Spin and orbital structure of $f\bar{f}$ in scalar and pseudoscalar decays

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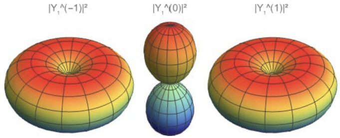
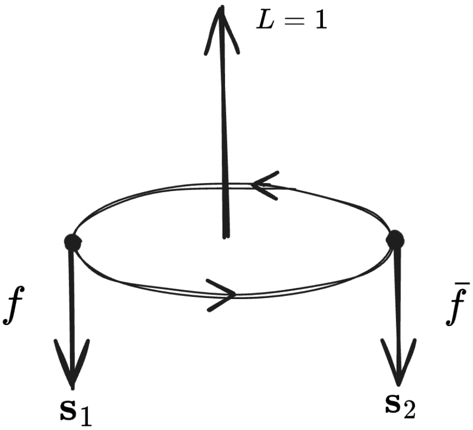
Motivation

- Spin-0 particles (e.g., Higgs-like bosons) may have uncertain CP parity (scalar 0^+, pseudoscalar 0^-, or a mixture).
- The CP nature determines the orbital angular momentum (OAM) and spin configuration of the fermion-antifermion ($f\bar{f}$) decay pair.
    - Scalar (0^+) $\to$ P-wave ($L = 1$) with spins correlated.
    - Pseudoscalar (0^-) $\to$ S-wave ($L = 0$) with spins anticorrelated.
- We introduce a method to determine CP parity by measuring **spin-orbital correlations** in the decay.
    - Predominantly **parallel spin** outcomes indicate a scalar.
    - Predominantly **antiparallel spin** outcomes indicate a pseudoscalar.
- The $L = 1$ orbital structure is encoded in **spherical harmonics**, making the angular dependence of the wavefunction a direct probe.

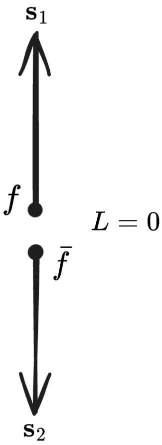


Scalar vs Pseudoscalar

Scalar



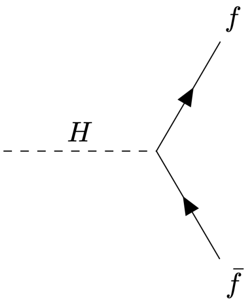
Pseudoscalar



Definitions

For a decay into a pair of fermion f and antifermion $\bar{f}$ with spin 3-vectors $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$, respectively.

$$H \to f + \bar{f}, \tag{1}$$



Final state in a momentum space

$$|f\bar{f}\rangle = \int \frac{d\boldsymbol{p}_1}{(2\pi)^3} \frac{d\boldsymbol{p}_2}{(2\pi)^3} \frac{\widetilde{\psi}_{r_1 r_2}(\boldsymbol{p}_1, \boldsymbol{p}_2)}{\sqrt{2E_{\boldsymbol{p}_1} 2E_{\boldsymbol{p}_2}}} |\boldsymbol{p}_1, r_1; \boldsymbol{p}_2, r_2\rangle, \tag{2}$$

Where two particle state

$$|\boldsymbol{p}_1, r_1; \boldsymbol{p}_2, r_2\rangle = \sqrt{2E_{\boldsymbol{p}_1} 2E_{\boldsymbol{p}_2}} a^\dagger_{r_1}(\boldsymbol{p}_1) b^\dagger_{r_2}(\boldsymbol{p}_2) |0\rangle, \tag{3}$$

Momentum-space wavefunction

Since the S-matrix is the evolution operator, projecting the quantum state $S|H\rangle$ onto the two-particle Fock state $\langle \boldsymbol{p}_1, r_1; \boldsymbol{p}_2, r_2|$ yields the wave function in momentum space:

$$\widetilde{\psi}_{r_1 r_2}(\boldsymbol{p}_1, \boldsymbol{p}_2) = \frac{\langle \boldsymbol{p}_1, r_1; \boldsymbol{p}_2, r_2|S|H\rangle}{\sqrt{2E_{\boldsymbol{p}_1}\, 2E_{\boldsymbol{p}_2}}} = (2\pi)^4 \delta^4(P - p_1 - p_2)\, \frac{i\mathcal{M}_{r_1 r_2}(\boldsymbol{p}_1, \boldsymbol{p}_2)}{\sqrt{2E_{\boldsymbol{p}_1}\, 2E_{\boldsymbol{p}_2}}}\,, \tag{4}$$

where P is the four–momentum of the Higgs boson. The plane–wave S–matrix element is calculated at leading order in the Standard Model.

$$i\mathcal{M}_{r_1 r_2}(\boldsymbol{p}_1, \boldsymbol{p}_2) = i\, \frac{m_f}{v}\, \overline{u}_{r_1}(\boldsymbol{p}_1)\, v_{r_2}(\boldsymbol{p}_2)\,, \tag{5}$$

Coordinate-space wavefunction

The wave function ψ in the configuration space can be found as the Fourier transform of $\widetilde{\psi}$:

$$\psi_{r_1 r_2}(\boldsymbol{x}_1, \boldsymbol{x}_2) = \int \frac{d\boldsymbol{p}_1}{(2\pi)^3} \frac{d\boldsymbol{p}_2}{(2\pi)^3} \widetilde{\psi}_{r_1 r_2}(\boldsymbol{p}_1, \boldsymbol{p}_2)\, e^{+i(\boldsymbol{p}_1 \cdot \boldsymbol{x}_1 + \boldsymbol{p}_2 \cdot \boldsymbol{x}_2)}. \tag{6}$$

Using Eqs. (6), (4), and (5), we obtain

$$\psi_{r_1 r_2}(\boldsymbol{r}) = \psi_{r_1 r_2}(\boldsymbol{x}_1, \boldsymbol{x}_2) = i\, \frac{m_f\, p}{16\pi^2 v} \int d\Omega_{\boldsymbol{n}}\, e^{ip\, \boldsymbol{n}\cdot\boldsymbol{r}}\, \overline{u}_{r_1}(p\boldsymbol{n})\, v_{r_2}(-p\boldsymbol{n})\,, \tag{7}$$

where $\boldsymbol{r} = \boldsymbol{x}_1 - \boldsymbol{x}_2$ is the relative coordinate, $E_p = m_H/2$, $p = \sqrt{E_p^2 - m_f^2}$, and the integral is taken over the unit sphere $d\Omega_{\boldsymbol{n}}$.

Coordinate-space wavefunction

Wave function as a rotated combination of the z-basis components

$$\psi_{\rho_1\rho_2}(\boldsymbol{r};\boldsymbol{s}_1,\boldsymbol{s}_2) = \sum_{r_1,r_2=\pm} D^{(1/2)}_{r_1\rho_1}(\boldsymbol{s}_1)\, D^{(1/2)}_{r_2\rho_2}(\boldsymbol{s}_2)\, \psi_{r_1 r_2}(\boldsymbol{r}), \tag{8}$$

where for $\boldsymbol{s}_i = (\sin\theta_i\cos\phi_i, \sin\theta_i\sin\phi_i, \cos\theta_i)$
SU(2) rotation that maps z to s (Euler angles $\phi, \theta, 0$)

$$\begin{aligned} D^{(1/2)}(\boldsymbol{s}_i) &= e^{-i\phi_i\sigma_z/2}\, e^{-i\theta_i\sigma_y/2} = \\ &= \begin{pmatrix} \cos\frac{\theta_i}{2} & -e^{-i\phi_i}\sin\frac{\theta_i}{2} \\ e^{i\phi_i}\sin\frac{\theta_i}{2} & \cos\frac{\theta_i}{2} \end{pmatrix}. \end{aligned} \tag{9}$$

Coordinate-space wavefunction

One gets the wave function in the configuration space (where $\hat{\boldsymbol{r}} = \boldsymbol{r}/r$):

$$\psi_{\rho_1\rho_2}(\boldsymbol{r};\boldsymbol{s}_1,\boldsymbol{s}_2) = \frac{m_f\, p^2}{2\pi v}\chi^\dagger_{\rho_1}(\boldsymbol{s}_1)\,(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}})\,\eta_{\rho_2}(\boldsymbol{s}_2)\,j_1(pr) \tag{10}$$

Thus,

$$|\psi_{\rho_1\rho_2}(\boldsymbol{r};\boldsymbol{s}_1,\boldsymbol{s}_2)|^2 = \frac{1}{2}\left(\frac{m_f\, p^2}{2\pi v}\right)^2 j_1^2(pr)\left[1 \;+\; \rho_1\rho_2\big(\boldsymbol{s}_1\cdot\boldsymbol{s}_2 - 2(\boldsymbol{s}_1\cdot\hat{\boldsymbol{r}})(\boldsymbol{s}_2\cdot\hat{\boldsymbol{r}})\big)\right]. \tag{11}$$

Results and Discussion

Using the identity

$$\boldsymbol{\sigma} \cdot \hat{\boldsymbol{r}} = \sqrt{\frac{4\pi}{3}} \sum_{m=-1}^{+1} Y_{1m}(\hat{\boldsymbol{r}})\, \sigma_m^{(1)}, \tag{12}$$

where

$$\sigma_0^{(1)} = \sigma_z, \quad \sigma_{\pm 1}^{(1)} = \mp \frac{\sigma_x \pm i\sigma_y}{\sqrt{2}}$$

the wave function in Eq.(28) can be written as

$$\psi_{\rho_1 \rho_2}(\boldsymbol{r}; \boldsymbol{s}_1, \boldsymbol{s}_2) \propto \sum_{m=-1}^{+1} Y_{1m}(\hat{\boldsymbol{r}})\, A_m(\boldsymbol{s}_1, \boldsymbol{s}_2), \tag{13}$$

where

$$A_m(\boldsymbol{s}_1, \boldsymbol{s}_2) = \chi^\dagger_{\rho_1}(\boldsymbol{s}_1)\, \sigma_m^{(1)}\, \eta_{\rho_2}(\boldsymbol{s}_2)$$

One can see that its spatial part is $L = 1$ wave. The coefficients A_m are purely spin factors; they fix the mixture over $m = -1, 0, +1$ for any chosen $\boldsymbol{s}_{1,2}$ and outcomes $\rho_{1,2} = \pm 1$.

Benchmark orientations and m-content

Pauli spinors:

$$
\begin{aligned}
\chi_+(\boldsymbol{s}) &= \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}, & \chi_-(\boldsymbol{s}) &= \begin{pmatrix} -\,e^{-i\phi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix} \\
\eta_+(\boldsymbol{s}) &= \begin{pmatrix} e^{-i\phi}\sin\frac{\theta}{2} \\ -\,\cos\frac{\theta}{2} \end{pmatrix}, & \eta_-(\boldsymbol{s}) &= \begin{pmatrix} \cos\frac{\theta}{2} \\ -\,e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}.
\end{aligned}
\tag{14}
$$

When $\boldsymbol{s}_1 = \boldsymbol{s}_2 \equiv \hat{\boldsymbol{s}}$ it is natural to choose the orbital quantization axis along $\hat{\boldsymbol{s}}$. In this basis

$$
A_m(\hat{\boldsymbol{s}}, \hat{\boldsymbol{s}}) = \chi^\dagger_{\rho_1}(\hat{\boldsymbol{s}})\,\sigma^{(1)}_m\,\eta_{\rho_2}(\hat{\boldsymbol{s}})
$$

obeys the simple selection rules

$$
A_0 \neq 0 \;\text{ if }\; \rho_1 = -\rho_2, \qquad A_{\pm 1} \neq 0 \;\text{ if }\; \rho_1 = \rho_2, \tag{15}
$$

Benchmark orientations and m-content (Case 1)

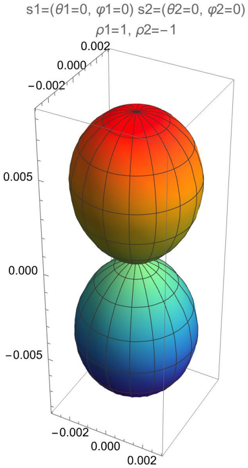
Evaluating at two extrema of the polar angle θ of $\hat{\boldsymbol{r}}$ with respect to $\hat{\boldsymbol{s}}$ gives:

1. $\boldsymbol{s}_1 = \boldsymbol{s}_2 = \hat{\boldsymbol{r}}$ $(\theta = 0)$: only the anti-parallel outcomes survive,

$$|\psi|^2 \propto (1 - \rho_1\rho_2),$$

and from (15) this projects onto the *pure* $m = 0$ component,

$$\psi \propto j_1(pr)\, Y_{1,0}(\hat{\boldsymbol{r}}; \hat{\boldsymbol{s}}).$$



Benchmark orientations and m-content (Case 2)

Evaluating at two extrema of the polar angle θ of $\hat{\boldsymbol{r}}$ with respect to $\hat{\boldsymbol{s}}$ gives:

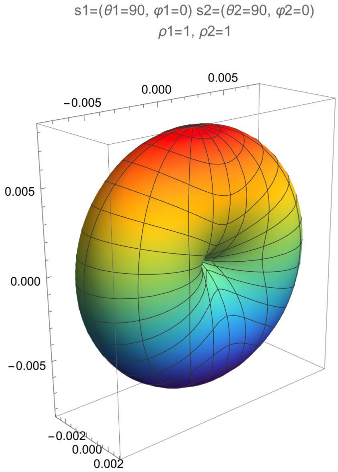
2. $\boldsymbol{s}_1 = \boldsymbol{s}_2 \perp \hat{\boldsymbol{r}}$ $(\theta = \pi/2)$: only the parallel outcomes survive,

$$|\psi|^2 \propto (1 + \rho_1\rho_2),$$

and (15) selects the *pure* $m = \pm 1$ sector,

$$\psi \propto j_1(pr)\left[\, Y_{1,+1}(\hat{\boldsymbol{r}}; \hat{\boldsymbol{s}}) \text{ or } Y_{1,-1}(\hat{\boldsymbol{r}}; \hat{\boldsymbol{s}})\,\right],$$

with the sign correlated with (ρ_1, ρ_2) by the chosen spinor phases.



Pseudoscalar (0^-) case

For a pseudoscalar coupling the amplitude is

$$i\mathcal{M}_{\rho_1\rho_2} = i\,\frac{m_f}{v}\,\overline{u}_{\rho_1}(p\boldsymbol{n},\boldsymbol{s}_1)\,\gamma^5\,v_{\rho_2}(-p\boldsymbol{n},\boldsymbol{s}_2), \tag{16}$$

The coordinate–space wave function is therefore

$$\psi_{\rho_1\rho_2}(\boldsymbol{r};\boldsymbol{s}_1,\boldsymbol{s}_2) = \frac{m_f E_p}{2\pi v}\,j_0(pr)\,\chi^\dagger_{\rho_1}(\boldsymbol{s}_1)\,\eta_{\rho_2}(\boldsymbol{s}_2), \tag{17}$$

a pure $L=0$ partial wave. Its modulus squared reads

$$|\psi_{\rho_1\rho_2}|^2 = \left(\frac{m_f E_p}{2\pi v}\right)^2 j_0^2(pr)\,\frac{1}{2}\Big[1-\rho_1\rho_2\,\boldsymbol{s}_1\cdot\boldsymbol{s}_2\Big]. \tag{18}$$

Mixed state of scalar and pseudoscalar parts

For $\varepsilon_1, \varepsilon_2 \in \mathbb{C}$ we can decompose $|\psi|^2$ using this identity

$$
\begin{aligned}
|\psi|^2 &= |\varepsilon_1 \psi_S + \varepsilon_2 \psi_P|^2 \\
&= |\varepsilon_1|^2 |\psi_S|^2 + |\varepsilon_2|^2 |\psi_P|^2 + 2\Re(\varepsilon_1 \varepsilon_2^* \psi_S \psi_P^*)
\end{aligned}
\tag{19}
$$

Full expression:

$$
\begin{aligned}
|\psi|^2 = \frac{m_f^2}{4\pi^2 v^2} \Big[& |\varepsilon_1|^2 p^4 j_1^2(pr) \left[1 + \rho_1 \rho_2 \left(s_1 \cdot s_2 - 2(\boldsymbol{s}_1 \cdot \hat{\boldsymbol{r}})(\boldsymbol{s}_2 \cdot \hat{\boldsymbol{r}})\right)\right] \\
& + |\varepsilon_2|^2 E_p^2 j_0^2(pr) \left[1 - \rho_1 \rho_2 \left(\boldsymbol{s}_1 \cdot \boldsymbol{s}_2\right)\right] \\
& + \varepsilon_1 \varepsilon_2^* j_0^2(pr) j_1^2(pr) \rho_1 \rho_2 \, \hat{\boldsymbol{r}} \cdot (\boldsymbol{s}_2 \times \boldsymbol{s}_1) \Big]
\end{aligned}
\tag{20}
$$

Conclusion

- We constructed the coordinate-space wave function of a fermion–antifermion pair produced in two-body decays of a spin-zero parent.
- For a scalar 0^+ coupling we obtained a pure $L = 1$ partial wave, $\psi(\boldsymbol{r}) \propto j_1(pr)\,\chi^\dagger(\boldsymbol{s}_1)\,(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}})\,\eta(\boldsymbol{s}_2)$,
- For a pseudoscalar (0^-) coupling the spinor bilinear is independent of the emission direction, yielding $\psi(\boldsymbol{r}) \propto j_0(pr)\,\chi^\dagger(\boldsymbol{s}_1)\eta(\boldsymbol{s}_2)$, a pure $L = 0$ (S-wave) with the standard singlet spin factor.
- For mixed CP parity states, interference terms appear that are sensitive to CP-violating phases.
- We clarified why the back-to-back momenta of the final particles do not contradict nonzero orbital angular momentum: L is a property of the relative coordinate.

Derivation of scalar wavefunction I

Using Eqs. (6), (4), and (5), we obtain

$$\begin{aligned}\psi_{r_1 r_2}(\boldsymbol{r}) &= \psi_{r_1 r_2}(\boldsymbol{x}_1, \boldsymbol{x}_2) \\ &= i\,\frac{m_f\,p}{16\pi^2 v}\int d\Omega_{\boldsymbol{n}}\; e^{\,ip\,\boldsymbol{n}\cdot\boldsymbol{r}}\,\overline{u}_{r_1}(p\boldsymbol{n})\,v_{r_2}(-p\boldsymbol{n})\,,\end{aligned} \tag{21}$$

where $\boldsymbol{r} = \boldsymbol{x}_1 - \boldsymbol{x}_2$ is the relative coordinate, $E_p = m_H/2$, $p = \sqrt{E_p^2 - m_f^2}$, and the integral is taken over the unit sphere $d\Omega_{\boldsymbol{n}}$.

With help of Wigner rotation one can define wave function in the configuration space assuming arbitrary directions of spin vectors:

$$\psi_{\rho_1\rho_2}(\boldsymbol{r}; \boldsymbol{s}_1, \boldsymbol{s}_2) = \sum_{r_1, r_2 = \pm} D^{(1/2)}_{r_1\rho_1}(\boldsymbol{s}_1)\, D^{(1/2)}_{r_2\rho_2}(\boldsymbol{s}_2)\, \psi_{r_1 r_2}(\boldsymbol{r}), \tag{22}$$

Derivation of scalar wavefunction II

where for $\boldsymbol{s}_i = (\sin\theta_i \cos\phi_i, \sin\theta_i \sin\phi_i, \cos\theta_i)$

$$D^{(1/2)}(\boldsymbol{s}_i) = e^{-i\phi_i\sigma_z/2}\, e^{-i\theta_i\sigma_y/2} = \begin{pmatrix} \cos\frac{\theta_i}{2} & -\,e^{-i\phi_i}\sin\frac{\theta_i}{2} \\ e^{i\phi_i}\sin\frac{\theta_i}{2} & \cos\frac{\theta_i}{2} \end{pmatrix}. \tag{23}$$

Therefore:

$$\psi_{\rho_1\rho_2}(\boldsymbol{r};\boldsymbol{s}_1,\boldsymbol{s}_2) = i\,\frac{m_f\,p}{16\pi^2 v}\int d\Omega_{\boldsymbol{n}}\; e^{\,ip\boldsymbol{n}\cdot\boldsymbol{r}}\,\overline{u}_{\rho_1}(p\boldsymbol{n},\boldsymbol{s}_1)\,v_{\rho_2}(-p\boldsymbol{n},\boldsymbol{s}_2), \tag{24}$$

where

$$u_\rho(p\boldsymbol{n},\boldsymbol{s}) = \sum_{r=\pm} D^{(1/2)}_{r\rho}(\boldsymbol{s})\,u_r(p\boldsymbol{n}) = \begin{pmatrix} \sqrt{E_p+m_f}\;\chi_\rho(\boldsymbol{s}) \\ \sqrt{E_p-m_f}\;(\boldsymbol{\sigma}\cdot\boldsymbol{n})\;\chi_\rho(\boldsymbol{s}) \end{pmatrix} \tag{25}$$

and

$$v_\rho(-p\boldsymbol{n},\boldsymbol{s}) = \sum_{r=\pm} D^{(1/2)}_{r\rho}(\boldsymbol{s})\,v_r(-p\boldsymbol{n}) = \begin{pmatrix} -\sqrt{E_p-m_f}\;(\boldsymbol{\sigma}\cdot\boldsymbol{n})\;\eta_\rho(\boldsymbol{s}) \\ \sqrt{E_p+m_f}\;\eta_\rho(\boldsymbol{s}) \end{pmatrix}. \tag{26}$$

Derivation of scalar wavefunction III

The Pauli spinors $\chi_\rho(\boldsymbol{s})$ and $\eta_\rho(\boldsymbol{s})$ are given by Eqs. (30). The product

$$\overline{u}_{\rho_1}(p\boldsymbol{n}, \boldsymbol{s}_1)\, v_{\rho_2}(-p\boldsymbol{n}, \boldsymbol{s}_2) = -2p\, \chi^\dagger_{\rho_1}(\boldsymbol{s}_1)\, (\boldsymbol{\sigma} \cdot \boldsymbol{n})\, \eta_{\rho_2}(\boldsymbol{s}_2)$$

under the integral determines the wave function in the configuration space.
Taking the angular integral in Eq. (24):

$$\int d\Omega_{\boldsymbol{n}}\, e^{ip\boldsymbol{n}\cdot\boldsymbol{r}}\, \boldsymbol{n} = 4\pi i\, j_1(pr)\hat{\boldsymbol{r}}, \qquad \hat{\boldsymbol{r}} = \boldsymbol{r}/r, \tag{27}$$

One gets the wave function in the configuration space:

$$\psi_{\rho_1\rho_2}(\boldsymbol{r}; \boldsymbol{s}_1, \boldsymbol{s}_2) = \frac{m_f\, p^2}{2\pi v} \chi^\dagger_{\rho_1}(\boldsymbol{s}_1)\, (\boldsymbol{\sigma} \cdot \hat{\boldsymbol{r}})\, \eta_{\rho_2}(\boldsymbol{s}_2)\, j_1(pr) \tag{28}$$

Let $\chi_r(\mathrm{s})$ be the normalized eigen-spinor of $\boldsymbol{\sigma}\cdot\mathrm{s}$ with eigenvalue $r = \pm 1$ (spin "up/down" along s), while $\eta_r(\mathrm{s}) = i\sigma^2\chi_r^*(\mathrm{s})$ is defined for the antifermion Pauli spinor. Then the polarization matrices read

$$\begin{aligned} \chi_r(\mathrm{s})\,\chi_r^\dagger(\mathrm{s}) &= \frac{1}{2}\Big(1 + r\,\mathrm{s}\cdot\boldsymbol{\sigma}\Big), \\ \eta_r(\mathrm{s})\,\eta_r^\dagger(\mathrm{s}) &= \frac{1}{2}\Big(1 - r\,\mathrm{s}\cdot\boldsymbol{\sigma}\Big) \end{aligned} \tag{29}$$

Pauli spinors read:

$$\begin{aligned} \chi_+(\boldsymbol{s}) &= \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}, \qquad \chi_-(\boldsymbol{s}) &= \begin{pmatrix} -\,e^{-i\phi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix} \\ \eta_+(\boldsymbol{s}) &= \begin{pmatrix} e^{-i\phi}\sin\frac{\theta}{2} \\ -\,\cos\frac{\theta}{2} \end{pmatrix}, \qquad \eta_-(\boldsymbol{s}) &= \begin{pmatrix} \cos\frac{\theta}{2} \\ -\,e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}. \end{aligned} \tag{30}$$

Derivation of $|\psi|^2$ II

Then,

$$
\begin{aligned}
|\psi_{\rho_1\rho_2}|^2 &\propto \mathrm{Tr}\Big[\chi_{\rho_1}(\boldsymbol{s}_1)\chi^\dagger_{\rho_1}(\boldsymbol{s}_1)(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}})\eta_{\rho_2}(\boldsymbol{s}_2)\eta^\dagger_{\rho_2}(\boldsymbol{s}_2)(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}})\Big] \\
&= \frac{1}{4}\mathrm{Tr}\big[(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}})^2 - \rho_1\rho_2(\boldsymbol{\sigma}\cdot\boldsymbol{s}_1)(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}})(\boldsymbol{\sigma}\cdot\boldsymbol{s}_2)(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}})\big] \\
&= \frac{1}{2}\left[1 - \rho_1\rho_2\Big((\boldsymbol{s}_1\cdot\hat{\boldsymbol{r}})(\boldsymbol{s}_2\cdot\hat{\boldsymbol{r}}) - (\boldsymbol{s}_1\times\hat{\boldsymbol{r}})\cdot(\boldsymbol{s}_2\times\hat{\boldsymbol{r}})\Big)\right] \\
&= \frac{1}{2}\left[1 + \rho_1\rho_2\Big((\boldsymbol{s}_1\cdot\boldsymbol{s}_2) - 2(\boldsymbol{s}_1\cdot\hat{\boldsymbol{r}})(\boldsymbol{s}_2\cdot\hat{\boldsymbol{r}})\Big)\right]
\end{aligned}
\tag{31}
$$

Derivation for a mixed state I

For $\varepsilon_1, \varepsilon_2 \in \mathbb{C}$ we can decompose $|\psi|^2$ using this identity

$$\begin{aligned} |\psi|^2 &= |\varepsilon_1 \psi_S + \varepsilon_2 \psi_P|^2 \\ &= |\varepsilon_1|^2 |\psi_S|^2 + |\varepsilon_2|^2 |\psi_P|^2 + 2\Re(\varepsilon_1 \varepsilon_2^* \psi_S \psi_P^*) \end{aligned} \tag{32}$$

For the interference term we obtain

$$\psi_S \psi_P^* \propto i\, j_0(pr) j_1(pr) \underbrace{\chi^\dagger_{\rho_1}(\boldsymbol{s}_1)(\boldsymbol{\sigma} \cdot \hat{\boldsymbol{r}}) \eta_{\rho_2}(\boldsymbol{s}_2)}_{S} \underbrace{\eta^\dagger_{\rho_2}(\boldsymbol{s}_2) \chi_{\rho_1}(\boldsymbol{s}_1)}_{P^*}$$

Using (29) we get

$$\begin{aligned} SP^* &= \chi^\dagger_{\rho_1}(\boldsymbol{s}_1)(\boldsymbol{\sigma} \cdot \hat{\boldsymbol{r}}) \frac{1}{2} \Big(1 - \rho_2\, \mathrm{s} \cdot \boldsymbol{\sigma} \Big) \chi_{\rho_1}(\boldsymbol{s}_1) = \\ &= \frac{1}{2} \chi^\dagger_{\rho_1}(\boldsymbol{s}_1)(\boldsymbol{\sigma} \cdot \hat{\boldsymbol{r}}) \chi_{\rho_1}(\boldsymbol{s}_1) - \frac{1}{2} \rho_2 \chi^\dagger_{\rho_1}(\boldsymbol{\sigma} \cdot \hat{\boldsymbol{r}})(\boldsymbol{\sigma} \cdot \boldsymbol{s}_2) \chi_{\rho_1}(\boldsymbol{s}_1) \end{aligned} \tag{33}$$

Some useful identities:

$$\begin{aligned}
\chi^\dagger_{\rho_1}(\boldsymbol{s}_1)(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}})\chi_{\rho_1}(\boldsymbol{s}_1) &= \rho_1(\hat{\boldsymbol{r}}\cdot\boldsymbol{s}_1) \\
(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}})(\boldsymbol{\sigma}\cdot\boldsymbol{s}_2) &= \hat{\boldsymbol{r}}\cdot\boldsymbol{s}_2 + i\boldsymbol{\sigma}\cdot(\hat{\boldsymbol{r}}\times\boldsymbol{s}_2) \\
\chi^\dagger_{\rho_1}(\boldsymbol{s}_1)(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{r}})(\boldsymbol{\sigma}\cdot\boldsymbol{s}_2)\chi_{\rho_1}(\boldsymbol{s}_1) &= \hat{\boldsymbol{r}}\cdot\boldsymbol{s}_2 + i\rho_1\boldsymbol{s}_1\cdot(\hat{\boldsymbol{r}}\times\boldsymbol{s}_2)
\end{aligned} \tag{34}$$

Taking (34) into account we get:

$$SP^* = \frac{1}{2}\rho_1(\hat{\boldsymbol{r}}\cdot\boldsymbol{s}_1) - \frac{1}{2}\rho_2(\hat{\boldsymbol{r}}\cdot\boldsymbol{s}_2) - \frac{i}{2}\rho_1\rho_2\boldsymbol{s}_1\cdot(\hat{\boldsymbol{r}}\times\boldsymbol{s}_2)$$

Thus

$$2\Re(\varepsilon_1\varepsilon_2^*\psi_S\psi_P^*) = \frac{m_f^2}{4\pi^2v^2}\varepsilon_1\varepsilon_2^* j_0^2(pr) j_1^2(pr)\rho_1\rho_2\boldsymbol{s}_1\cdot(\hat{\boldsymbol{r}}\times\boldsymbol{s}_2)$$

Derivation for a mixed state III

Full expression:

$$|\psi|^2 = \frac{m_f^2}{4\pi^2 v^2}\Big[|\varepsilon_1|^2 p^4 j_1^2(pr)\left[1 + \rho_1\rho_2\left(s_1 \cdot s_2 - 2(\boldsymbol{s}_1 \cdot \hat{\boldsymbol{r}})(\boldsymbol{s}_2 \cdot \hat{\boldsymbol{r}})\right)\right]$$
$$+ |\varepsilon_2|^2 E_p^2 j_0^2(pr)\left[1 - \rho_1\rho_2(\boldsymbol{s}_1 \cdot \boldsymbol{s}_2)\right]$$
$$+ \varepsilon_1\varepsilon_2^* j_0^2(pr) j_1^2(pr)\rho_1\rho_2 \boldsymbol{s}_1 \cdot (\hat{\boldsymbol{r}} \times \boldsymbol{s}_2)\Big] \tag{35}$$