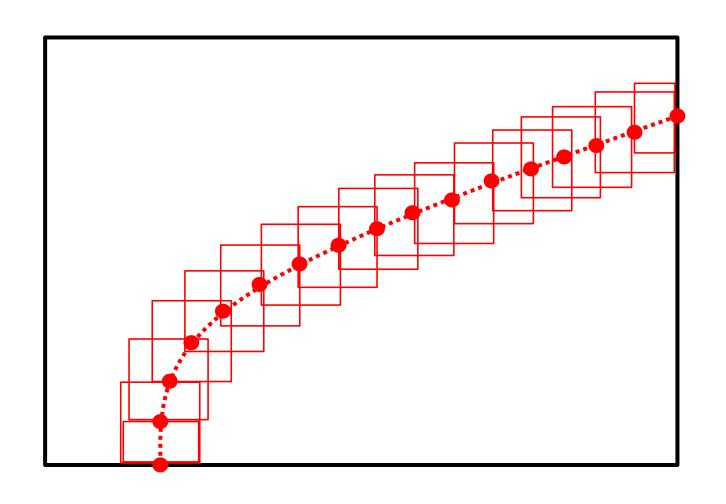
Calculation of the integral with GSL Monte Carlo VEGAS



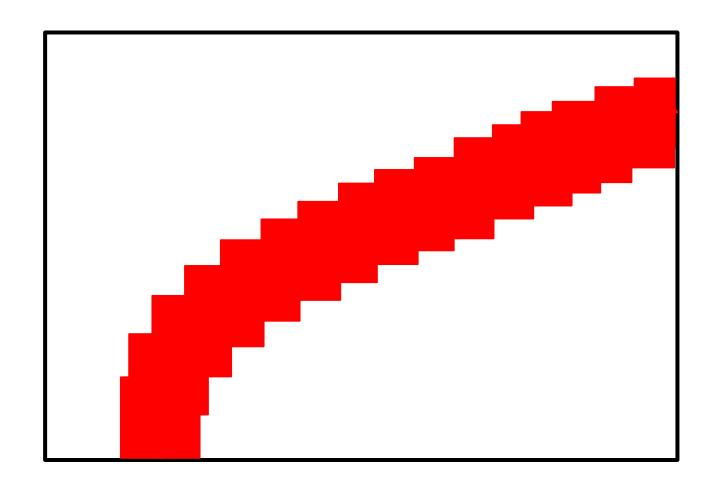
Make singular areas



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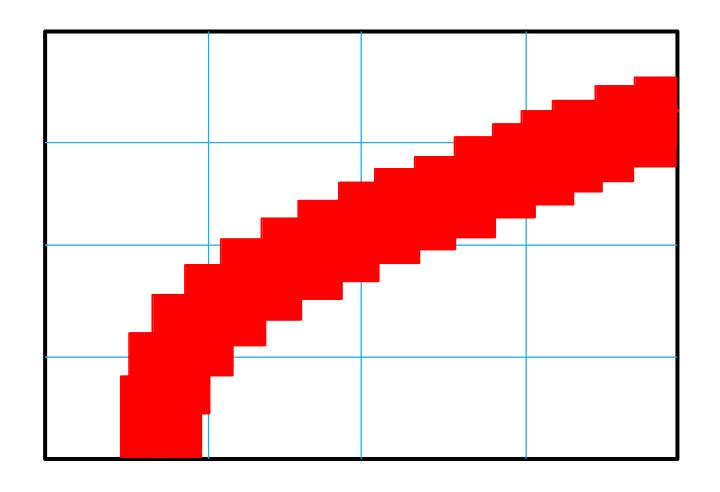
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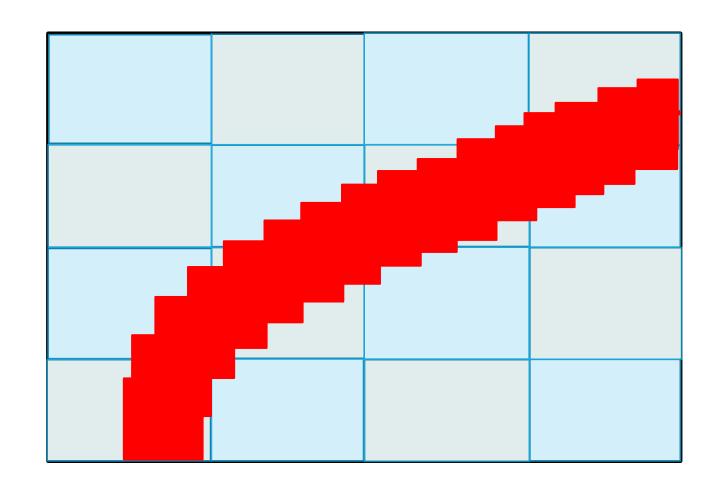
Add multithreading in calculation process

1. Construct a singular region of size delta around each singular point Set the integrand value to zero if the 2. Split the initial integration domain into segments generated point falls within the singular region 5. Pass integrand with integration domain and calculate it taking into account singular 3. For each segment, create a list of singular regions regions with which there are intersections 6. Sum results of all segments 4. Calculate each segment separately 7. Return calculation result

Calculate subsegments



Calculate subsegments



Test

$$\iint \frac{1}{y - x - 1} dx dy$$

$$x = [0:10]$$

$$y = [0:12]$$

 Ξ - comparison to not duplicate singular points

Intel Core i5-12500H Cores: 4 + 8 x 2.5 GHz + 1.8 GHz

 $\boldsymbol{\Delta}$ - exclusion area around singular points

Numerical answer: 2.0557374

 $\boldsymbol{\beta}\,$ - areas size around singular point in searching algorithm

						1 thread. 1 segment	16 threads. 36 segments
GSL MC VEGAS error	Ξ	Δ	В	Points number	Result	Time	Time
0.05	0.001	0.1	0.01	1001	2.056268	25 sec	4 sec
0.5	0.001	0.1	0.01	1001	2.054786	19 sec	4 sec
0.05	0.001	0.1	0.1	101	2.056857	8 sec	< 1sec
0.5	0.001	0.1	0.1	101	2.05573	5 sec	< 1 sec

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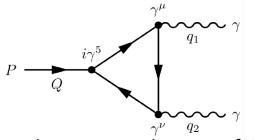
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Calculation of the decay width $\pi^0 \rightarrow \gamma \gamma$

The calculation of the two-photon pion decay width was performed in the framework of the Nambu-Jona-Lasinio model.

The decay amplitude is calculated as

$$T^{(1)}(P^2; q_1^2, q_2^2) = Q_H \int \frac{dp}{(2\pi)^4} \text{tr}\{\Gamma_H S_3 \gamma_\nu S_2 \gamma_\mu S_1\}$$



The Feynman diagram of the $\pi^0 o \gamma \gamma$ decay

with the pion vertex function $\Gamma_H \equiv \Gamma_\pi = (i\gamma_5)g_{\pi qq}$ and $Q_H = N_c(e_u^2 - e_d^2)e^2$ with e_u = 2/3 and e_d = -1/3 and after taking the trace, the amplitude can be written as:

$$T^{(1)}(P^2; q_1^2, q_2^2) = \epsilon_{\mu\nu\alpha\beta} \epsilon_1^{\mu} \epsilon_2^{\nu} q_1^{\alpha} q_2^{\beta} N_c \frac{4e^2}{3} g_{\pi qq} m I_3(P, q_1, q_2)$$

The width of the photoproduction is calculated by the simple relation:

$$\Gamma(\pi^0 \longrightarrow \gamma \gamma) = \frac{M_\pi^3}{64} |32\alpha \pi g_{\pi qq} m I_3(0, 0, P^2)|^2$$

$$I_3(q_1^2, q_2^2, P^2) = \int \frac{dp}{(2\pi)^4} \frac{1}{(p_0^2 - E^2)((p_0 - q_{10})^2 - E_1^2)((p_0 + q_{20})^2 - E_2^2)}$$

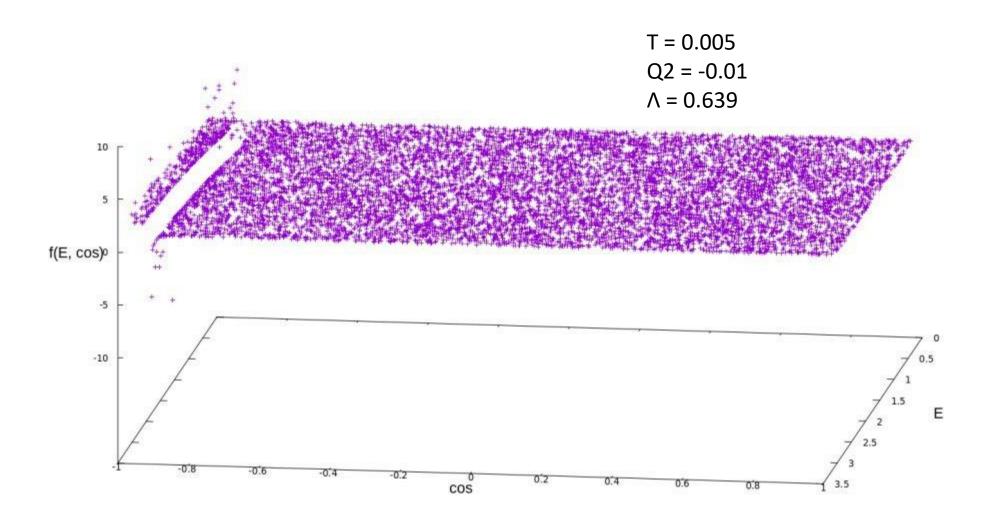
Integral I_3 at finite temperature

The main feature of the NJL model is the possibility to introduce finite temperature using the Matsubara summation. The integral I_3 at finite temperature splits into real and imaginary parts, and real part of the integral has a view:

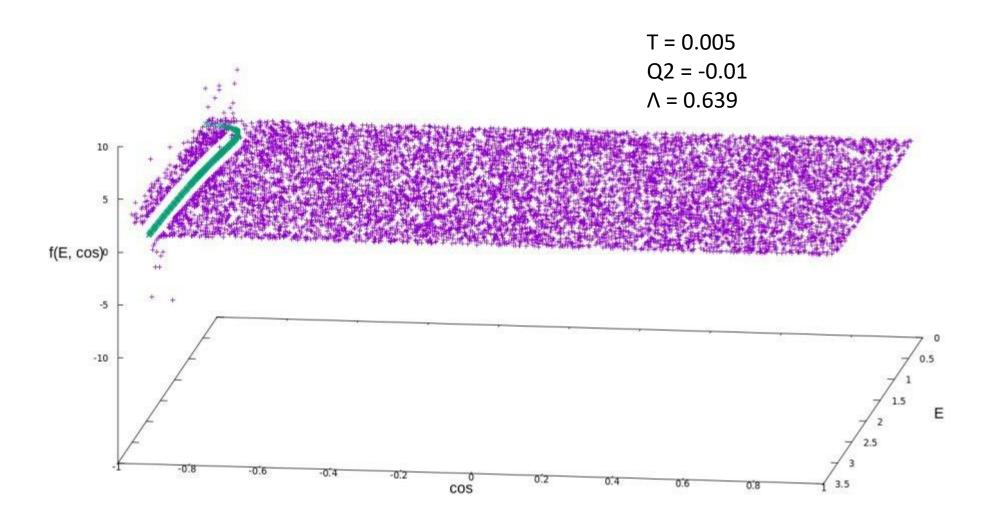
$$Rel_{3}$$

$$= \frac{1}{8\pi^{2}} \int_{m}^{\sqrt{\Lambda^{2}+m^{2}}} \sqrt{E^{2}-m^{2}} \int_{-1}^{1} d\cos\theta \left\{ f(E-\mu) \left[-\frac{1}{2Eq_{10}-2\bar{p}q_{10}\cos\theta} \frac{1}{2EP_{0}+q_{2}^{2}} - \frac{1}{2Eq_{20}+2\bar{p}q_{10}\cos\theta} \frac{1}{2EP_{0}+q_{2}^{2}} + \frac{1}{2Eq_{10}-2\bar{p}q_{10}\cos\theta} \frac{1}{2EP_{0}+P_{0}^{2}} \right] \right\}$$

Calculation of the integral $I_3(P^2, 0, 0; T)$



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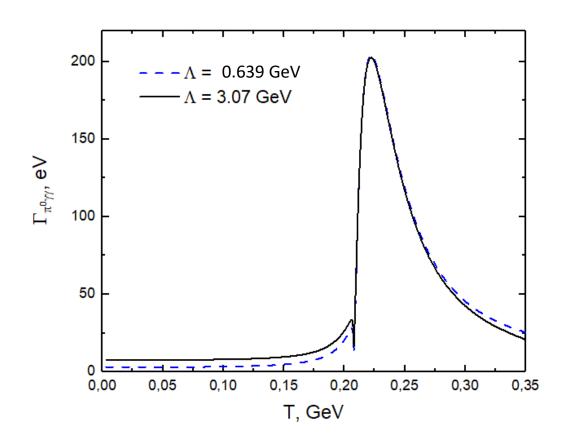


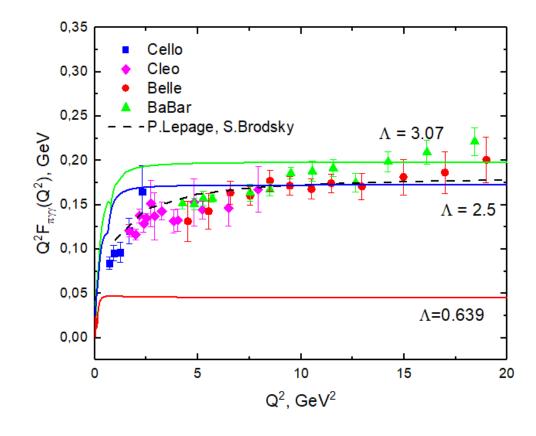
The decay width and transition form factor of $\pi^0 \to \gamma \gamma$

$$\Gamma(\pi^0 \longrightarrow \gamma \gamma) = \frac{M_\pi^3}{64} |32\alpha \pi g_{\pi qq} m I_3(0, 0, P^2)|^2$$

$$|F_{\pi\gamma}(0)|^2 = \frac{64\pi}{(4\pi\alpha)^2} \frac{\Gamma_{\pi^0\gamma\gamma}}{M_{\pi}^3}$$
 $F_{\pi\gamma}(Q^2) = 8g_{\pi qq} m I_3(0, Q^2, P^2)$

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Summary

- Algorithm works and gives quite accurate results
- Parallelization allows to significantly reduce the calculation time
- The method was applied to calculate $\pi^0 \rightarrow \gamma \gamma$ problem
- The method will be applied for further calculations